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Explicit analytical continuation of the inhomogeneous Epstein zeta function

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Abstract

The two-dimensional inhomogeneous zeta-function series (with homogeneous part of the most general Epstein type):

$$\sum_{m,n\in\mathbf{Z}}{}'(am^2+bmn+cn^2+q)^{-s},$$

is analytically continued in the variable s by using zeta-function techniques. The result shows the pole structure explicitly, and is given in terms of a series in one index only. This extends a previous formula by S. Chowla and A. Selberg, while preserving for any $q>0$ the good convergence properties of the final series, which led these authors to obtain important applications of the formula in number theory and in the theory of elliptic functions. Further applications to several unsolved problems of mathematical physics are described.

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1 Introduction

In a paper by S. Chowla and A. Selberg published some years ago [1], these authors considered several applications of an interesting formula they had found for the analytical continuation of the general Epstein zeta function in two indices [2]

$$\sum_{m,n\in\mathbf{Z}}{}'(am^2+bmn+cn^2)^{-s} \tag{1}$$

(the prime means that the term with $m=n=0$ is to be excluded from the sum). The formula was given in that article without explicit derivation (which was promised to appear in a subsequent paper), and became to be named after their authors (see, for instance, [3]). As remarked by Chowla and Selberg, the extremely good convergence properties of the series of Bessel functions which appears in the formula, make it both simple to apply and very useful. In particular, it was employed in [1] to construct an easy proof of the famous conjecture by Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant (an alternative to the derivation of Heilbronn [4], which was based on earlier work by Deuring [5]), to demonstrate the positiveness of certain Dirichlet L-functions at $s=1/2$, and in a classical problem of the theory of elliptic functions, where the range of computability in finite terms of the standard K elliptic integral was extended considerably.

Guided by these results and by the necessity to obtain a similar formula for the case of the 'Epstein like' zeta function corresponding to a quadratic form plus a constant term —in order to deal with several open problems in mathematics and mathematical physics— we will here consider the double, doubly infinite series

$$E(s;a,b,c;q)\equiv\sum_{m,n\in\mathbf{Z}}{}'(am^2+bmn+cn^2+q)^{-s}. \tag{2}$$

With $q\neq 0$ (in general), the parenthesis in (2) is the inhomogeneous quadratic form

$$Q(x,y)+q,\qquad Q(x,y)\equiv ax^2+bxy+cy^2, \tag{3}$$

restricted to the integers. In the general theory that deals with the homogeneous case, one assumes that $a,c>0$ and that the discriminant

$$\Delta=4ac-b^2>0 \tag{4}$$

(see [1]). Here we will impose the additional condition that q be such that $Q(m,n)+q\neq 0$, $\forall m,n\in\mathbf{Z}$. In the usual applications of the theory, those conditions are indeed satisfied.

Before we attack this problem, and as a more simple application of the general formalism, we will first consider the case (also interesting for its many applications) of the analytical continuation of the series

$$G(s;a,c;q) \equiv \sum_{n=-\infty}^{+\infty} \left[a(n+c)^2+q\right]^{-s}. \tag{5}$$

The result for this case can be considered as a by-product of our general approach.

For the benefit of the reader, the derivation of the extended formula will be given in full detail. The paper concludes with two sections where the new formula is applied to a couple of specific, well-formulated mathematical problems of mathematical-physics origin.

2 The particular case of a one-dimensional Epstein-Hurwitz series

While deriving this case we shall already introduce the general procedure. Actually, the analytical continuation can be done in different ways, but maybe the most direct method is by using Jacobi's fundamental identity for the theta function θ_3,

$$\theta_3(z\,|\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q = e^{\pi i\tau}, \quad |q|<1, \quad \tau\in\mathbf{C}, \tag{6}$$

that is

$$\theta_3(z\,|\tau) = \frac{1}{\sqrt{-i\tau}} e^{z^2/(\pi i\tau)}\theta_3\left(\frac{z}{\tau}\,\middle|\,\frac{-1}{\tau}\right), \tag{7}$$

or, in other words

$$\sum_{n=-\infty}^{+\infty} e^{n^2\pi i\tau+2niz} = \frac{1}{\sqrt{-i\tau}}\sum_{n=-\infty}^{+\infty} e^{(z-n\pi)^2/(\pi i\tau)} \tag{8}$$

