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Dynamic Binding of Holes in Doped Antiferromagnets

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Abstract.

We study the low-energy part of the spectrum of a version of the $t - J - V - V'$ model with a strongly anisotropic antiferromagnetic exchange term. We perturb around the half filled state and work in the large $\frac{J}{t}$ limit. We argue that one can suppress phase separation by tuning the parameters in such a way as to exactly balance the attractive part of the exchange interaction and to escape the conclusions of Kivelson, Emery and Lin. We rigorously show that in the energetically most favoured states, two holes of opposite spin form a bound state. The binding mechanism is dynamical and relies on the interplay between the kinetic and magnetic terms in the Hamiltonian. Phase separation is suppressed because the leading effective interaction between holes depends on the hopping term and attraction is enhanced by delocalization. We also propose a model of a valence bond solid for which the attractive part of the exchange interaction is not balanced, holes bind in pairs and phase separation is suppressed for purely geometrical reasons. Our mathematical results are proven by explicitly block diagonalizing the Hamiltonians by means of a rigorously controlled cluster expansion algorithm not yet used in this context.

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1. Introduction.

The discovery of high-T_c superconductors triggered renewed interest in the problem of the motion of holes in an antiferromagnetic background. Of the models retaining the physically significant features of the general problem, the $t-J$ model and its variants have been singled out [1], [2] as among the most basic models describing the behaviour of hard core electrons on a lattice. The Hamiltonian we choose has the following form:

$$\begin{aligned}\mathbb{H}^{\Lambda}_{tJ}(\lambda,t) = & \sum_{|x-y|=1} n_x\, n_y \left[\frac{1}{2}\left(J\overset{(3)}{\sigma_x}\overset{(3)}{\sigma_y} + V\right) + \lambda\left(\overset{(1)}{\sigma_x}\overset{(1)}{\sigma_y} + \overset{(2)}{\sigma_x}\overset{(2)}{\sigma_y}\right)\right] \\ & - \sum_{\sigma,\|x-y\|_1=1} t\left(c^{+}_{x\sigma}\, c_{y\sigma} + \text{h.c.}\right) + V' \sum_{\|x-y\|_1=2} n_x n_y. \qquad (1.1)\end{aligned}$$

The plain $t-J$ model corresponds to the case $V=-J$ and $V'=0$ and can be obtained by extrapolating to arbitrary ratios of $\frac{t}{J}$ a Hamiltonian directly derived from the Hubbard model at strong coupling. Expansions around the classical limit $t=\lambda=0$ are particularly appealing because of the apparent simplicity of this case. However, this approach requires a careful tuning of the parameters in the Hamiltonian in order to avoid the phase separation of the ground state into hole rich and hole poor regions which can occur even in the dilute limit. The difficulty of the problem is well illustrated in the paper [3] of Kivelson, Emery and Lin, who show, by means of a perturbative calculation truncated at the leading order, that if t and λ are small enough and if $V'=0$ then various kinds of phase separation occur. These authors consider all values of V, with the exception of the critical value $V=J$. Their result gave rise to an as yet unresolved controversy with various other groups performing numerical work with high temperature cluster

expansions and with exact diagonalizations, and whose results suggest the existence of both a phase separated and a superconducting regime for $\frac{J}{t} < 1$; see Putikka et al. [5], Dagotto et al. [6], Prelovsek and Zotos [7]. The last group also observes an intermediate kind of phase separation which occurs in a precursor phase to superconductivity and in which holes line up and form domain walls. Both kinds of phase separation have been experimentally observed [8], [9].

The main body of this paper contains a technical discussion of a cluster expansion algorithm which is well suited to a rigorous as well as to a numerical analysis of this class of problems and allows one to systematically block diagonalize the Hamiltonian. Each block can be interpreted as the effective Hamiltonian controlling the interaction between a certain set of quasiparticles. This allows us to compute properties of excited states and of holes such as the dispersion laws for isolated quasiparticles and to study the interaction between quasi-holes and spin-wave excitations. We illustrate this formalism by studying the problem of deriving the reduced Hamiltonian for a pair of holes and proving the existence of a bound state. Although the analysis of this algorithm is our main concern, we cannot avoid a heuristic discussion of the problem of phase separation. In fact, if the effective interaction between holes is of a kind that favours phase separation, then bubbles of holes form even in the diluted limit, and a two-particle calculation is not sufficient to describe the relevant physical properties. Since we are not able to substantiate our arguments on phase separation with rigorous proofs, yet, we present our discussion of this question in Appendix B. Our conclusion is that if one chooses the coupling V to lie within a cusp in parameter space such that $|V - J| \leq O(t^2)$, then the occurrence of phase separation is decided by the next neighbour coupling V'; see Fig. 2 in Appendix B. In the limit of small t and small λ we can distinguish four different phases. In the first and in the fourth phase, holes phase separate in bubbles, in the second one domain walls appear, and in the third phase a balance of opposite effects is reached which stabilizes a superfluid state of tightly bound hole pairs. Phase separation in bubbles occurs because the leading interaction between holes is static, i.e. velocity independent, and, as a consequence, holes are expelled from the antiferromagnetic insulator. In our scenario, bubbles and domain walls evaporate in

