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DUALITY, INTEGRABILITY AND W-ALGEBRAS VIA LAGRANGIAN MANIFOLDS

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ABSTRACT

We show that the duality property for a two-dimensional field theory(2DFT)can be naturally associated to the corresponding Gauss map . This ,in turn,is Lagrangian(totally real)i.e. with vanishing symplectic form and trivial normal bundle.Owing on these properties we get a new integrability condition for 2DFTs,(higher dimensional)classical solutions(that generalize Witten's black-hole solution)as well as a novel interpretation for the W-algebras(as affine algebras attached to special symplectic(Lagrangian)manifolds.)We point out that the Calabi-Yau manifolds are Lagrangian,too, and this allows to interpret the mirror symmetry as a duality(in our sense).

A new approach to conformal invariance(via focal manifolds)and several additional aspects are also discussed.

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DUALITY, INTEGRABILITY AND W-ALGEBRAS

VIA LAGRANGIAN MANIFOLDS

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1.Introduction

The purpose of this paper is to present a new approach to duality /1/ for two-dimensional field theories(thereafter 2DFTs).This,in turn, will suggest a new and rigorous view on integrability of 2DFTs /2/ as well as a generalization of the (black-hole)solution proposed by Witten's /3/ some time ago.

More precisely,we shall show that the duality -viewed as a property of the target space of a 2DFT - can be associated to the corresponding Gauss map /4/. and this implies the appearance of the so-calle Lagrangian manifolds/5/ (special symplectic manifolds with vanishing symplectic form).The integrability will be connected with the fact that a Lagrangian immersion has a flat normal curvature(i.e. a Lax system),while all the classical solutions of 2DFTs are Lagrangian manifolds.(The precise definitions of Lagrangian manifolds,respectively Lagrangian immersions will be presented in the next Section .)

The same Lagrangian manifolds picture will allow a new approach to W-algebras /6/.We will show that the W-algebras naturally appear

as the affine algebra of the corresponding (special)symplectic manifolds.

As additional byproducts of our approach we will briefly discuss the following facts :i) observing that the Kaehler form is proportional to the symplectic form it results that all the Calabi-Yau manifolds are actually Lagrangian manifolds.;this will allow to interpret the Calabi-Yau mirror symmetry/ 7 / as a duality thereof (see also / 8 / for a partial discussion),ii) on a symplectic manifold the corresponding metric ,say $\langle \cdot , \cdot \rangle$, can be written as

$$\langle \cdot , \cdot \rangle = (\cdot , \cdot) + \Omega (\cdot , J \cdot) \qquad (1.1)$$

where $(\cdot , \cdot)$ is the usual Euclidean metric and Ω the symplectic form ;above J denotes the complex structure .Therefore,a Lagrangian manifold is totally real ,i.e. it is the boundary of an appropriately defined (complex)manifold.This suggests a possible solution of the (black-hole)information problem / 9 /.We affirme that to completely describe a black-hole one has to consider both the boundary and the core ,i.e. a mixed state.(A detailed discussion of this point will be presented in a separate paper .)

After this introduction we will proceed now to present our results.The layout of the (rest of the) paper is the following.In the Sect.2 we will introduce our approach to duality and we will show how it naturally suggests the appearance of Lagrangian manifolds.Sect.3. is devoted to integrabilty and classical solutions for 2DFTs in the context of the formalism we proposed in the previous Section.Our approach to W-algebras will be presented in the Sect.4.

A summary of the obtained results concludes the paper(Sect.5).

2a. Duality and Gauss Maps

Here we will present our novel approach to duality for 2DFTs. As it is well-known / 1 / up to now the following forms of duality have been suggested :

i)duality under the Hodge star - $*$ - operator.This type of duality allo (complemented with additional assumptions) to get an action functional the considered 2DFT;however, the conformal and inegrability properties (of the involved theory) are by no means evident.

