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Is There Hair in Einstein-Higgs Theory?

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Abstract

We give a proof of the fact that there are no nontrivial black holes with regular horizon in Einstein-Higgs theory with any number of scalar fields and arbitrary potential. We also give a brief discussion of the contrast between this theory and the Einstein Yang Mills theory that is responsible for the difference in the set of solutions allowed by one and the other.

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Recently, there has been a surge of interest in studying static black hole solutions for gravity coupled to nonlinear matter fields. This is due in part to the discovery that, in spite of the widely held belief to the contrary, nontrivial static black hole solutions exist in Einstein Yang Mills Theory[1], Einstein Skyrme Theory[2],and Einstein Yang Mills Higgs Theory[3]. These discoveries have shown the falsehood of the "no hair conjecture" which was introduced by Wheeler[4] as a consequence of the black hole uniqueness theorems in Einstein-vaccum and Einstein-Maxwell theories[5, 6] and to the Beckenstein[7, 8] type results that proof the non existence of nontrivial stationary black holes for various theories with simple matter fields coupled with Einstein's gravity.

Among these "no hair" results, there is a proof[7] that in the case where the matter consist of a single scalar field with a convex potential, that is a potential term that is a convex function of the scalar field as exemplified by the usual mass term, there exists no nontrivial black hole solution with regular horizon. The case in which the potential is nonconvex, as exemplified by the usual double well potential, is the simplest case not covered by these theorems, and although no "no-hair" proof exists in the literature, it is widely believed that this theory yields no such black hole solutions either. This belief is reinforced by the failure of numerical attempts to find them[9]. In the case of multiple scalar fields, the situation seemed complicated enough that no systematic study of the question has been carried out. Moreover, there is even a recent suggestion[3] that one might be able to elude the no hair conjecture with a simple model of two scalar fields if the potential is appropriately chosen.

In this paper, we provide a proof that, if we restrict ourselves to spherically symmetric configurations, no such solutions exist in a theory of any number of scalar fields, minimally coupled to gravity provided only that the potential is positive semidefinite (as needed to satisfy the Weak energy condition) and that the minima of the potential is zero (as needed to allow asymptotically flat solutions). The restriction to spherically symmetric configurations is not expected to be a serious one, as it is natural to expect that the *staticity implies spherical symmetry* result of [5] could be generalized to this theory.

We consider then the theory specified by the action:

$$S = \int \sqrt{(-g)}[(\alpha)^{-1}R - (1/2)\nabla^{\mu}\phi_i\nabla_{\mu}\phi_i - V(\phi)] \tag{1}$$

Where R is the Ricci scalar, $\alpha = 16\pi G$, ∇^μ is the covariant derivative, ϕ_i are the scalar fields and $V(\phi)$ the scalar potential.

Let's consider the equations that would describe a nontrivial, static, spherically symmetric black hole solution in this theory, if one existed.

The most general static spherically symmetric metric can be expressed in terms of the line element:

$$ds^2 = -(1-2m/r)e^{-2\delta}dt^2 + (1-2m/r)^{(-1)}dr^2 + r^2(d\theta^2 + \sin(\theta)^2 d\varphi^2) \tag{2}$$

where m and δ are functions of r only. The scalar fields ϕ_i are also functions of r only.

Einstein's equations then reduce to

$$\delta' = -\alpha(r/4)\sum_i(\phi_i')^2 \tag{3}$$

and

$$m' = \alpha(r^2/4)[(1/2)(1-2m/r)\sum_i(\phi_i')^2 + V] \tag{4}$$

where the prime stands for $\frac{d}{dr}$.

The conditions for regular black hole horizon at $r = r_H$ are

$$m(r_H) = r_H/2, \quad m(r) > r/2 \quad \forall r > r_H, \quad and\ \delta(r_H) = finite \tag{5}$$

The conditions for an asymptotically flat metric are that $m(r)$ and $\delta(r)$ converge to a finite limit as $r \to \infty$. This requires that $(\phi_i')^2 = o(r^{-3})$ and $V = o(r^{-3})$ for large r.

The equations for the scalar fields are

$$\nabla^\mu\nabla_\mu\phi_i = \frac{\partial V}{\partial \phi_i} \tag{6}$$

In our case they become

$$((1-2m/r)\phi_i')' + (2/r - \delta')(1-2m/r)\phi_i' = \frac{\partial V}{\partial \phi_i} \tag{7}$$

We note that a solution to the equation above is completely specified by the choice of the value of the ϕ_i's at r_H, since at that point the coefficient of ϕ_i'' vanishes and therefore

eq.7 specifies the value of ϕ_i' at r_H in terms of $\frac{\partial V}{\partial \phi_i}$. In particular, if the value of ϕ_i at r_H were to be an extrema of the potential, the solution would be $\phi_i = constant$ which would not be an acceptable solution unless the extrema of V was actually its minima 0 in which case the solution would be trivial.

These equations are quite similar (in particular, if we took a single scalar field with $V(\phi) = \lambda(\phi^2 - v^2)^2$) to the equations that describe static spherically symmetric black hole solutions in Einstein Yang Mills Theory which are known to have nontrivial solutions, the so called "colored black holes":

$$\delta' = -(2/r)(w')^2 \tag{8}$$

$$m' = (1 - 2m/r)(w')^2 + 2V(w)/r^2 \tag{9}$$

and

$$r^2(1 - 2m/r)w'' + 2(m - 2V(w)/r)w' = \frac{\partial V(w)}{\partial w} \tag{10}$$

where

$$V(w) \equiv (1/4)(1 - w^2)^2 \tag{11}$$

What is the essential difference between these equations that result in such a different set of solutions?

