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CALORIMETRIC MEASUREMENTS AND JET COUNTING[#1]

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The clustering algorithms used for counting hadronic jets map a continuum of final states into the discrete set of integer numbers and, therefore, are discontinuous and cannot be stable for all multiparticle configurations with respect to data and rounding errors, changes in the geometry of detector cells, and—on the theoretical side—small corrections in general. Stability of obsevables is ensured by their continuity with respect to an appropriate "calorimetric" topology in the continuum of final states, which is seen to be a stronger requirement than perturbative IR safety. A sequence of shape observables (*jet discriminators*) are derived that: (1) are continuous in the calorimetric topology; (2) effectively measure the qualitative feature of final states popularly known as the "number of jets"; (3) have a form of multiparticle correlators that is natural in the context of quantum field theory. Within the formalism of jet discriminators, the issue of Sudakov corrections becomes marginal, while taking into account hadronization is reduced to studying logarithmic and power corrections which can be done within the formalism of asymptotic operation similarly to the case of total cross sections.

1. Jet counting as the problem of measurement

This talk is about the problem that is central in physics—central because of its role as a bridge between raw experimental data and abstract theoretical formalisms. It is the problem of MEASUREMENT. Concretely, I would like to present a summary of a logical analysis of the measurement process that takes place in calorimeter-based detectors that are used in high-energy experiments. In such experiments the problem of measurement takes a new twist. Indeed, the average multiplicity hadronic final states at supercolliders is $O(100)$ with a wide dispersion so that events with as many as 400 particles in the final state are not uncommon. In such a situation it is impossible to keep track of the particles individually and one uses calorimeter cells to measure the "energy flow" resulting from the collision. However, a collection of raw data from such a detector that tells one what is the energy deposited in each calorimeter cell, cannot be really considered to be a result of a *physically meaningful* measurement—at least to no higher degree than a digitized photograph can be regarded a measurement. Indeed, the amount of data for each event is quite large. Therefore, the measurement process is incomplete without an additional—and absolutely essential—step that can be characterized as *reduction of information*. This step is needed in order to obtain a few numbers that could both be meaningfully used to characterize the essential qualitative features of the final states and be amenable to a theoretical analysis.

A physical basis for such a reduction is provided by two considerations. First, it is the experimental evidence that hadrons in a majority of final states are clustered so that particles in each cluster (jet) go roughly in the same direction [3]. Second, it is the QCD-based picture of hadronic energy flow inheriting main features of the partonic energy flow in the underlying hard process, with each jet associated with an underlying hard parton [7].

At present the jet paradigm forms the basis of practically all high-energy physics [24]. But the problem of adequate numerical description of multijet structure of multihadron final states ("What is a jet?" [38,43]) proved to be theoretically subtle, its apparent simplicity turned out deceptive, while its satisfactory solution, elusive.

The currently prevailing practice (which concerns a lot of interesting physics both about QCD and the

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Standard Model) is to study cross sections for a fixed number of jets in final states (cf. e.g. [38,39]). The number of jets is usually found using the so-called cluster algorithms (the JADE algorithm and its derivatives—for a review and comparison see e.g. [28,29]), wherein one attempts to reconstruct the pattern of underlying partons' momenta by, in effect, inverting the hadronization process.

The latter is non-perturbative in nature, is ill-inderstood, and presents a fascinating field of study.[#3] Complex formalisms loaded with physical meanings have a hypnotizing effect, and there exists a point of view that the problem of adequate description of mutlijet structure is a "physics issue" that concerns the "underlying jet dynamics" rather than "kinematics" and "geometry" (which have, of course, nothing to do with The Profound Physics) and that "the main point is that one would like to identify a class of <...> observables that are in closer correspondence with the underlying jet dynamics" [1]. Such a point of view, however —despite the indisputable importance of understanding hadronization—makes hardly more sense than choosing rulers and clocks to reflect "the underlying dynamics" of gravity when studying the laws governing the trajectories of falling bodies. Instead, one prefers one's rulers to be straight, and clocks, precise. Therefore, it makes a lot of sense to examine the jet counting as a problem of measurement, and to try—instead of attempting to solve the mathematically ill-posed problem of inverting the hadronization process—to find "straight rulers" for measuring the multijet structure of multihadron final states.

Let me emphasize once again: *the problem of an adequate quantitative description of multijet structure is the problem of measurement*—first and foremost. It thus deals more with kinematics than dynamics, and measurements of the multijet structure must be done in such a way as to avoid assumptions on the underlying dynamics. The clean data thus obtained would be neutral with respect to the physics of jets and thus not distorted by any artefacts due to an incomplete understanding of the latter.

Such clean measurements are theoretically possible. The possibility is based on a computation from data of the so-called *jet discriminators* $\mathbf{J}_m$, $m = 2,3...\infty$, that are shape observables given by explicit formulae with rather regular structure (explained below). For a given multiparticle final state $\mathbf{P}$, the m-th jet discriminator $\mathbf{J}_m(\mathbf{P})$ gives a value between 0 and 1, which is interpreted, by construction, as follows (cf. Fig. 1). The crosses are the values of jet discriminators for a final state which can be described as having 3 jets with a 4th jet about to branch off. Conventional jet counting is equivalent to replacing the continuous jet discriminators with dichotomic functions taking values 0 or 1 (shown as blobs). While the values of jet discriminators change a little when the final state configuration is deformed a little, the number of jets may jump and, therefore, is unstable with respect to data errors or small corrections for some final states and some values of $y_{\rm cut}$. If its value is close to 0 then there are less than m clusters of almost collinear hadrons; if its values differ from 0 appreciably then there are no less than m such cluster.