(for a popular reference see, for instance, Wittaker & Watson [6], p. 476). Here z and τ are arbitrary complex, $z,\tau\in\mathbf{C}$, with the only restriction that $\mathrm{Im}\,\tau>0$ (in order that $|q|<1$). For subsequent application, it turns out to be better to recast the Jacobi identity as follows (with $\pi i\tau\to -t$ and $z\to\pi z$):

$$\sum_{n=-\infty}^{+\infty} e^{-n^2t+2\pi inz} = \sqrt{\frac{\pi}{t}}\sum_{n=-\infty}^{+\infty} e^{-\pi^2(n-z)^2/t}, \tag{9}$$

equivalently

$$\sum_{n=-\infty}^{+\infty} e^{-(n+z)^2t} = \sqrt{\frac{\pi}{t}}\left[1+\sum_{n=1}^{\infty} e^{-\pi^2n^2/t}\cos(2\pi nz)\right], \tag{10}$$

where $z,t\in\mathbf{C}$, $\mathrm{Re}\,t>0$. This last expression will be the first ingredient in our calculation. Instead, we could have choosen to proceed by expanding in power series the exponent of the Mellin transform of the series above, and then by interchanging the order of summation of the two series, with the known prescription of adding the contribution of the corresponding contour integration on the complex plane —see Elizalde [8], where this general method was extensively applied (even in the case of arbitrary, non-negative exponents), and was shown to yield the Jacobi identity as a particular case (see Ref. [9] for a more extensive account). Actually, that both procedures should yield the same result in the quadratic case is well known from the direct proof given by Landsberg [7] of the Jacobi identity.

Another ingredient for our derivation is the gamma function identity. Applied to the abreviated expression $\sum(Q+q)^{-s}$ (which precise meaning corresponding to (2) or (5) can be left here undefined, for the moment), it yields

$$\sum(Q+q)^{-s} = \frac{1}{\Gamma(s)}\sum\int_0^{\infty} du\, u^{s-1}e^{-(Q+q)u}. \tag{11}$$

When $q>0$ and under the conditions that have been stated for the quadratic form Q, the integral and the sum commute. Before considering the general case, by setting first $Q=a(n+c)^2$, $a>0$, $c\in\mathbf{C}$, $c\neq 0,-1,-2,\ldots$ (of course, this is not a quadratic form and the series is now one-dimensional, but the same considerations about commutation of series and integral apply), we obtain the following expression (a consequence of the Jacobi and gamma function identity)

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}\left[a(n+c)^2+q\right]^{-s} &= \sqrt{\frac{\pi}{a}}\frac{\Gamma(s-1/2)}{\Gamma(s)}q^{1/2-s}+\frac{4\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}\\ &\quad\times\sum_{n=1}^{\infty} n^{s-1/2}\cos(2\pi nc)K_{s-1/2}(2\pi n\sqrt{q/a}),\end{aligned} \tag{12}$$

where K_ν is the modified Bessel function of the second kind. This equation is very useful, since it provides the analytic continuation of what can be called inhomogeneous, generalized Epstein series in one index, in terms of a series that turns out to be very quickly convergent for extense ranges of values of the parameters, in particular, when $\mathrm{Im}\,c<\sqrt{q/a}$, a case that frequently appears in the applications. An example of a result that can only be obtained by explict, non-trivial interchange of the summation indices and subsequent contour integration —and that it is *not* a consequence of the Jacobi identity— is the following:

$$\begin{aligned}F(s;a,c;q) \equiv \sum_{n=0}^{\infty}\left[a(n+c)^2+q\right]^{-s} &\sim \frac{q^{-s}}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+s)}{m!}\left(\frac{q}{a}\right)^{-m}\zeta_H(-2m,c)\\ &+\sqrt{\frac{\pi}{a}}\frac{\Gamma(s-1/2)}{2\Gamma(s)}q^{1/2-s}+\frac{2\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}\sum_{n=1}^{\infty} n^{s-1/2}\cos(2\pi nc)K_{s-1/2}(2\pi n\sqrt{q/a}).\end{aligned} \tag{13}$$

This function is studied in detail in Ref. [10] (with numerical tables, plots, and a couple of explicit applications). Notice, however, that this is not a convergent series but an asymptotic one (see Elizalde [8]). By observing that $\zeta_H(-2m,c)+\zeta_H(-2m,1-c)=0$, for $m\in\mathbf{N}$, and that $G(s;a,c;q)=F(s;a,c;q)+F(s;a,1-c;q)$, it is very easy to obtain (12) as a particular case of (13).