ii) " $\frac{1}{r}$ "- duality , primarily discussed in solid state physics

iii) the operational duality - via a Lagrange transform as discussed i

Our approach to duality is intimately connected with the the type ii) of duality. Essentially, the " $\frac{1}{r}$ " - duality connects the speec trum of the Laplacian (of the considered Riemannian manifold) with the length spectrum of the same manifold (i.e. with the lengths of the periodic geodesics of the same manifold). For instance, for a torus $T^n =$ (Γ a discrete subgroup of $\mathbb{R}^n$, $n \geq 1$) the duality can be ana lytically expressed via the Poisson summation formula

$$\sum_{\lambda \in Sp_{\Delta}(\mathbb{R}^n/\Gamma)} e^{-\lambda t} = \frac{Vol(\mathbb{R}^n/\Gamma)}{(4\pi t)^{n/2}} \sum_{\ell \in LSp(\mathbb{R}} e^{-\frac{\ell^2}{4t}} \qquad (2.1)$$

Here Δ is the Laplacian of $\mathbb{R}^n/\Gamma$ with spectrum Sp_Δ (and λ a generic element of Sp_Δ). Analogously, $L\text{-}Sp(\mathbb{R}^n/\Gamma)$ denotes the lengths spectrum of T^n (with volume $Vol(\mathbb{R}^n/\Gamma)$). The l.h.s. of (2.1) is formally a "zeta functional", while the r.h.s. is a "theta - functional" for the same manifold. For a general Riemannian manifold M the generalization of (2.1) is

$$Z_{\lambda \in Sp M}(\lambda, t) = \Theta_{\ell \in L\text{-}Sp M}(\ell, t) \qquad (2.2)$$

This summation formula has been further generalized (for locally symmetric speces) to the celebrated Selberg trace formula /10/.

For our purposes we quote the following known fact /11/. Let TM_1 be the tangent bundle of the Riemannian manifold M and $G: TM_1 \to TM_1$ the corresponding geodesic flow. (Here TM_1 is the unit tangent bundle, i.e. gent vectors of unit length). Then, ther is a natural correspondence between TM_1 and the closed periodic geodesics of M. Therefore, it seems natural to generalize the duality from closed periodic geodesics on M (the target space of a 2DFT) to the tangent space TM_1 of M. (It is also a bit easier to work with tangent vectors.) Formallly , a 2DFT means

$$\phi : M_2 \to M$$
$$\dim M = d \geq 2 \qquad (2.3)$$

with an appropriately (model-dependent) action functional. Then, the duality means

$$M \longrightarrow TM_1 \qquad (2.4)$$

and this, in turn, defines an "additional "2DFT (for generality sake we wor with TM)

$$\gamma : M_2 \to TM , \quad \gamma = \partial\phi \qquad (2.5$$

A bit more in-depth view to (2.4), (2.5) shows that attaching a point, say x, of M to the corresponding tangent vector TM_x, means to immer se M into the Grassmannian $Gr(M, M_2) = \frac{O(d)}{O(2)\,O(d-2}$

Therefore, to (2.3) we will attach the additional model

$$\gamma : M_2 \to Gr(M, M_2) \qquad (2.6$$

and duality means

$$M \to Gr(M, M_2) \qquad (2.7$$

The following diagram illustrates the facts we introduced above

$$\begin{tikzcd} & M \arrow[d, "\text{Duality}"] \\ M_2 \arrow[ur, "\phi"] \arrow[r, "\gamma"'] & Gr(M, M_2) \end{tikzcd} \qquad (2.8)$$

Now, (2.6) is known as the Gauss map /4/ of (2.3) and our approach to duality is summarized by the following proposition:

For 2DFTs duality means to pass from the target space M to the corresponding Grassmannian via the associated Gauss map.

To support this proposition we shall additionally present the following proof. Let us consider a two-dimensional manifold minimally immersed in a Euclidean space $\mathbb{R}^n$ with metric $ds^2 = dr^2 + r^2 d\theta^2$. Here r (θ) is the radial (angular) variable. Under the transform $r \to 1/r$, the above metric becomes

$$dr^2 \to \frac{1}{r^4} \cdot dr^2$$

$$ds^2 \to d\tilde{s}^2 = \frac{1}{r^4} \cdot ds^2 \qquad (2.9)$$

Denoting $\frac{1}{r^2} = k$ -principal curvature

$\frac{1}{r^4} = k^2 = K$ - Gauss curvature

(2.9) becomes

$$d\tilde{s}^2 = K\, ds^2 \qquad (2.10)$$

Now, such a formula defines the Gauss map (2.6) associated to (2.3) (See e.g. / 4 /) (Precisely, for minimal immessions, i.e. when $\sum k = 0$ It is easy to see that the metrics for spheres and hyperbolic spaces (respectively, complex projective spaces and complex balls) are invariant with respect to the $\frac{1}{r}$ -transform. Therefore, we got once again that 2DFTs in target spaces with constant curvature (space forms) duality means to pass from M to the corresponding Grassmannian $Gr(M, M_2)$.