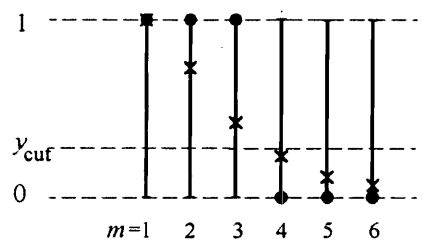

Fig. 1. Jet discriminators *vs.* jet counting.

The jet discriminators provide a logically precise quantitative characterization of the qualitative feature of multiparticle final states popularly known as the number of jets.

The jet discriminators are derived from first principles with very few assumptions which I tried to formulate explicitly below. The arbitrariness in their form corresponds, basically, to a different parametrization of the interval (0,1) and is, thus, inessential.

Although it is possible to use jet discriminators for jet counting in the naive sense (using a cutoff as shown in Fig. 1; such a procedure was called "Moscow sieve" in the preliminary publication [45]), I emphasize that, logically, a more sound approach seems to be simply to compute values of jet discriminators for each event, and then compute their average values for each process. Indeed, the m-the jet discriminator is interpreted as a measure of events that have no less than m jets. Its average value, therefore, is interpreted as $\sigma_{\rm tot}^{m\ \rm jets} + \sigma_{\rm tot}^{m+1\ \rm jets} + \sigma_{\rm tot}^{m+2\ \rm jets} + \ldots$, so that the sequence of jet discriminators provides no less information than the sequence of conventional cross sections for a fixed number of jets. In fact, the information from jet discriminators is more detailed, as should be clear from Fig. 1. In particular, the tail of the distribution—the values of

[#3] Among the issues that plague the standard approach are e.g. the ambiguities due to the arbitrary order in which the hadrons are clustered. Also, the notion of jets' 4-momenta is fully meaningful only in the leading order of perturbation theory; the more important are the radiative corrections, the less satisfactory is assigning particles to a particular jet with a well-defined 4-momentum.

the discriminators for m larger than the perceived number of jets—provides a quantitative description of the width of jets.

But most important of all, the jet discriminators are *continuous* (in the sense appropriate for calorimetric measurements—see below) functions of the multiparticle states.

This should be contrasted with the fundamental pathology inherent in the naive jet counting wherein one attempts to assign an integer "number of jets" to each event. The pathology is that such an assignment represents a *discontinuous* mapping of a continuum of multiparticle final states into a discrete set of integer numbers. Such a mapping cannot be stable with respect to data errors, small corrections etc. for all values of its arguments. As a result, there occurs a randomized smearing between cross sections for adjacent numbers of jets—a smearing that is hard to control. Although this pathology is somewhat masked off by the averaging over events, the issue is of a most fundamental nature and as such—irrespective of immediate numerical significance for the specific applications that are currently discussed—must be elucidated.

2. Measurements and continuity

Although Sterman and Weinberg [7,11,12] suggested a theoretical explanation of jets within the framework of perturbative QCD, the connection of their criterion of IR finiteness of observables with calorimetric measurements remained eclipsed by the calculations-related problems of IR finiteness [11,12,14,15].

In order to understand the issue, recall first the basic model of measurement as represented by simplest measurements of length (in the model considered here statistical errors are neglected). One chooses a unit of length (say, metre), and then constructs rulers graduated with marks indicating parts of the metre. The measurement proper consists in matching, say, a stick against a ruler, determining which marks of the ruler the stick's length falls between, and counting the number of marks on the ruler. The result thus obtained is a pair of rational numbers expressing the length in terms of fractions of the unit of length. An important point is that using different rulers results in different pairs of rational numbers. But the corresponding intervals overlap (or are embedded), and can be made (at least in principle) as small as needed for practical purposes. The latter fact allows one to introduce an abstract notion of the stick's "length" represented by a real number—a limit of pairs of rational numbers obtained in measurements with finite-precision rulers.

Such an abstraction has two aspects. First, one now has to deal with a single abstract object (length as a real number) that can be manipulated according to the rules derived from its definition which turn out to be very similar to the rules governing the rational numbers, and the issue of measurement errors is separated from the fundamental theories based on such a notion of length. The second aspect is the notion of continuity with respect to which the pairs of rational numbers obtained in measurements can be regarded as approximations for the underlying "exact" length.

I emphasize that the two aspects are intimately and inseparably related. This is best seen from the mathematical construction of the space of real numbers as a completion of the space of rational numbers with respect to the topology[#4] that is natural to (in fact, historically derived from) the measurements of length etc. One can see that the crucial element here is the notion of closeness (the topology)—without it, constructing abstract objects to represent "length" is meaningless.

Note that pairs of rational numbers are interpreted as "neighbourhoods" in the space of rational and real numbers that define the corresponding topology.

It is also important to understand that the usefulness of such abstractions as real numbers representing length etc. is determined, in the final respect, by practice—but via the measuring instruments one chooses to use.

Let us apply the same logic to calorimeter-based jet measurements. It is often said that calorimetric detectors measure "energy flow". What is the precise nature of the corresponding abstract objects (analogous to real numbers in the case of the length measurements)? And, first of all, what is the corresponding "calorimetric topology"?

3. Calorimetric (C-) topology and mathematical nature of energy flow

It is important to follow the general pattern exhibited above for the length measurements in as straightforward manner as possible, in order to formalize WHAT IS as precisely as possible[#5].