Those formulas give explicit answers to many situations that remained unsolved in zeta-function regularization. Only the case $c=0$ had been dealt with satisfactorily (through the corresponding expression obtained just putting $c=0$ in (12)). As a particular case of eq. (12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left[a(n+1/2)^2+q\right]^{-s} &= \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}q^{1/2-s}+\frac{2\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}\\ &\quad\times\sum_{n=1}^{\infty}(-1)^n n^{s-1/2}K_{s-1/2}(2\pi n\sqrt{q/a}).\end{aligned}\tag{14}$$

More involved multiple series of this kind, such as the diagonal, inhomogeneous, generalized Epstein (or Epstein-Hurwitz) multiple series

$$E_k(s;a_1,\ldots,a_k;c_1,\ldots,c_k;c)\equiv\sum_{n_1,\ldots,n_k\in\mathbf{Z}}\left[a_1(n_1+c_1)^2+\cdots+a_k(n_k+c_k)^2+c\right]^{-s},\tag{15}$$

can be treated in a recurrent way, starting from (12). The general recurrence is

$$\begin{aligned}E_k(s;a_1,\ldots,a_k;c_1,\ldots,c_k;c) &= \sqrt{\frac{\pi}{a_k}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_{k-1}(s;a_1,\ldots,a_{k-1};c_1,\ldots,c_{k-1};c)\\ &+\frac{4\pi^s}{\Gamma(s)}a_k^{-s/2-1/4}\sum_{n_1,\ldots,n_{k-1}\in\mathbf{Z}}\left[\sum_{j=1}^{k-1}a_j(n_j+c_j)^2+c\right]^{-s/2+1/4}\sum_{n_k=1}^{\infty}n_k^{s-1/2}\cos(2\pi n_k c_k)\\ &\times K_{s-1/2}\left(\frac{2\pi n_k}{\sqrt{a_k}}\sqrt{\sum_{j=1}^{k-1}a_j(n_j+c_j)^2+c}\right).\end{aligned}\tag{16}$$

This exact recurrence is very appropriate for numerical computation, since from the second term on the rhs, owing to the rapid exponential convergence of the Bessel function, only a few first terms need to be taken into account to achieve a good approximation. The recurrence is then implementable in any of the algebraic computational packages commonly available.

3 Derivation of the general formula

To deal with the case when Q is a two-dimensional quadratic form and the sum $\sum$ a double, doubly infinite series, we need a further ingredient, i.e., casting the quadratic form

$Q(n_1,n_2)$ as the sum of two squares

$$Q(n_1,n_2)=a\left[\left(n_1+\frac{bn_2}{2a}\right)^2+\frac{\Delta}{4a^2}n_2^2\right].\tag{17}$$

and then to proceed by considering first the summation over n_1 (this will be the sum to which the Jacobi identity (9) will be applied), while treating n_2 as a parameter. By doing so, performing then the change of variables

$$u=\frac{2\pi n_1}{\sqrt{\Delta}\,n_2}t,\tag{18}$$

and taking advantage of the common idea of rewriting the double sum as a sum over the product $n=n_1n_2$ and (a finite one) over the divisors of the product:

$$\sum_{n_1,n_2}\left(\frac{n_1}{n_2}\right)^{s-1/2}=\sum_{n_1,n_2}(n_1n_2)^{s-1/2}n_2^{1-2s}=\sum_n n^{s-1/2}\sum_{d|n}d^{1-2s},\tag{19}$$

the following expression is obtained

$$\begin{aligned}E(s;a,b,c;q) &= {\sum_{m,n\in\mathbf{Z}}}'[Q(m,n)+q]^{-s}={\sum_{m,n\in\mathbf{Z}}}'(am^2+bmn+cn^2+q)^{-s}\\ &= 2\zeta_{EH}(s,q/a)\,a^{-s}+\frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\Gamma(s-1/2)\zeta_{EH}(s-1/2,4aq/\Delta)\\ &+\frac{2^{s+3/2}\pi^s}{\Gamma(s)\Delta^{s/2-1/4}\sqrt{a}}\sum_{n=1}^{\infty}n^{s-1/2}\cos(n\pi b/a)\sum_{d|n}d^{1-2s}\int_0^{\infty}dt\,t^{s-3/2}\\ &\times\exp\left\{-\frac{\pi n\sqrt{\Delta}}{2a}\left[\left(1+\frac{4aq}{d^2\Delta}\right)t+t^{-1}\right]\right\},\end{aligned}\tag{20}$$