the superconducting phase because the leading interaction term in the effective Hamiltonian depends on the hopping and binding between two holes is enhanced by the delocalization of the pair. This is the "dynamic binding mechanism" referred to in the title of this article. The superconducting phase is located near the crossover region between two phases corresponding to two different types of long range order in the phase separated bubble. The resulting frustration effect makes the otherwise subleading interaction term giving rise to dynamic binding become the dominant term. An alternative mechanism to escape phase separation which has been proposed in [10] involves long range interactions. Long range Coulomb interactions cannot be neglected in a model that is designed to give a realistic account of the physical situation and they certainly might play a major role in the problem. However, an explanation of high temperature superconductivity based on the frustration of phase separation due to long range interactions is problematic because of the competition between superconducting and charge density wave instabilities [11]. We thus think that it is worth investigating whether the superfluid phase we find at large $\frac{J}{t}$ and large $\frac{J}{\lambda}$, and which is controlled by dynamic binding, can be continued to the physically relevant parameter region. We are presently investigating this question numerically with the help of the algorithm described in this article and plan to discuss the results in [12]. A second direction worth exploring is illustrated by a Valence Bond Solid (VBS) model of strongly correlated electrons that we also study in this article. In this model, phase separation is suppressed by the special geometry of the lattice. This model is also interesting from the technical point of view because it gives us a chance to illustrate our zero temperature expansion at work in a situation with a "bulk" sign problem. That holes bind in spin-singlet pairs in models of this type has been argued previously in [22].

The model Hamiltonian in (1.1) is defined on a large cube $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions. The Hilbert space is $\mathcal{H}^{\Lambda}_{t,J} = \mathbb{P}_S \otimes_{x\in\Lambda} \mathbb{C}^3$, where $\mathbb{P}_S$ is the projection onto the subspace which is invariant with respect to global spin flips. As a basis for $\mathbb{C}^3$ we choose $|\uparrow\rangle$, $|\downarrow\rangle$, $|e\rangle$, where e means "empty". Doubly occupied sites are thus excluded. Fixing the energy units in such a way that $J = 1$, the parameter region we explore has both the transverse coupling λ and the hopping amplitude t small compared to J. The number of electrons is taken

to be equal to $|\Lambda|$ minus one or two, i.e., we study the problem of the propagation of one hole in the background of a half-filled antiferromagnetic insulator and the interaction between two holes. For a system with two holes, the results depend on whether the spins of the two holes are opposite or parallel. For holes with antiparallel spins we show that binding occurs.

Because we choose $V \approx J$, there is no explicit attractive potential of order J. Hence, even in the limit of small λ and t, binding is due to an interplay between the kinetic and the magnetic terms. The reason why holes tend to bind is related to the obstruction of independent motions by an antiferromagnetic background. In fact, as a single hole moves it leaves behind a trace of frustrated spins unless it performs rather complicated paths to reshuffle the disturbed spins back into place; see for instance [19]. Within a small t expansion, the amplitude for hopping of an isolated hole is of order t^6. However, a pair of holes has an amplitude for hopping of order t^2, provided the two holes simply follow each other. The understanding of these string effects and the construction of mathematical algorithms for their analysis are important for the development of quantitative methods to study such magnetic systems. The Brinkman-Rice approximation [13] in which one sums up only self-retracing paths and extrapolates to $J = 0$, and variations of it [14], have been instrumental in approximately computing Green's functions in closed form. The numerical evaluation of the exact series expansions would be useful to validate the conclusions of these approximate treatments. It would also be interesting to be able to gather information on the quasiparticle structure that can be used to enhance the signal of Quantum Montecarlo calculations which suffer from the infamous "sign problem" and from the smallness of the wave function renormalization factor; see for instance [15]. In this paper, we develop a cluster expansion whose convergence can be proven rigorously and which gives a method for constructing the effective reduced Hamiltonians for any finite number of holes by integrating out the spin degrees of freedom. With the same method, we can also produce an expansion for the exact creation operator of dressed quasiparticles. A rigorous analysis of such methods is presented in this article. We explicitly compute the series expansions to leading order, prove their convergence and show that, in the two-holes-sector, these quasiparticles are bound in the ground state.