A calorimetric detector $\boldsymbol{D}$ consists of a finite number of calorimeter cells D_i. The only important thing about such a cell is which part of the unit sphere around the collision point it covers. So, D_i can be thought of simply as a part (subset) of the unit sphere. It is convenient (but not necessary) to assume that the cells cover the entire unit sphere. Whether or not they are non-intersecting is inessential.

[#4] Topology in the mathematical sense is a logically self-consistent description of the usual notion of closeness of rational numbers—or other objects.

[#5] because otherwise one would simply end up with a formalization of one's prejudices!

Each calorimeter cell measures the total energy deposited in it by the final state particles that passed through it. The measured energy ought to be represented as a pair of (non-negative) numbers similarly to the case of length measurements. So, a data record obtained by a calorimetric detector is $\{D_i,(E_i^-,E_i^+)\}_i$ where the index i takes a finite number of values while $E_i^+ > E_i^- \geq 0$ describe the measured energy. As in the case of length measurements, this is a sufficient foundation to construct the abstract objects corresponding to the notion of "energy flow", and this automatically defines a topology in the collection of all such objects. Without going into tedious but essentially simple details, I summarize the results.

The deposited energy is a function that is additive on disjoint subsets (cells), and it is almost a tautology to say that the abstract objects describing "energy flow" are, in the mathematical sense, *measures*[#6] (for a general mathematical background on measures see the fine textbook [2]; an excellent complementary source is [10]; the abstract theory of measure is also discussed in any advanced textbook on the theory of probability).

However, the topology in the space of measures—as in any functional space—can be introduced in different ways (cf. e.g. [2]). The calorimetric topology (C-topology everywhere below) turns out to be the weak topology which can be described as follows. Let $E^{(n)}$ be a sequence of such measures (=energy flows) and let $E^{(n)}(D)$ be their values on any subset D of the unit sphere (i.e. the energies deposited in the detector cell D). Then $E^{(n)}$ converges in the C-topology if and only if the numerical sequences $E^{(n)}(D)$ converge for any D, no correlation being necessary between rates of convergence for different D. (An important technical point here is the compactness of the unit sphere and the finite number of the detector cells in any detector as described in the preceding paragraph.)

The energy flows corresponding to the multiparticle states with a finite number of particles (this is the most important case if one studies hadronic final states) are described as finite linear combinations of δ-functions on the unit sphere, the coefficients and positions being the energies and the directions of the particles. If the state contained collinear particles then their energies are summed in the energy flow. This is discussed in more detail below.

4. Reduction of information in physical measurements

Unlike real numbers that quantify length, abstract measures that represent energy flows of multiparticle states are elements of an infinitely-dimensional functional space and are much less tangible objects than real numbers, and they are also somewhat amorphous for practical purposes. Therefore, the information from hundreds of calorimeter cells —although providing an approximate description of energy flows—can only be regarded as raw data but not as the final result of a physically meaningful measurement.

This is similar to a situation where one cannot use rulers to measure some kind of special sticks directly but rather has to photograph those sticks first. An intermediate representation in the form of a digitized photograph would be crucial for precision measurements but it does not by itself give one the length or width or volume of the stick. First one has to decide what one wishes to measure, and what the length (width, volume) of such sticks is, exactly. Only after that one will be able to measure[#7] the needed parameter by performing its calculation from the digitized photograph.

Therefore, the following axiom suggests itself:

Axiom of reduction of information. *For such measurements to be physically meaningful they must include the step of a reduction of information, i.e. a transition from raw data to a few numbers that quantify the physical features of the objects studied.*

More formally, one would like to have numeric-valued functions that map the abstract objects that our measurement devices measure, into numbers that reflect the features of those objects that one wishes to study. Specifically, we would like to have functions on the collection of energy flows (abstract measures) that quantify the feature of multihadronic states called the number of jets.

Another aspect of the same problem is that it is a typical problem of *algebraization*—i.e. how to express in numbers the qualitative feature (the

[#6] An abstract measure is a function on subsets of the unit sphere whose value on a subset which is a union of two non-intersecting subsets is the sum of its values on the two subsets. The name reflects the fact that, say, the surface of a part of the unit sphere is precisely such a function which, thus, "measures" the surface. Then an energy flow "measures" how much energy passed through the part of the unit sphere covered by a detector cell. Therefore, strictly speaking, a calorimetric detector measures measures called energy flows that measure the flow of energy through detector cells.

[#7] It appears that the confusion with jet counting is partly due to the fact that measurement here takes the form of a computation—and is, thus, not perceived as measurement which it essentially is. The counting of marks on the ruler when measuring sticks is an integral part of the measurement process and cannot be excluded from the latter on the grounds that it is a "computation". Appearences are misleading.

number of jets in our case) that one wishes to study quantitatively.[#8]

5. Physical observables and C-continuity

One wishes to deal with numeric-valued functions on a *topological* space of abstract measures. Where the topology enters the game? A natural answer is that the functions have to be continuous with respect to that topology (C-continuous in the case of C-topology in the space of energy flows). Let us understand why.

Recall that there are data errors due to replacement of measured energy intervals with a single number, and there are rounding errors due to non-ideal nature of computers. One can also, in principle, measure the same event using detectors with different geometry of cells. Nevertheless, we would like that results of our measurements *always* remained approximate descriptions of the measured objects—within the precision of measuring devices, of course. One half of this goal was achieved when the C-topology was identified as a natural topology in the space of the objects one measures. The second half will be achieved if one ensures that the reduction of information is performed using functions that are C-continuous. This would ensure stability of the measured numbers with respect to data and rounding errors, change of the detector's geometry, and small corrections in theoretical calculations.