The notion of Gauss map can be extended to higher dimensional manifolds, i.e. to higher dimensional field theories; however, in such a case there is no more the above defined duality !

As a further argument for our approach let us point out that Griffiths and Harris / 12 / defined an algebraic-geometric duality that, in case the target space is a hypersurface (as usually assumed) coincide with the (Gauss-map based) duality we introduced above.

To illustrate our formalism, we shall consider a 2DFT immersed in a 4-dimensional Euclidean space (the case of a Minkowski space or a target space with constant non-zero curvature, goes analogously), the immersion

$$\phi : M_2 \to \mathbb{R}^4 \qquad (2.1$$

The M_2 metric is as usual $ds^2 = \lambda^2(z)^2$, $z = x_1 + i x_2$, x_i - coordinates on M_2) and the associated Gauss map is

$$\gamma : M_2 \to \frac{SO(4)}{SO(2) \cdot SO(2)} \cong \frac{SO(3) \cdot}{SO(2) \cdot S}$$

$$\cong S^2 \cdot S^2 \cong P^1(\mathbb{C}) \cdot P$$

$P^i(\mathbb{C})$ = i-th dim. complex projective space (2

Analitically,

$$\gamma = \partial\phi / \partial z, \quad z = x_1 + i x_2 \qquad (2$$

and the target space is the complex quadric

$$Q_2 = (P^1(\mathbb{C}) \cdot P^1(\mathbb{C})) = \left(\sum_{i=1}^{4} y_i^2 = 0\right.$$

(For $\mathbb{R}^n$ one has $Q_n = SO(n) / SO(2) \cdot SO(n-2)$

More simply, if w_1, w_2 are the homogeneous coordinates on the two P^1 (see (2.12)) then one can define on Q_2 the following (Fubini-

Study)metric

$$ds^2_{Q_2} = \frac{2|dw_1|^2}{(1+|w_1|^2)^2} + \frac{2|dw_2|^2}{(1+|w_2|^2)^2} \qquad (2.15)$$

With

$$F_i = \frac{\frac{\partial w_i}{\partial z}}{1+|w_i|^2} \qquad i=1,2$$

$$\hat{F}_i = \frac{\frac{\partial w_i}{\partial \bar{z}}}{1+|w_i|^2} \qquad (2.16)$$

one can introduce the currents

$$J_i = \frac{2}{\lambda^2}\left(|\hat{F}_i|^2 - |F_i|^2\right) \qquad (2.17)$$

so that

$$J_1 + J_2 = K \quad \text{– Gauss curvature} \qquad (2.18)$$

$$J_1 - J_2 = K_N \quad \text{normal curvature} \qquad (2.19)$$

The corresponding σ-nonlinear lagrangian ("dual" to the Fubini-Study metric) is

$$L = \frac{1}{\lambda^2}\left(|F_i|^2 + |\hat{F}_i|^2\right) \qquad (2.20)$$

We will exploit thse results in the next Section.(Notice that F_i, $\hat{F}_i$ actually correspond to the Gauss map (2.13).)

2b.Gauss Maps and Lagrangian Manifolds.

To begin we recall that the Gauss maps have already been used by one of us /13/ to obtain classical solutions (instantons) for the nonlinear models.It is easy to observe that there is an intimate connec between analyticity properties of the map (2.3) and the conformal properties of the corresponding Gauss map. (2.6)

Here we shall point out another property of the Gauss maps that will play the essential role for the rest of the paper.