where the function $\zeta_{EH}(s,p)$ (the one dimensional Epstein-Hurwitz or inhomogeneous Epstein series) is given by

$$\begin{aligned}\zeta_{EH}(s;p) &= \sum_{n=1}^{\infty}\left(n^2+p\right)^{-s}=\frac{1}{2}{\sum_{n\in\mathbf{Z}}}'\left(n^2+p\right)^{-s}\\ &= -\frac{p^{-s}}{2}+\frac{\sqrt{\pi}\,\Gamma(s-1/2)}{2\,\Gamma(s)}p^{-s+1/2}+\frac{2\pi^s p^{-s/2+1/4}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-1/2}K_{s-1/2}(2\pi n\sqrt{p}),\end{aligned}\tag{21}$$

as a particular case of eq. (12) (or (13)).

Expression (20) —as (12)— can be written also in terms of modified Bessel functions of the second kind, to yield finally

$$E(s;a,b,c;q)=2\zeta_{EH}(s,q/a)\,a^{-s}+\frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\Gamma(s-1/2)\zeta_{EH}(s-1/2,4aq/\Delta)\tag{22}$$

$$+\frac{2^{s+5/2}\pi^s}{\Gamma(s)\Delta^{s-1/2}\sqrt{a}}\sum_{n=1}^{\infty} n^{s-1/2}\cos(n\pi b/a)\sum_{d|n} d^{1-2s}\left(\Delta+\frac{4aq}{d^2}\right)^{s/2-1/4} K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\right).$$

Eq. (22) provides the analytical continuation of the inhomogenous Epstein series, in the variable s, as a meromorphic function in the complex plane. Its pole structure is explicitly given in terms of the well-known pole structure of $\zeta_{EH}(s,p)$ (see, for instance, [10]).

Eq. (22) constitutes the fundamental result of this paper. I propose to call it the *extended Chowla-Selberg* formula, since it certainly contains the Chowla-Selberg formula as the particular case $q=0$, i.e.,

$$E(s;a,b,c;0) = 2\zeta(2s)\,a^{-s} + \frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\Gamma(s-1/2)\zeta(2s-1) + \frac{2^{s+3/2}\pi^s}{\Gamma(s)\Delta^{s/2-1/4}\sqrt{a}}$$
$$\times\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos(n\pi b/a)\int_0^{\infty} dt\, t^{s-3/2}\exp\left[-\frac{\pi n\sqrt{\Delta}}{2a}(t+t^{-1})\right], \qquad (23)$$

where

$$\sigma_s(n) \equiv \sum_{d|n} d^s, \qquad (24)$$

namely the sum over the s-powers of the divisors of n. (Notice that there is an error in the transcription of formula (23) in Ref. [3]).

Formula (22) has never appeared in the mathematical literature. The good convergence properties of expression (23), that were so much prised by Chowla and Selberg, are shared by its non-trivial extension (22). This renders the use of the formula quite simple. In fact, the two first terms are still rather nice —under the form (21)— while the last one (impressive in appearence) is even more quickly convergent than in the case of eq. (23), and thus absolutely harmless in practice. Only a few first terms of the three series of Bessel functions in (22), (21) need to be calculated, even if one demands good accuracy. We should also notice that the pole of (23) at $s=1$ appears through $\zeta(2s-1)$ in the second term, while for $s=1/2$, the apparent singularities of the first and second terms cancel each other and no pole is formed. Analogously, the pole at $s=1/2$ in (22) appears only from the first term. It is remarkable that eq. (22) also has these good properties, *for any non-negative value of* q. In fact, for large q the convergence properties of the series of Bessel functions are clearly enhanced, while for q small we get back to the case of Chowla and Selberg. Notice, however, that this is not obtained through the high-q expansion (12), but using the low-q, binomial expansion

$$\sum_{n=0}^{\infty}\left[a(n+c)^2+q\right]^{-s} = a^{-s}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+s)}{\Gamma(s)\,m!}\left(\frac{q}{a}\right)^m\zeta_H(2s+2m,c), \qquad (25)$$

which is convergent for $q/a \le 1$. For $q\to 0$ it reduces to $a^{-s}\zeta_H(2s,c)$. Actually, formula (22) is still valid in a domain of negative q's, namely for $q > -\min(a,c,a-b+c)$.