We can see the answer by studying the problem using as guidance the equivalent problem of a particle in a potential well (as done in pedagogical expositions of simple solitons, see [10]), with the added complication that the mass will be variable and there will be friction terms. We proceed to derive what would be the equivalent of the work energy theorem. Multiplying eq. 7 by ϕ_i', and summing over i we obtain:

$$[(1/2)\mu \sum_i (\phi_i')^2]' + [(1/2)\mu' + (2/r - \delta')\mu] \sum_i (\phi_i')^{2\prime} = \sum_i \frac{\partial V}{\partial \phi_i} \phi_i' \tag{12}$$

where $\mu \equiv (1 - 2m/r)$. We then define $U(\phi) = -V(\phi)$ and the "Energy" E of the system as:

$$E = 1/2\mu \sum_i (\phi_i')^2 + U \tag{13}$$

Then we can write the eq. 12 as:

$$E' = -b\sum_i(\phi_i')^2 \tag{14}$$

Where we see that the "mass" μ is variable and there is friction specified by the coefficient b given by

$$b = [(1/2)\mu' + (2/r - \delta')\mu] = (2/r)(1-(3m/2r)) - m'/r - \delta'(1-2m/r) \tag{15}$$

In the limit $\alpha \to 0$, we have $m' = \delta' = 0$ and then is clear that $b > 0$ but in general b has no definite sign and its behavior is obtained substituting Einsteins equations 3 and 4, which yields:

$$b = \alpha(r/4)E + (2/r)(1-(3m/2r)) \tag{16}$$

We write eq.14 introducing the integrating factor which turns out to be just $e^{-\delta}$, thus we have

$$\frac{d}{dr}(Ee^{-\delta}) = -(2/r)(1-(3m/2r))e^{-\delta}\sum_i(\phi_i')^2 \tag{17}$$

We note now that $E(r_H) = U_{|r_H} = -V(\phi(r_H)) < 0$ so $Ee^{-\delta}(r_H) < 0$ and since the R.H.S of 14 is negative semidefinite for $r > r_H$ on account of eq. 5 it follows that $Ee^{-\delta}$ is a decreasing function of r. It is therefore always more negative than $Ee^{-\delta}(r_H)$. However, in order to have an asymptotically flat solution, $Ee^{-\delta}$ should approach 0 as $o(r^{-3})$ when $r \to \infty$. Thus, in this theory there is no nontrivial solution representing a static spherically symmetric black hole with regular horizon.

In the Einstein Yang Mills case, a similar calculation yields:

$$\frac{d}{dr}(Ee^{-\delta}) = -[3m - r]e^{-\delta}(w')^2 \tag{18}$$

where $E \equiv [1/2\mu(w')^2 - V(w)]$ with $\mu = r^2(1-2m/r)$. So in this case the term in the R.H.S.is positive for $r > 3m$ allowing the final value (considering r as the "evolution parameter") of $Ee^{-\delta}$ to be larger than the initial one, thus providing for the possibility of nontrivial solutions.(This also shows that substantial part the Y-M structure must be

present beyond the 3/2 of the radius of the horizon for any nontrivial black hole solution in this theory).

We have found a way to analyze the possibilities of nontrivial static spherically symmetric black hole solutions in Einstein Higgs and Einstein Yang Mills theories that serves to proof the non existence of such solutions in the first and shows how the argument is eluded in the second. We expect that this type of analysis should proof useful when applied to other theories to uncover those in which there is potentially new classical hair and to understand some of their properties.

References

[1] P. Bizon, Phys. Rev. Lett. **64** (1990), 2844; M. S. Volkov and D. V. Gal'tsov, Sov. J. Nucl. Phys. **51** (1990), 1171; H. P. Kunzle and A. K. M. Masood-ul-Alam, J. Math. Phys. **31** (1990), 928.

[2] P. Bizoń and T. Chmaj, Phys.Lett. **B297** (1992); 55. S. Droz, M. Heusler and N. Straumann, Phys.Lett. **B268** (1992)

[3] B. R. Greene, S. D. Mathur and C.M. O'Niell, Phys. Rev. D **47**, 2242 (1993).

[4] For a discussion see "Gravitating Solitons and Hairy Black Holes" P. Bizon Jagellonian Univeresity Preprint. January 1994

[5] W.Israel, Comm.Math.Phys. **8** (1968) 245;

[6] S. W. Hawking and G.F.R. Ellis in *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973); B. Carter, in *Black Holes*, Les Houches 1972, eds. B.,C. de Witt (Gordon and Breach, New York, 1973); D.C. Robinson, Phys.Rev.Lett.**34** (1977) 905; P.O.Mazur, J.Phys. **A15** (1982) 3173; D. Sudarsky and R. M. Wald, Phys. Rev. D **46**, 1453 (1992).

[7] J.D. Bekenstein, Phys.Rev.Lett.**28** (1972) 452; Phys.Rev.**D5** (1972) 1239; 2403;

[8] J.E.Chase, Comm.Math.Phys.**25** (1970) 276; C.Teitelboim, Phys.Rev.**D5** (1972) 2941; J.B. Hartle, in *Magic without Magic*, ed. J. Klauder (Freeman, San Francisco, 1972).

[9] P. Bizon (private communication).

[10] R. Rajamaran in *Instantons and Solitons*, (North Holland Publishing Company, Amsterdam, New York, Oxford 1982).