It should be emphasized that whenever such a stability is lacking (e.g. at the points of discontinuity of the would-be observables), then one sometimes is likely to measures data errors enhanced by the instability rather than the characteristics of the objects one studies. This, of course, should be avoided, whence the following

Axiom of continuity. *Physical observables—the functions used for reduction of information in physical measurements—must be continuous with respect to the topology determined by the measuring devices in order to be stable against small variations of the input (data errors etc.).*

The role of the observables continuous in the above sense may be said to consist in replacing the rather complex (although, I stress, completely physical) topology in the functional space of abstract measures/energy flows with the familiar continuity of real numbers.

One may argue that in real life one often recurs to descriptions in terms of discrete values (good and bad; green, red and blue; etc). But one should not forget that one can make those descriptions more precise whenever needed (not-so-good; light green etc.)—very much as in the case of length measurements. One can view the continuous jet discriminators as a construction that allows one to formalize notions like "a final state with two and a half jets".

The two immediate general questions are:

— What are the C-continuous functions?

— How IR-safety relates to C-continuity?

6. C-continuous functions on abstract measures

Functions defined on functional spaces are traditionally called functionals in mathematics. Although there seems to exist no formal studies of general non-linear continuous functionals on the space of abstract measures, it is not difficult to describe a family of such functionals. Let $\hat{q}$ denote a generic element of the unit sphere S around the collision point ($\hat{q}$ can be regarded as a unit vector). Let $\mathrm{d}\hat{q}$ denote the standard integration over S, so that $\int \mathrm{d}\hat{q} = 4\pi$. Let $E(\hat{q})$ be an energy flow (=abstract measure on the sphere), so that its value on the subset $D \subset S$ (=the energy deposited in the calorimeter cell D) is $E(D) = \int_D \mathrm{d}\hat{q}\, E(\hat{q})$.

A simplest—linear—functional is

$$\mathsf{F}_1(E) = \int_S \mathrm{d}\hat{q}\, E(\hat{q})\, \varphi(\hat{q}), \tag{1}$$

where $\varphi(\hat{q})$ is continuous on S. More general—m-linear—functionals are as follows:

$$\mathsf{F}_m(E) = \int_S \mathrm{d}\hat{q}_1 \ldots \mathrm{d}\hat{q}_m\, E(\hat{q}_1) \ldots E(\hat{q}_m)\, \varphi(\hat{q}_1, \ldots, \hat{q}_m) \tag{2}$$

where $\varphi(\hat{q}_1, \ldots, \hat{q}_m)$ is a continuous function on a direct product of m copies of the unit sphere. C-continuity of such functionals is a trivial consequence of the general theory of measure [2].

7. C-continuous functions on multiparticle final states

Let E_i be the energy of the i-th particle and $\hat{p}_i$, its direction (a unit 3-vector). Then a multiparticle state consisting of n particles can be described as

$$\mathsf{P} = \{E_1, \hat{p}_1; \ldots; E_n, \hat{p}_n\}. \tag{3}$$

The energy flow for such a state is

$$E(\hat{q}) = E_1\, \delta(\hat{q}, \hat{p}_1) + \ldots + E_n \delta(\hat{q}, \hat{p}_n), \tag{4}$$

where the δ-functions on the unit sphere are defined as

$$\int_S \mathrm{d}\hat{q}\;\; \delta(\hat{q}, \hat{p})\, \varphi(\hat{q}) = \varphi(\hat{p}), \tag{5}$$

for any continuous φ. The functionals (2) on such energy flows can be regarded as functions on multiparticle states:

[#8] Such an algebraization may require highly non-trivial mathematical means, as is the case e.g. with the characterization of manifolds in terms of cohomology groups etc. The case of jet counting, however, is not that difficult.

$$\mathbf{F}_m(\mathbf{P}) = \mathbf{F}_m(E_1, \hat{p}_1; \ldots; E_n, \hat{p}_n)$$
$$= \sum_{1 \le i_1 \le n} \cdots \sum_{1 \le i_m \le n} E_{i_1} \ldots E_{i_m} \varphi_m(\hat{p}_{i_1}, \ldots, \hat{p}_{i_m}). \tag{6}$$

It is easy to recognize here an m-particle correlator, especially if it is rewritten in the following form with combinatoric coefficients that are immediately reminiscent of the Bose-Einstein statistics:

$$\mathbf{F}_m(E_1, \hat{p}_1; \ldots; E_n, \hat{p}_n) = \sum_{1 \le i_1 \le \ldots i_m \le n} \frac{m!}{m_1! \ldots m_k!}$$
$$\times E_{i_1} \ldots E_{i_m} \varphi_m(\hat{p}_{i_1}, \ldots, \hat{p}_{i_m}), \tag{7}$$

where $m_1, \ldots, m_k$ are the numbers of elements in groups of identical indices among $i_1, \ldots, i_m$ ($k \le m$ and $m_1 + \ldots + m_k = m$). Recall that such correlators are a standard instrument of the formalism of second quantization (for a systematic exposition see e.g. [23]) that has been specifically developed for studying multiparticle systems with a varying number of particles, which is exactly what is needed in the case of multihadron final states.

8. C-continuity *vs* IR safety

IR safety within perturbation theory was established for a wide class of functions on multiparticle states [14,15]. It turns out that C-continuity is a more stringent restriction on the form of the observables than the perturbative IR safety.