4 Critical behaviour of a field theory at non-zero temperature

Let us consider the effective potential of a quantum theory that has been studied recently, the phase-transition structure of which we want to analyze, in terms of the dimension D of the space where we are working. D is one of the variables of the theory, ranging from $D=2$ to $D=3$ (see, for instance, Muta and Inagaki [12]). For what concerns us here, we just need to know that the effective potential is given by the expression

$$V^{\beta\mu}(\sigma) = \frac{\sigma^2}{2\lambda} - \frac{\sqrt{\pi}}{\beta\,(2\pi)^{D/2}}\Gamma\left(\frac{1-D}{2}\right) \qquad (26)$$
$$\times\sum_{n=-\infty}^{+\infty}\left\{\left[\left(\frac{2n+1}{\beta}\pi - i\mu\right)^2\right]^{(D-1)/2} + \left[\left(\frac{2n+1}{\beta}\pi - i\mu\right)^2+\sigma^2\right]^{(D-1)/2}\right\}.$$

By direct application of the first of the equations considered above, eq. (12), we obtain

$$\sum_{n=-\infty}^{+\infty}\left[\left(\frac{2n+1}{\beta}\pi - i\mu\right)^2+\sigma^2\right]^{(D-1)/2} = \frac{\beta}{2\sqrt{\pi}}\frac{\Gamma(-D/2)}{\Gamma((1-D)/2)}\sigma^D$$
$$+\frac{\beta^{1-D/2}(2\sigma)^{D/2}}{\sqrt{\pi}\,\Gamma((1-D)/2)}\sum_{n=1}^{\infty}(-1)^n n^{-D/2}\left(e^{\beta\mu n}+e^{-\beta\mu n}\right)K_{-D/2}(\beta\sigma n), \qquad (27)$$

and

$$\sum_{n=-\infty}^{+\infty}\left[\left(\frac{2n+1}{\beta}\pi - i\mu\right)^2\right]^{(D-1)/2} \qquad (28)$$
$$= \left(\frac{2\pi}{\beta}\right)^{D-1}\left[\zeta\left(1-D,\frac{1}{2}-i\frac{\beta\mu}{2\pi}\right)+\zeta\left(1-D,-\frac{1}{2}+i\frac{\beta\mu}{2\pi}\right)-\left(\frac{1}{2}-i\frac{\beta\mu}{2\pi}\right)^{D-1}\right].$$

The effective potential becomes

$$V^{\beta\mu}(\sigma) = \frac{\sigma^2}{2\lambda} - \frac{\sqrt{\pi}}{\beta\,(2\pi)^{D/2}}\left\{\Gamma\left(\frac{1-D}{2}\right)\left(\frac{2\pi}{\beta}\right)^{D-1}\left[\zeta\left(1-D,\frac{1}{2}-i\frac{\beta\mu}{2\pi}\right)\right.\right.$$
$$\left.+\zeta\left(1-D,-\frac{1}{2}+i\frac{\beta\mu}{2\pi}\right)-\left(\frac{1}{2}-i\frac{\beta\mu}{2\pi}\right)^{D-1}\right]+\frac{\beta}{2\sqrt{\pi}}\Gamma\left(\frac{-D}{2}\right)\sigma^D$$
$$\left.+\frac{\beta^{1-D/2}(2\sigma)^{D/2}}{\sqrt{\pi}}\sum_{n=1}^{\infty}(-1)^n n^{-D/2}\left(e^{\beta\mu n}+e^{-\beta\mu n}\right)K_{D/2}(\beta\sigma n)\right\}. \qquad (29)$$

The equation for the extrema of the potential leads us to the critical points (phase transitions), i.e.

$$\frac{\partial V^{\beta\mu}(\sigma)}{\partial\sigma} = 0. \qquad (30)$$

This yields

$$\frac{\sigma}{\lambda}-(2\pi)^{-D/2}\left[\frac{1}{2}\Gamma\left(\frac{-D}{2}\right)D\sigma^{D-1}+\beta^{-D/2}D(2\sigma)^{D/2-1}\sum_{n=1}^{\infty}(-1)^n n^{-D/2}\left(e^{\beta\mu n}+e^{-\beta\mu n}\right)K_{D/2}(\beta\sigma n)\right.$$
$$\left.+\beta^{1-D/2}(2\sigma)^{D/2}\sum_{n=1}^{\infty}(-1)^n n^{1-D/2}\left(e^{\beta\mu n}+e^{-\beta\mu n}\right)K'_{D/2}(\beta\sigma n)\right]=0. \qquad (31)$$