If the target space M (of a 2DFT) is of constant curvature(as we throughly assume) ,then there is the isomorphism $TM \cong TM^*$, where TM^* is the cotangent bundle of M .On TM^* onn defi a symplectic(closed) form,say Ω.The above isomorphism implies a symplectic involution i ,so that for a 2DFT

$$r : M_2 \longrightarrow TM^*/i$$

$$r^*\Omega = 0 \qquad (2.21)$$

An immersion with this property will be called Lagrangian immersion, while the submanifold with a vanishing Ω will be called Lagrangian (sub)manifold / 5 /.We acknowledge that this property has

already been discussed -via another line of reasoning- by Arnold et al /14 /. (This suggests that our approach is also connected with the singulatity theory ,i.e. with the Landau-Ginzburg method /15,16/.

In a recent paper /17 / it is affirmed-without proof-that all the relevant geometries for 2DFTs are Legendrian,i.e. with vanishing contact form.It is well-known /5, 18 /,that there are deep relationships between symplectic and contact geometries,i.e. between Lagrangian and Legendrian manifolds.However,contact forms can essentially be defined only for odd dimensions and,on the other hand ,they can not account for Calabi-Yau manifolds.Additionally,as Duistermaat /19 / showed the main contribution to an action functional (of a 2DFT) is basically provided by the Lagrangian manifolds .(And,to illustrates his ideas he used double point singularities ,lated suggested by Martinec ,/15 /.This can be seen as the "missing link" between Landau- Ginzburg and Calabi-Yau approaches.)

To conclude this Section we shall present a particular class of Lagrangian manifolds,the so-called Lagrangian planes $\Lambda_n = \frac{U(n)}{O(n)}$. Although not explicitely specified ,it seems that Λ_n can play an eminent role in supergravity theories with antisymmetric tensor fields/ 20 /

For the interested reader we shall also reveal that the branched covering of the quadric Q_2 (see (2.14)) is Lagrangian,precisely a Calabi-Yau(K-3) surface.(This can be used as starting point for an investigation with the title,e.g."Strings and K-3 Surfaces".)

Observe also that our definition of duality is completely independent of the fact that it is an exact symmetry or not.

3.Integrability and classical solutions for 2DFTs

In this Sectiom we shall show that the Lagrangian manifolds we introduced in the previous Section can provide a rigorous approach to the integrabilty of 2DFTs.

We start with the 2DFT immersed in R^4 ,we already discussed above.

We shall take

$$K_N = 0 \qquad (3.1)$$

i.,e. (see (2.19))

$$J_1 = J_2$$

$$|F_1|^2 - |F_2|^2 = |\hat{F}_1|^2 - |\hat{F}_2|^2 = 0 \qquad (3.2$$

This implies

$$\frac{\partial w_1}{\partial z}\,\overline{\frac{\partial w_1}{\partial z}} - \frac{\partial w_2}{\partial z}\cdot\overline{\frac{\partial w_2}{\partial z}} = 0 \qquad (3.3$$

These commutation relations can be interpreted as a Lax system and this suggests the following integrability condition (I C).

IC : A 2DFT is integrable if the corresponding normal curvature $K_N = 0$

i.e.if it corresponds to a Lagrangian immersion.

If we rewrite the M_2 metric as $ds^2 = 4e^{2\omega}|dz|^2$ (ds^2 was initially defined after (2.11)),then it is a simple matter to see that our integrability condition (3.3) leads to

$$\omega_{z\bar{z}} + \frac{1}{2}\sinh(2\omega) = 0 \qquad (3.4)$$

i.e. the sinh-Gordon equation.If in place of $M_2 \to R^4$ we take $M_2 \to G$ (G -compact semisimple Lie group) and use the fact that homotopically G is a produc of spherees,i.e. $G \cong \prod_i S^{k_i}$, $\sum_i (2k_i + 1) = r$ rank G) ,then our IC leads to the well-known Toda equation

$$\omega^i_{z\bar{z}} + \exp\left(\sum_{j=1}^{r} K_{ij}\,\omega^j\right) = 0 \quad i=1,\dots,r \qquad (3.5)$$

(Here K_{ij} is the Cartan matrix defined interms of the "spins" k_i.)