In general, a function $\mathbf{F}(\mathbf{P})$ on multiparticle states $\mathbf{P}$ is as series of component functions F_n—one for each n, the number of particles in $\mathbf{P}$:

$$\mathbf{F}(\mathbf{P}) = \mathbf{F}(E_1, \hat{p}_1; \ldots; E_n, \hat{p}_n)$$
$$\equiv F_n(E_1, \hat{p}_1; \ldots; E_n, \hat{p}_n). \tag{8}$$

For such an observable to be IR safe it should, as a minimum, be invariant with respect to fragmentations (replacement of a particle with two collinear particles with the same total energy):

$$F_n(E_1, \hat{p}_1; \ldots; E_n, \hat{p}_n)$$
$$= F_{n+1}(E_1, \hat{p}_1; \ldots; zE_n, \hat{p}_n; (1-z)E_n, \hat{p}_n),$$
$$0 \le z \le 1, \tag{9}$$

as well as some natural regularity conditions at the points where the fragmentation invariance is imposed [14]. Eq. (9) is, in fact, a boundary condition for construction of F_{n+1} once F_n is defined. F_{n+1} is otherwise arbitrary.

To compare this with C-continuity, consider the states $\mathbf{P}'$ whose first n particles are the same as on the l.h.s. of (9), and that contain additional particles that are almost collinear to $\hat{p}_n$ and whose total energy is $(1-z)E_n$. C-continuity implies that for any $\delta > 0$ there should exist a small neighbourhood of $\hat{p}_n$ on the unit sphere such that as long as all the additional particles are within that neighbourhood, the observable will deviate from its value before fragmentation by no more than δ, i.e. $|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{P}')| < \delta$. Note that there is no restriction on the number of particles in the fragmented state $\mathbf{P}'$. This means that C-continuity imposes a sort of uniformity bounds on the rate of change (on the derivatives in the case of smooth functions) of the component functions F_n —*for all n* simultaneously. One can see that by making successive F_n vary arbitrarily fast while maintaining fragmentation invariance, one can make impossible existence of such a neighbourhood for some small δ.

Problem. *If the C-continuity is more stringent than the perturbative IR safety, what would be a purely quantum-field-theoretic interpretation of the C-continuity? (Calorimeters, in the final respect, are described by quantum field theory, too!).*

Note that abstract measures of the form (4) are dense in the space of all abstract measures, i.e. any abstract measure can be approximated in the C-topology by discrete measures of the form (4). Therefore, any C-continuous function on multiparticle states can be extended by C-continuity to a (non-linear) functional defined on all abstract measures. It is not clear, however, whether there exist convenient forms of such generalized C-continuous functionals other than (2), and whether such general functionals can be useful in applications. For the purposes of counting jets, it will be sufficient to consider functionals of the form of multiparticle correlators (6) on energy flows corresponding to states with a finite number of particles, (4).

9. Discriminating purely m-jet states

Once an appropriate notion of closeness has been found, we can use it to compare multiparticle states with regard to similarity of their structure. To reemphasize: using the notion of C-closeness for this purpose ensures that the comparison will be stable against data errors and rounding errors, variations in the geometry of detector cells, and small corrections in theoretical calculations.

We describe a state as having m jets if it is similar to a state that contains exactly m particles such that none of them is soft, and such that no pair has almost collinear momenta. A straightforward attempt to count jets this way (based on C-continuity) would result in prescriptions similar to the Snowmass Accord (see e.g. [43]). But a naive counting of jets, as discussed above, is unsatisfactory in principle, and we would like to find (C-) continuous

numeric-valued observables to characterize the "number of jets".[#9]

The starting point of jet counting is the above identification of the "clearly m-jet states". The essence of the problem is that one would like to discriminate between those states and other states in the continuum with C-topology which is not a particularly simple object to deal with—there is no simple and natural norm or distance or other single-valued measure of closeness for the C-topology. Therefore, we should first consider what are the most general options available to us.

The wisdom of general topology (see e.g. [4]) tells us that the single most general option for discriminating elements of a topological space is to watch values of continuous functions that take special values (e.g., 0) on the chosen elements. Therefore, let us try to construct C-continuous functions that are exactly zero on the "clearly m-jet states". A little further thought reveals that such states are, essentially, the states with exactly m particles, and it should be technically easier to construct a function that is exactly zero on such states irrespective of whether they have exactly m jets or some of the m particles are soft or almost collinear etc. Automatic fragmentation invariance of C-continuous functions dictates that such a discriminator function will be zero on all states with less than m particles because any such state can be turned by fragmentations into a state with m particles. The conclusion is that a reasonable goal is to construct, for each m, a C-continuous *jet discriminator* $\mathbf{J}_m(\mathbf{P})$ that is exactly zero on any state with less than m particles. Then one will be able to say that if $\mathbf{J}_m(\mathbf{P}) \sim 0$ then $\mathbf{P}$ should have less than m jets. Note that one should then also expect that $\mathbf{J}_M(\mathbf{P}) \sim 0$ for all $M > m$.

Another restriction is that one would prefer to look for jet discriminators among the observables that have the form of multiparticle correlators (6). This would leave open the option of their possible study by field-theoretic methods.

We are all set up for a decisive step.