The series are convergent when $|\mu|\leq\sigma$, for any D. In the first approximation, we have just

$$\frac{\sigma}{\lambda}-\frac{\Gamma\left(\frac{-D}{2}\right)D}{2\,(2\pi)^{D/2}}\sigma^{D-1}=0, \qquad (32)$$

that is

$$\sigma_0=\left[-\frac{2\,(2\pi)^{D/2}\Gamma(D/2+1)\sin(D\pi/2)}{\pi\,D\,\lambda}\right]^{1/(D-2)}. \qquad (33)$$

In the particular case when the number of dimensions is 5/2 (an intermediate situation), we obtain

$$\sigma_0|_{D=5/2}=\frac{\Gamma(1/4)^2\sqrt{\pi}}{2^{5/2}\lambda^2}=\frac{4.1187}{\lambda^2}. \qquad (34)$$

The self-consistency condition is $\sigma_0\geq|\mu|$, namely,

$$|\mu|\leq\left[-\frac{2\,(2\pi)^{D/2}\Gamma(D/2+1)\sin(D\pi/2)}{\pi\,D\,\lambda}\right]^{1/(D-2)}. \qquad (35)$$

(Notice that, for $D=5/2$, this means $|\mu|<4.12$ for $\lambda=1$).

The second approximation yields, with $\sigma=\sigma_0+\sigma_1$

$$\frac{\sigma_1}{\lambda}-\frac{D(D-1)\Gamma(-D/2)}{2\,(2\pi)^{D/2}}\sigma_0^{D-2}\sigma_1+\frac{\sigma_0^{(D-3)/2}}{\sqrt{2}\,(\pi\beta)^{(D-1)/2}}\left(\frac{D}{2\beta}-\sigma_0\right)e^{-\beta(\sigma_0-|\mu|)}\simeq 0. \qquad (36)$$

The consistency check to second order is $\sigma_1<<\sigma_0$. We have

$$\frac{\sigma_1}{\sigma_0}\simeq\frac{\lambda}{(2-D)\sqrt{2}}\left(1-\frac{D}{2\beta\sigma_0}\right)\left(\frac{\sigma_0}{\pi\beta}\right)^{(D-1)/2}\frac{1}{\sigma_0}e^{-\beta(\sigma_0-|\mu|)}<<1. \qquad (37)$$

This is satisfied at high temperature T, if $\lambda\sim\beta=(kT)^{-1}$. In particular, the exponential is then $\sim\exp(-2kT)$, which is very good for the quick convergence of the series of Bessel functions. For the particular case $D=5/2$, substituting σ_0 into the last expression, we get

$$\begin{aligned}\left.\frac{\sigma_1}{\sigma_0}\right|_{D=5/2} &\simeq -\left(1-\frac{5\sqrt{2/\pi}\lambda^2}{\Gamma(1/4)^2\beta}\right)\frac{\lambda^{3/2}}{(2\pi)^{7/8}\sqrt{\Gamma(1/4)}\beta^{3/4}}e^{-\beta(\sigma_0-|\mu|)}\\ &= -0.1052\left(1-0.3035\,\frac{\lambda^2}{\beta}\right)\frac{\lambda^{3/2}}{\beta^{3/4}}e^{-\beta(4.1187/\lambda^2-|\mu|)}. \qquad (38)\end{aligned}$$

It is interesting to observe that the constants that appear in this example look very much like the ones obtained by Chowla and Selberg in a very different situation.

5 Aplication to quantizing through the Wheeler-De Witt equation

In a recent publication [11], dealing with the approach to (2+1)-dimensional quantum gravity which consists in making direct use of the Wheeler-De Witt equation, Carlip has come across a rather involved mathematical problem. (None of the aproaches that have been employed for the quantization of gravity is simple, for different reasons.) Here we will concentrate only in the specific mathematical point of the whole question, which concerns the calculation of the determinant that appears in his method for the case of the torus topology. It corresponds to a differential operator, D_0, which has the following set of eigenfunctions and eigenvalues

$$|mn>=e^{2\pi i(mx+ny)}, \qquad l_{mn}=\frac{4\pi^2}{\tau_2}|n-m\tau|^2+V_0, \qquad (39)$$

where m and n are integers, and τ and τ_2 are the usual labels corresponding to the standard two-dimensional metric for the torus

$$d\bar{s}^2=\tau_2^{-1}|dx+\tau dy|^2, \qquad (40)$$

with x and y angular coordinates of period 1 and $\tau=\tau_1+i\tau_2$ the modulus (a complex parameter). V_0 is the spatial integral of the relevant potential function.