Of course,our IC does not imply that the considered model is conformal variant.A more detailed analysis (precisely,for Lagrangian manifolds with constant principal curvatures,the so-called isoparametric manifolds /21/) shows that a 2DFT is conformally invariant(minimal) if the manifold attached via duality are focal manifolds.Precisely,to any Gauss map one can attach a scalar curvature $\tilde{K}$;when

$$\frac{1}{K} \pm \frac{1}{\tilde{K}} = \frac{1}{K_T} \qquad (3.6)$$

then the associated manifolds are focal.Here K_T is the scalar curvature of the target.For instance,,for $\gamma: M_2 \to P^i(\mathbb{C}),\ i \geq 1$

$$\tilde{K} = \frac{k(k+1)}{6}, \quad k = 1,2,\dots \qquad (3.7)$$
$$K_T = 1$$

and therefore

$$\frac{1}{\tilde{K}} = 1 - \frac{6}{k(k+1)} \qquad (3.8)$$

We get simultaneously an interpretation for the central charge ($c = \frac{1}{\tilde{K}}$) as well as for the minimal integrable 2DFTs.According to the picture above mal models correspond to focal Lagrangian manifolds,while the integrable(non minimal)2DFTs are "ordinary" Lagrangian manifolds.There is also a definition of focal manifolds in the context of singularity theory/22/ and this suggests once again the deep relationship between Calabi-Yau(i.e.Lagrangian approach and the Landau-Ginzburg (i.e.the singularity) approach.

We shall proceed now to get classical solutions for 2DFTs;as already alluded to we shall look for classical solutions that are Lagrangian manifolds.To do this we shall re-analyse the solution proposed by Witten's /3/ some years ago and then we shall generalize it in a Lie-theoretic framework.

Essentially,Witten's solution can be viewed as the Lagrangian immersion

$$\frac{SO(2,1)}{SO(1,1)} \longrightarrow \frac{SO(2,1)}{SO(2)} \cdot U(1) \qquad (3.9)$$

(Notice that we are in the target space !)Locally,it is the upper half-plane of the complex plane(or,via the well-knon Cayley transform,the unit complex disc)and the corresponding geometry is non-Euclidean(hyperbolic).Observe that $SO(1,1)$ is the parabolic subgroup of $SO(2,1)$ while $SO(2)$ is the compact subgroup of the same group.(For a definition of the parabolic,resp. compact subgroups of a Lie group as well as of the subsequent Lie-theoretic notions,see e.g. /23/.This suggests the following generalization(see also /24/)

Let G be a semi-simple non-compact Lie group with Lie algebra $\mathfrak{g}$. Further, let K, P be the maximal compact, resp. parabolic subgroup ~~subgroup~~ of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. (Here $\mathfrak{k}$ is the Lie algebra corresponding to K, while $\mathfrak{p}$ is a linear vector space of dimension $n = \dim \mathfrak{p} = \dim G - \dim K$.) Then a Lagrangian (totally real) manifold can be defined via

$$G/P \to S^{n-1} \subset \mathfrak{p} \qquad (3.10)$$

(Here S^{n-1} is the $(n-1)$-dimensional unit sphere in $\mathfrak{p}$.) Taking into account the Iwasawa decomposition of G and P, i.e. /23/

$$G = KAN$$
$$P = MAN$$

with A -Abelian subgroup of G, N nilpotent subgroup and G, M centralizer of A in K, it results that our solutions possess the symmetry $\frac{K}{M} \cdot SO(n)$ (analogous to the Kazama-Suzuki model /25/).

The structure of these solutions is extremely rich and a detailed discussion requires a separate paper. However, we shall briefly present some essential points :

a) it is relatively simply to see that Witten's solution is a Lagrangian manifold with one principal curvature (with multiplicity one, $SO(n,1) \cdot U(1)$ leads to one "family" with multiplicity $(n-1)$). Using (3.10) one can obtain two generations (i,e. a solution with two principal curvatures) for $G = SO(q_1, q_2)$, $q_1, q_2 \geq 1$ (The corresponding multiplicities are $q_1 - 1$, $q_2 - 1$). This follows from

$$\frac{K}{M} \simeq \left(\frac{G}{P}\right) \to \frac{SO(q_1)\, SO(q_2)}{SO(q_1-1)\, SO(q_2-1)} \simeq S^{q_1-1}.$$