Consider a function of the form (6) that is exactly zero on all states with less than m particles or the states that can be obtained from those by fragmentations. It is sufficient to ensure that such a function is exactly zero on any state with m particles two of which are exactly collinear. This is achieved if φ_m is zero whenever any pair of its arguments (that are particles' directions) coincides. General theorems of analysis [2] tell one that a sufficiently smooth function that has such a property can be usually represented as a product of a factor (a polynomial) that describes the geometry of zeros and a non-zero function. In our case, $\varphi_m(\hat{p},\hat{p},\ldots) = 0$ implies that $\varphi_m(\hat{p}_1,\hat{p}_2,\ldots) = (1-\hat{p}_1\cdot\hat{p}_2)\times\tilde{\varphi}_m(\hat{p}_1,\hat{p}_2,\ldots)$ where the form of the nullifying factor is chosen from considerations of rotational invariance. Since φ_m is symmetric in its arguments, there is a factor of this form for each pair of particles. Taking φ_m to be a product of only such factors, one arrives at the following expression for the m-th jet discriminator ($\mathbf{J}_1(\mathbf{P}) \equiv 1$):

$$\mathbf{J}_m(\mathbf{P}) = \mathbf{J}_m(E_1,\hat{p}_1;\ldots;E_n,\hat{p}_n) = N_m\, E_{\text{tot}}^{-m} \times \sum_{1\le i_1<\ldots<i_m\le n} E_{i_1}\ldots E_{i_m}\, j_m(\hat{p}_{i_1},\ldots,\hat{p}_{i_m}), \tag{10}$$

where

$$j_m(\hat{p}_1,\ldots,\hat{p}_m) = \prod_{1\le i<j\le m}(1-\hat{p}_i\cdot\hat{p}_j) = \prod_{1\le i<j\le m}(1-\cos\theta_{ij}), \tag{11}$$

and N_m is a normalization. The power of the total energy of the multiparticle state, $E_{\text{tot}} = \sum_{1\le i\le n} E_i$, is introduced to make the discriminators dimensionless. Note that the total energy is C-continuous, and the ratio of two C-continuous functions is again C-continuous. The combinatoric coefficients are gone because the corresponding terms in the sum are automatically zero due to the properties of φ_m.

10. Properties of the jet discriminators

It turns out that the jet discriminators exhibit a behaviour that seems to make them an adequate tool for studying multijet structure.

(i) Numerical experimentation shows that the jet discriminators reach their global maximal values on the configuration with the energy uniformly spread over the unit sphere, $E_{\mathbf{P}_\infty^{\text{sym}}}(\hat{q}) = E_{\text{tot}}(4\pi)^{-1}$. Then the normalizations can be defined so that the discriminators vary in the interval (0,1). One finds: $N_2 = 2$, $N_3 = \frac{27}{4} = 6.75$, $N_4 = 36$, $N_5 = \frac{9375}{32} = 292.96875$, $N_6 = \frac{455625}{128} = 3559.5703125$, etc[#10]. Note that this asymptotic maximum is also reached on some highly symmetric configurations—but is never exceeded.

(ii) Some special values of the jet discriminators are as follows. For the state $\mathbf{P}_3^{\text{sym}}$ consisting

[#9] One should not, of course, expect *a priory* that there exists, say, a single function whose values would be close to an integer n on multiparticle states that are close to those with exactly n particles. Such implicit assumptions may be very misleading—in dealing with such an unusual and subtle problem as jet counting it is essential that one should keep all options open by trying to understand and follow the internal logic of the problem.

[#10] I thank B. B. Levtchenko for a numerical check of the normalizations.

of 3 symmetrically arranged particles, $\mathbf{J}_3(\mathbf{P}_3^{\rm sym}) = \frac{27}{32} \approx 0.84$. For the symmetric state of 4 particles (tetrahedron), $\mathbf{J}_3(\mathbf{P}_4^{\rm sym}) = 1$ and $\mathbf{J}_4(\mathbf{P}_4^{\rm sym}) = \frac{64}{81} \approx 0.8$. For the symmetric state of six particles (octahedron) one has $\mathbf{J}_3(\mathbf{P}_6^{\rm sym}) = \mathbf{J}_4(\mathbf{P}_6^{\rm sym}) = 1$. It turns out that the jet discriminators exhibit a threshold behaviour—their values become close to the maximal value, 1, rather fast and remain stable.

(iii) The discriminators exhibit a region of stability with respect to almost collinear fragmentations—this is due to the $O(\theta^2)$ behaviour of the factors at small θ.

(iv) The jet discriminators exhibit a monotonic decrease for the values of m larger than the perceived number of jets in the state. This is explained as follows.[#11] Let the state $\mathbf{P}$ consist of two clusters of particles, each concentrated in a cone of a small angular diameter θ. Then, as is easily seen from the above explicit expressions, $\mathbf{J}_3(\mathbf{P}) = O(\theta^2)$ (because each triplet of particles contains a pair belonging to the same cluster), $\mathbf{J}_4(\mathbf{P}) = O(\theta^4)$, etc.

(v) The observables of the above form can be used to study other characteristics of multijet states. (Note e.g. that one can use one C-continuous observable as a parameter of another—which gives one much flexibility in constructing new observables.) In particular, the tail of the distribution of values of the jet discriminators describes the width of jets: if one insists on assigning a definite number of jets m to a state, then the $(m+1)$-th discriminator provides a satisfactory numerical characterization of the average angular width of jets in the state (one should take the square root because of the $O(\theta^2)$ behaviour of the basic factors in the jet discriminators).

(vi) The formulae and normalizations presented above are, in principle, valid for any process, but they are most useful for the process $e^+e^- \to$ hadrons. One may need to introduce modifications in order to make them more convenient for studying other processes. This can be done similarly to how the clustering algorithms are modified in such cases. (Note that it is only at this point that one should make the assumption of masslessness of particles in the final states and the Lorentz-covariance is mentioned the first time.)