At that point, the difficulty has boiled down to a well formulated mathematical problem which has a straight solution by means of the expressions derived above, through the calculation of the corresponding zeta function, ζ_{D_0}. In fact, after simplifying the notation a little, we easily recognize that we have to deal here with a series of the form (2). For $s=1$ (the value of interest) the analytic continuation hits a pole and, therefore, it must be conveniently defined [13].

The quantization of gravity in a spacetime with the topology $\mathbf{R}\times\mathbf{T}^2$, $\mathbf{T}^2$ being the two-dimensional torus with the standard metric (40), proceeds through the calculation of the zeta function corresponding to the basic differential operator D_0, which has the spectral decomposition given above (39). In terms of the function $E(s;a,b,c;q)$, the zeta function of D_0 is

$$\zeta_{D_0}(s)=F\left(s;4\pi^2/\tau_2,-8\pi^2\tau_1/\tau_2,4\pi^2(\tau_1^2+\tau_2^2)/\tau_2;V_0\right). \qquad (41)$$

One has, in particular, $\Delta=64\pi^4$ and, using Eq. (22),

$$\zeta_{D_0}(s) = \frac{2^{-2s+1}\pi^{-2s}}{\tau_2^{-s}}\zeta_{EH}\left(s,V_0\tau_2/(4\pi^2)\right)$$

$$+\frac{2^{-2s+1}\pi^{-2s+1/2}\Gamma(s-1/2)}{\tau_2^{s-1}\Gamma(s)}\zeta_{EH}\left(s-1/2, V_0/(4\pi^2\tau_2)\right)$$
$$+\frac{2^{-2s+2}\pi^{-s}\sqrt{\tau_2}}{\Gamma(s)}\sum_{n=0}^{\infty} n^{s-1/2}\cos(2n\pi\tau_1)\sum_{d|n} d^{1-2s}\int_0^{\infty} dt\, t^{s-3/2}$$
$$\times \exp\left\{-n\pi\tau_2\left[\left(1+\frac{V_0}{4\pi^2 d^2\tau_2}\right)t+t^{-1}\right]\right\}. \qquad (42)$$

The quantity of interest is the determinant of the operator D_0. This is most conveniently computed by means of its zeta function. In particular:

$$\det{}^{1/2} D_0 = \exp\left[-\frac{1}{2}\zeta'_{D_0}(0)\right]. \qquad (43)$$

Thus, we must now calculate the derivative of (22) at $s = 0$. We have, for the general function $E(s; a, b, c; q)$

$$F'(0; a, b, c; q) = \ln a + 2\zeta'_{EH}(0, q/a) - \frac{2\pi\sqrt{\Delta}}{a}\zeta_{EH}(-1/2, 4aq/\Delta)$$
$$+4\sum_{n=1}^{\infty} n^{-1}\cos(n\pi b/a)\sum_{d|n} d\,\exp\left[-\frac{\pi n}{a}\left(\Delta+\frac{4aq}{d^2}\right)^{1/2}\right], \qquad (44)$$

where

$$\zeta'_{EH}(0; p) = -\pi\sqrt{p}+\frac{1}{2}\ln p + 2p^{1/4}\sum_{n=1}^{\infty} n^{-1/2}K_{1/2}(2n\pi\sqrt{p}), \qquad (45)$$

while for $\zeta_{EH}(-1/2; p)$ the principal part prescription (PP) gives

$$PP\zeta_{EH}(-1/2; p) = -\frac{p}{4}-\frac{\sqrt{p}}{2}-\frac{\sqrt{p}}{\pi}\sum_{n=1}^{\infty} n^{-1}K_1(2n\pi\sqrt{p}). \qquad (46)$$