Getting solutions with three principal curvatures, i.e. three generations is a bit more complicated. The main difficulty is to determine the corresponding G's; nevertheless we get ($\frac{K}{M} \to \mathbb{R}^n$, $n = 3d+2$, $d = 1, 2, 4, 8$

$$\frac{K}{M} \to \frac{SU(3)}{SO(3)} \qquad d = 5$$

$$\frac{K}{M} \to \frac{SU(3) \cdot SU(3)}{SU(3)} \qquad d = 8$$

$$\frac{K}{M} \to \frac{SU(6)}{Sp(3)} \qquad d = 14$$

$$\frac{K}{M} \to \frac{E_6}{F_4} \qquad d = 26$$

(3.11)

Observe that these solutions are connected with the division algebras (real, complex, quaternionic and respectively Cayley).

b) Of course, the solutions we obtained do not exhaust all the possi (constant principal curvature) solutions $G/P \to \mathfrak{p}$; moreover, there m exist solutions that are not homogeneous at all. Note also the isomorphism

$$M/\mathfrak{p} \simeq \mathfrak{a}/W$$

(W is the Weyl group of K and $\mathfrak{a}$ the Lie algebra of the Abelian sub group A),i.e. these solutions possess additionally a Weyl symmetry.(This suggests a possible connection with braiding and the quantum symmetry; however,we shall pursuit this point here.)

4. Lagrangian Manifolds and W-Algebras

Here we shall show that the formalism we introduced above permits a natural definition of the W-algebras.

Firstly, we shall consider the case of homogeneous Lagrangian manifolds;then,we shall extend our discussion to arbitrary (Lagrangian)manifolds.The first part of our analysis has some contact points with the approac of Gervais et al / 26 / ;however,there only the Kaehler case is considere (Note that manifolds of the type $G/P \to \mathfrak{p}$ need not to be Kaehler.)

To be specific we consider a Lagrangian immersion i.e. an integrable 2DFT (see also (3.10))

$$\mathcal{F}: M_2 \to G/P \to S^{n-1} \subset \mathfrak{p} \qquad (4.1)$$

Then

$$f = \mathcal{F}^{-1}: S^{n-1} \to M_2 \qquad (4.2)$$

is a polynomial map (a so-called Cartan polynomial/ 21 /)and the commutator of W's , $W = \partial f$ is a linear combination of W's.Observe that W is actually defined on the Grassmannian attached to S^{n-1} and

therefore it can be interpreted as the principal symbol of an appropriately defined differential operator.Hence,one can write the following (equal time commutator (x_1, x_1' are spatial variables)

$$[W_i, W_j] = \mathcal{P}_{ij}(W)\,\delta(x_1 - x_1') \qquad i,j = 1,2,\dots \qquad (4.3)$$

Therefore,the W-algebras are the affine algebras attached to isoparametric (Lagrangian) manifolds.The first commutator of the series (4.3) is the usual Virasoro algebra and at the same time -from a Lie -theoretic point of view- the well-known Bochner -Lichnerowicz formula /23 /($[\nabla_i, \nabla_j] = \alpha_{ij}\nabla_{i+j} + \mathcal{R}$, $\mathcal{R}$ scalar curvature)This observation supports once again the geometric interpretation of the central charge we proposed above(see (3.2)).Precisely, c is the (inverse of the involved)scalar curvature, while the level k is the order of the corresponding eigenvalue.

To determine W's we propose the following algorithm.One can show that $\mathcal{F}$ is the volume(squared)of the corresponding symmmetric domain $K/M \subset \mathfrak{p}$ and, using standard results one determines f and $W = \partial f$.