In various processes involving hadron beams one would like to separate the beam jet consisting of the remnants of the initial hadrons from the true transverse jets (see e.g. [42,47]). This is easily achieved by introducing a factor $(1-(\hat{p}_i\hat{q})^2)$ for each particle in the m-plet in (11), where $\hat{q}$ describes the beam direction.

For the deeply inelastic electron-nucleon scattering one prefers to view the process in the Breit reference frame (see [47] and refs. therein). To this end it is sufficient to rewrite the above formulae in a Lorentz-covariant form. Let $p_i^\mu = (E_i, E_i\hat{p}_i)$ be each particle's 4-momentum and let P be a 4-vector whose rest frame determines the coordinate system in which jets are observed (for the detector rest frame $P^\mu = (1,0)$). Then it is sufficient to replace $E_i \to P\cdot p$ and $E_iE_j(1-\cos\theta_{ij}) = p_i\cdot p_j$.

(vii) Computation of jet discriminators from data is, in principle, straightforward—one simply treats each calorimeter cell as an individual particle in eq. (10) (note e.g. that one does not even need to determine the momentum and direction of, say, a π-meson that lit up a cluster of neighbouring cells). But it can require considerably more computer resources than clustering algorithms for final states with $\sim 100-400$ particles and $m \sim 4-6$. However, the starting point here is the well-defined explicit analytical expressions for jet discriminators with a very regular structure that allows a number of optimizations. The two important optimizations are as follows. One is to perform the summation a la Monte-Carlo with each particle receiving a probability equal to its share of the total energy. The second optimization consists in a preclustering similar to the usual clustering algorithms but with very small $y_{\rm cut}$—it is sufficient to reduce the number of particles and protojets to $\sim 30-50$, then further explicit summations could be done even on a lower-end PC. Note that preclustering here is only an auxiliary trick at intermediate stages of the computation which only needs to be done in definite cases when one expects such a preclustering not to distort the values of the discriminators. The exact details of such a preclustering can be determined from the quality of the resulting approximation for jet discriminators and, thus, are easily controlled.[#12]

(viii) *Lumps vs. jets.* As was discussed e.g. in [28,29], clustering algorithms may sometimes count "lumps" instead of jets. A lump is a group of soft particles ($E_i \sim 0$) with large relative angles but erroneously identified as a jet by a clustering algorithm. The jet discriminators are insensitive to lumps by construction because the contribution to

[#11] I thank Bruce Straub for this important question.

[#12] A study of feasibility of computation of jet discriminators from data is currently under way in collaboration with B. B. Levtchenko (Institute of Nuclear Physics of the Moscow State University, boris@npi.msu.su). Members of jets-related experimental collaborations are encouraged to contact us.

jet discriminators from particles whose total relative energy is $O(\varepsilon)$, is always $O(\varepsilon)$.

(ix) Higher-order corrections to jet discriminators in the case of 4π detectors can be computed as in the case of the total cross sections except that one cannot perform the Wick rotation—one has to deal with Minkowskian diagrams and phase-space integrations. But this still is a simpler task than in the case of traditional approach because the jet discriminators differ from the total cross section by a weight that has a natural form of scalar products. Therefore, my opinion based on an experience of developing methods and algorithms [18–19,21,25, 26,32] that made possible the recent crop of NNL calculations [20,27,30,31,33–37], is that it should be possible (if hard) to calculate the NNL corrections to the third jet discriminator, and the 5-th discriminator at the tree level for the process $e^+e^- \to$ hadrons.

(x) The importance of small Sudakov corrections in the usual clustering algorithms is entirely due to the discussed instability of clustering algorithms (discontinuous yes/no decisions at intermediate steps) with respect to small corrections. With jet discriminators, one deals with a weighted cross section similar to the total cross section, so that the calculations are greatly simplified as compared with the usual jet calculations, and the corrections are adequately resummed using the standard renormalization group methods. Therefore, for measuring the number of jets *per se* within the formalism of jet discriminators, the issue of Sudakov corrections is marginal.

But Sudakov-type corrections remain crucial for studying the tail of the distribution of the values of jet discriminators (see Fig. 1 above).

(xi) *Power corrections.* The importance of studying power corrections in the context of jet counting was discussed e.g. in [41]. Because of the similarity of jet discriminators and total cross sections pointed out above, studying power corrections can also be done similarly to the latter case. For instance, in the case of $\sigma_{\rm tot}(e^+e^- \to hadrons)$ the power corrections are governed by non-zero vacuum condensates of local operators (cf. the method of QCD sum rules; see e.g. [17]). They can be seen (but not computed!) within perturbation theory by performing a correct expansion in light quark masses [22]. The same mathematical techniques (known as the theory of asymptotic operation [40, 44,46]) can be applied to the case of jet discriminators to derive the structure of power corrections. But this requires additional work.

(xii) Lastly, I would like to point out that the jet discriminators are closely related to the usual shape observables—thrust [5], spherocity [6], acomplanarity [9], and the non-rotationally invariant moments and their algebraic combinations of [13,16]—which are all C-continuous, as well as to the series of observables including energy-energy correlations [8]—which are *not* C-continuous. Refs. [13,16] are particularly instructive because the failure to construct jet finding observables was due to a few subtle imprecisions in formalization of the problem including an unmotivated and incorrect use of the L_2 topology in the space of energy flows represented as continuous functions on the sphere.