Finally, for the determinant of D_0, we obtain

$$\det{}^{1/2} D_0 = \frac{1}{\sqrt{V_0}}\exp\left\{-\frac{V_0}{8\pi}-\sqrt{\frac{2}{\pi}}(V_0\tau_2)^{1/4}\sum_{n=1}^{\infty} n^{-1/2}K_{1/2}\left(n\sqrt{V_0\tau_2}\right)\right. \qquad (47)$$
$$\left.-\frac{1}{\pi}\sqrt{\tau_2 V_0}\sum_{n=1}^{\infty} n^{-1}K_1\left(n\sqrt{\frac{V_0}{\tau_2}}\right)-2\sum_{n=1}^{\infty} n^{-1}\cos(2n\pi\tau_1)\sum_{d|n} d\,\exp\left[-2n\pi\tau_2\left(1+\frac{V_0}{4\pi^2\tau_2 d^2}\right)^{1/2}\right]\right\}.$$

We observe again that the final formula is really simple for practical purposes, since it provides a very good approximations with just a few terms, which are, on its turn, elementary functions of the relevant variables and parameters. This is so, because the infinite series that appear converge quickly (terms exponentially decreasing with n). In an asymptotical approach to the determinant, only the first terms in Eq. (47) are relevant.

From the detailed analysis in [11], it follows that the quantity to be calculated is the derivative with respect to V_0 of the above determinant, since this quantity vanishes precisely at the solutions of the Hamiltonian constraint (always in the language of quantization

through the corresponding Wheeler-De Witt equation). In other words, the solutions of the equation

$$\frac{\partial}{\partial V_0}\det{}^{1/2} D_0 = 0, \qquad (48)$$

will yield the conditions that the quantized magnitudes and parameters are bound to satisfy as a consequence of the Wheeler-De Witt equations. After some work, Eq. (48) can be written as

$$\det{}^{1/2} D_0 \times \left[-\frac{1}{8\pi}-\frac{1}{2V_0}-\frac{\tau_2(\tau_2 V_0)^{-3/4}}{2\sqrt{2\pi}}\sum_{n=1}^{\infty} n^{-1/2}K_{1/2}\left(n\sqrt{V_0\tau_2}\right)\right.$$
$$-\frac{1}{2\pi}\sqrt{\frac{\tau_2}{V_0}}\sum_{n=1}^{\infty} n^{-1}K_1\left(n\sqrt{\frac{V_0}{\tau_2}}\right)-\frac{\tau_2(\tau_2 V_0)^{-1/4}}{\sqrt{2\pi}}\sum_{n=1}^{\infty} n^{1/2}K_{1/2}{}'\left(n\sqrt{V_0\tau_2}\right)-\frac{1}{2\pi}\sum_{n=1}^{\infty} K_1{}'\left(n\sqrt{\frac{V_0}{\tau_2}}\right)$$
$$\left.+\frac{1}{2\pi}\sum_{n=1}^{\infty}\cos(2n\pi\tau_1)\sum_{d|n}\left(d^2+\frac{V_0}{4\pi^2\tau_2}\right)^{-1/2}\exp\left(-2n\pi\tau_2\sqrt{1+\frac{V_0}{4\pi^2\tau_2 d^2}}\right)\right]=0, \qquad (49)$$

where the primes mean derivatives of the Bessel functions. The analysis can be carried out without further problem. In principle —the consistency of the approximation is to be checked *a posteriori*— Eq. (49) can be reduced to the a simple expression which includes, at most, the first of the terms involving Bessel functions (same treatement as described in detail for the previous problem).

To summarize, two problems were singularized out in [11] as the main difficulties that appear in the quantization of (2+1)-dimensional gravity through the Wheeler-De Witt equation: (a) to give grounds for the choice of the specific operator ordering of the Hamiltonian constraint which leads to the Wheeler-De Witt equation of the quantized system, and (b) to understand the functional dependence of the determinant $\det^{1/2} D_0$ in terms of the relevant variables and to obtain its extrema as a function of the potential V_0. With the use of zeta-function techniques, we have been able to solve problem (b) by means of a consistent approximation. We should finally mention that an alternative approximation for this problem was developed in [11] by using Eisenstein series theory (see, for instance, [14] and [15]). The advantages of the present method with respect to that one are quite conclusive.

Apart from the applications that we have here considered —directly dealing with the generalization of the Chowla-Selberg formula derived above— in general our new expression will be certainly useful in many situations involving massive theories, finite temperatures or a chemical potential in a compactified spacetime. This is the meaning to be attributed to the constant q in the more usual mathematical-physics problems. As for the possible mathematical applications, the introduction of the constant q into the Epstein function

extends the scope of the applications already envisaged by Chowla and Selberg, in the sense that now one should be able to develop a sort of classification of the problems according to the different values of q.

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