We shall extend now our discussion to the case when the implied Lagrangian manifold is not homogeneous. We recall /27 / that for any Lagrangian manifold one can define a so-called Monge-Ampère equation

$$\det \begin{pmatrix} \partial - i\alpha_1 & & 0 \\ & \ddots & \\ 0 & & \partial - i\alpha_n \end{pmatrix} \Psi = 0 \qquad (4.4)$$

The potential ψ above is connected with the integrability condition in the sense that $\Sigma \psi \chi$ (χ -the conjugate solution to ψ) satisfies the Toda equation.From (4.4) we can write

$$\left(\partial^n - \sum_{i=2}^{n} W_i \partial^i \right) \psi = 0 \qquad (4.5)$$

The constants α_i are connected with the "spins" h_i (see the derivation of (3.5));they were absorbed in our definition of W's (4.5)..A direct calculation leads once again to

$$[W_i, W_j] = \mathcal{P}_{ij} (W) \delta(x_1 - x_1')$$

i.e. to the W-algebra (4.3) we get above.

To conclude this Section we shall briefly comment two interesting points :i) it is well-known /24/ that a semi-simple non-compact Lie algebra g admits a Z-gradation,i.e.

$$g = g_{-1} + g_0 + g_1$$
$$[g_{-1}, g_1] \subseteq g_0 \qquad (4.6)$$
$$[g_0, g_{\pm 1}] \subseteq g_{\pm 1}$$

This can give a rationale for the approach suggested in /28/ (to derive W-algebras via the embedding sl_2 into g .)

ii)Our approach presented above shows that classifying the W-algebras essentially means to classify the underlying Lagrangian manifolds.Now,there is an infinite set of such manifolds and a "complete " classification of W-algebras is obviously illusory.(A bit more complicated is the situation for W_∞

i.e. in case when the corresponding Lagrangian manifold is immersed infinite dimensional space.)Nev3ertheless, a further study of the pr ties of the W-algebras via the Lie-theoretic and geometric methods we above is worth doing.

5.Discussion

Here we shall present a succint discussion of the results we get previous Sections.In essence,we focused our attention on the properti of the target manifold of a 2DFT starting with a generalized duality rement we have proved that the target space is a Lagrangian manifold totally real).We pointed out that Lagrangian implies (in the Kaehler Calabi-Yau-

This allowed us to get a generalized integrability condition as classical solutions a la Witten for the considered 2DFT.Notice, th due to the factoring of the parabolic subgroup (see (3.10)) these sol tions automatically avoid closed time curves ,i.e. they satisfy the ca condition .Of course,this class of solutions (which generalize the ins tons-like solutions) for which we propose the name symplectons-do exhaust the possible solutions of the considered 2DFT.

As a third result we (re)obtained the W-algebras as affine alg of the underlying Lagrangian manifolds.The full implications of the Mon Ampère equation remains to be further studied;it appears that in some cases the Monge-Ampère operator is identical to the Dirac operator.

As byproducts we get :i) a new view to conformal invariance.A 2 is conformal invariant (minimal)if the underlying target space satisfies "focality"condition.ii)Several connections between the singularity thory

Lagrangian (focal)manifolds,i.e. relationships between Calabi-Yau and Landau-Ginzburg approaches.iii)We alluded to a new view to mirror symmetry/7/ (i.e. the symmetry that commutes the Kaehler and Complex structures,i.e. the tangential and normal cohomologies.)The mirror symmetry as a duality has recent been discussed in a very special case in /8/.Briefly,we suggest the following line of attack(a detailed discussion will be presented elsewhere).For a Lagrangian manifold M one can write $TM \oplus NM = TM \oplus J(TM)$ (Here NM is the normal bundle and J the complex structure.)Therefore, a manifold dual to (i.e. according to our picture defined via the corresponding Gauss map) commutes TM and $J(TM)$,i.e. the tangent and normal cohomologies.

Obviously,the Lagrangian manifolds(or isoparametric,if the principal curvatures are constant) are not an universal remedy for all the problems of 2DFTs (which according to our geometric and Lie-theoretic picture include the string-inspired models ,topological or not).However,although the formalism we suggested opens new questions(e.g. a) what means the "core" of a domain a Lagrangian manifold bounds, b)does a W-algebra uniquely determines its underlying 2DFT or one has to look for another "better" formalism,c)is there a relationship between time and symplectic forms (we have in mind the formal similarity between $(1,1)$ and $\tau + it$),etc) it has at least the merit to organize various facts in a logical way and to propose new lines of investigation.

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