11. Summary

The formalism of jet discriminators is based on very few assumptions (correlator-type C-continuous observables and rotational invariance). It offers a comprehensive alternative to the clustering approach to jet counting and is devoid of the fundamental flaws of the latter (which are all due, in the final respect, to the discontinuous nature of the naive jet counting and, thus, instability and sensitivity to otherwise negligible details).

Apart from the natural inertia, and in view of the flaws inherent to the clustering approach I see no fundamental reason why the practice of clustering algorithms should continue in the problems where the concept of jets is simply used to tag the events (e.g. in studies of the Standard Model), except as an intermediate preclustering optimization trick in approximate computation of the jet discriminators from data, or as a means of a preliminary analysis of data. The advantages of the formalism of jet discriminators should be particularly non-negligible at lower energies and whenever higher precision and quality of data as well as a better control of theoretical uncertainties (due to higher order and higher twist corrections) are needed.

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References

[1] (anonymous)
[2] L. Schwartz, *Analyse Mathématique*. Vol. 1. Paris: Hermann, 1967
[3] G. Hanson et al., Phys. Rev. Lett. 35 (1975) 1609
[4] R. Engelking, General Topology. Warszawa, PWN, 1977
[5] E. Farhi, Phys. Rev. Lett. 39 (1977) 1587
[6] H. Georgi, M. Machacek, Phys. Rev. Lett. 39 (1977) 1237
[7] G. Sterman, S. Weinberg, Phys. Rev. Lett. 39 (1977) 1436
[8] C. C. Basham, L. S. Brown, S. D. Ellis, S. T. Love, Phys. Rev. Lett. 41 (1978) 1585
[9] A. De Rujula, J. Ellis, E. G. Floratos, M. K. Gaillard, Nucl. Phys. B138 (1978) 387
[10] R. D. Richtmyer, *Principles of Advanced Mathematical Physics*. Vol. 1. Springer-Verlag, 1978
[11] G. Sterman, Phys. Rev. D17 (1978) 2773
[12] G. Sterman, Phys. Rev. D17 (1978) 2789
[13] G. C. Fox, S. Wolfram, Nucl. Phys. B149 (1979) 413
[14] G. Sterman, Phys. Rev. D19 (1979) 3135
[15] G. Tiktopoulos, Nucl. Phys. B147 (1979) 371
[16] G. Fox, T.-Y. Tse, S. Wolfram, Nucl. Phys. B165 (1980) 80
[17] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B174 (1980) 378
[18] F. V. Tkachov, Phys. Lett. B100 (1981) 65
[19] K. G. Chetyrkin, F. V. Tkachov, Nucl. Phys. B192 (1981) 159
[20] K. G. Chetyrkin, S. G. Gorishny, S. A. Larin, F. V. Tkachov, Phys. Lett. B132 (1983) 351
[21] S. G. Gorishny, S. A. Larin, F. V. Tkachov, Phys. Lett. B124 (1983) 217
[22] F. V. Tkachov, Phys. Lett. B125 (1983) 85
[23] N. N. Bogoliubov, N. N. Bogoliubov (jr.), *Introduction to Quantum Statistical Mechanics*. Moscow, 1984
[24] S. D. Ellis, *Lectures on perturbative QCD*, preprint NSF-ITP-88-55
[25] S. G. Gorishny, Nucl. Phys. B319 (1989) 633
[26] S. G. Gorishny, S. A. Larin, L. R. Surguladze, F. V. Tkachov, Comp. Phys. Commun. 55 (1989) 381
[27] L. R. Surguladze, F. V. Tkachov, Nucl. Phys. B331 (1990) 35
[28] S. Bethke, Z. Kunszt, D. E. Soper, W. J. Stirling, preprint CERN-TH.6222/91, 1991
[29] S. Catani, preprint CERN-TH.6281/91
[30] S. G. Gorishny, A. L. Kataev, S. A. Larin, Phys. Lett. B259 (1991) 144
[31] S. G. Gorishny, A. L. Kataev, S. A. Larin, L. R. Surguladze, Phys. Lett. B256 (1991) 81
[32] S. A. Larin, F. V. Tkachov, J. A. M. Vermaseren, preprint NIKHEF-H/91-18, 1991
[33] S. A. Larin, F. V. Tkachov, J. A. M. Vermaseren, Phys. Lett. B272 (1991) 121
[34] S. A. Larin, F. V. Tkachov, J. A. M. Vermaseren, Phys. Rev. Lett. 66 (1991) 862
[35] S. A. Larin, J. A. M. Vermaseren, preprint NIKHEF-H/91-17, 1991
[36] S. A. Larin, J. A. M. Vermaseren, Phys. Lett. B259 (1991) 345
[37] L. R. Surguladze, M. A. Samuel, Phys. Rev. Lett. 66 (1991) 560
[38] S. D. Ellis, in [39]
[39] Z. Kunszt, ed, *New Techniques for Calculating Higher Order QCD Corrections*. ETH. Zürich, 1992
[40] F. V. Tkachov, in [39]
[41] B. R. Webber, in [39]
[42] S. Catani, preprint CERN-TH.6895/93
[43] S. D. Ellis, preprint CERN-TH.6861/93
[44] G. B. Pivovarov, F. V. Tkachov, Int. J. Mod. Phys. A8 (1993) 2241
[45] F. V. Tkachov, preprint INR-F5T/93-01
[46] F. V. Tkachov, Int. J. Mod. Phys. A8 (1993) 2047
[47] B. R. Webber, preprint CERN-TH.6871/93, 1993

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