Dressing transformations [16],[17] are the technical tool we use to carry out our program. This technique is also useful in the study of quantum spin systems such as the doped $t - J - V - V'$ model for which there is a sign problem, i.e. for which the ground state wavefunction Ψ_0 and the statistical weights in the partition function are not positive. In such a situation, one cannot write Ψ_0 as $\exp(-h)$, where $h = -\ln \Psi_0$ can be interpreted as the short ranged Hamiltonian of a statistical physics problem, in order to unravel the clustering properties of Ψ_0. A more general [16] way of representing the ground state wavefunction is to write it in the form $\Psi_0 = \exp(\mathbb{R})|0\rangle$, where $|0\rangle$ is the unperturbed ground state, the Néel state in the case of the Heisenberg model, and $\mathbb{R}$ is an operator which is short ranged and is translation invariant if so is the Hamiltonian. In a sense, $\mathbb{R}$ plays the role of the logarithm of the ground state wavefunction, but the added degree of flexibility deriving from the use of operator-valued, rather then $c-$valued, functions allows one to accomodate the troublesome negative signs. Refining this idea [17], one can allow the operator $\mathbb{R}$ to be skewsymmetric, so that the resulting ground state wavefunction does not require normalization factors, and a natural basis for finite energy excitations orthogonal to the ground state is readily available. In this paper, we present a new streamlined derivation of some of the methods of [16] and [17] and also show how to incorporate a finite number of holes in this scheme and how to compute the reduced Hamiltonian in the sectors with one and two holes.

This paper is organized as follows: In the rest of this introductory section we give precise statements of our results. In section 2 we consider the $t - J - V - V'$ Hamiltonian in the Ising limit and compute the reduced Hamiltonians. We then incorporate spin fluctuations, in section 3, by means of dressing transformations. In section 4 we consider a variation of the VBS model in which holes have an attractive interaction of order J, due to the exchange term, but which nonetheless does not appear to suffer from the pathology of phase separation in the large $\frac{J}{t}$ limit. In Appendix A we discuss the relation of our cluster expansion to the one used by Gelfand, Singh and Huse in their numerical investigations. Finally, in Appendix B we comment on the problem of phase separation. We attempt to determine the region of parameters $(t > \lambda, V, V')$ for which one obtains pairing of holes without phase separation. A numerical implementation of our algorithm will be discussed in [12].

REMARK: All constants appearing in the estimates throughout this paper are independent of Λ. From now on, we fix an arbitrary bounded square Λ and drop subscripts Λ. Although our methods and results are valid in any number of dimensions, we specialize to $d = 2$ which is the physically relevant case and is also technically the simplest case to carry out explicit calculations.

If $t = \lambda = 0$, the spectrum of the $t - J$ Hamiltonian consists of even and positive integers ≥ 4. If one or two holes are present, the ground state is infinitely degenerate. As we turn on t and λ, the lowest energy band remains isolated from the rest of the spectrum. In fact, we have

THEOREM 1. (LOWEST ENERGY BAND). *Let us suppose that there are $n = 0, 1, 2$ holes. There are constants $c_1, c_2 \geq 0$ such that if $|t| \leq c_1$, $|\lambda| \leq c_2$, then the eigenvalues of $\mathbb{H}_{tJ}$ bifurcating off zero energy do not cross the other eigenvalues from the bands of higher energy. Moreover, there is a projection operator $\mathbb{P}_n(\lambda, t)$ analytic in t and λ in a small double disc, such that $\mathbb{P}_n(0,0)$ is the eigenprojection onto the zero-energy eigenspace of $\mathbb{H}_{tJ}(0,0)$, and we have*

$$\left[\mathbb{H}_{tJ}(\lambda, t),\ \mathbb{P}_n(\lambda, t)\right] = 0. \tag{1.2}$$

If $E_{tJ}(\lambda)$ is the ground state energy of $\mathbb{H}_{tJ}(\lambda, t)$ in the half-filled sector then the restriction of the Hamiltonian to the lowest energy band can be defined as follows:

$$h_n(\lambda, t) = \left(\mathbb{H}_{tJ}(\lambda, t) - E_{tJ}(\lambda)\right) \mathbb{P}_n(\lambda, t) \tag{1.3}$$

on the reduced Hilbert space

$$\mathcal{H}_n(\lambda, t) \equiv P_n(\lambda, t)\, \mathcal{H}_{tJ}. \tag{1.4}$$

We remark that one must subtract the ground state energy $E_{tJ}(\lambda)$, in order to produce an operator $h_n(\lambda, t)$ which is well defined and whose spectrum is bounded from below, uniformly in the volume $|\Lambda|$ of the region Λ. In fact, the vacuum energy is an extensive quantity proportional to the volume $|\Lambda|$ and thus diverges in the thermodynamic limit. The restricted Hamiltonians $h_n(\lambda, t)$, $n = 0, 1, 2$ commute with translations $\mathcal{T}_x$, $x \in \Lambda$. Hence, the reduced Hilbert space $\mathcal{H}_n(\lambda, t)$ can be written as the direct sum of generalized eigenspaces $\mathcal{H}_n(\lambda, t, k)$ such that

$$\mathcal{H}_n(\lambda, t) = \oplus_k\, \mathcal{H}_n(\lambda, t, k) \tag{1.5}$$

where the direct sum indexed by k ranges over the points in the first Brillouin zone which are allowed by the finite volume cutoff, and $\mathcal{H}_n(\lambda, t, k)$ is given by

$$\mathcal{H}_n(\lambda, t, k) \;=\; \text{span}\Big\{\; |\psi\rangle \in \mathcal{H}_n : \; \mathcal{T}_x \, |\psi\rangle \;=\; e^{ik\cdot x} \, |\psi\rangle, \; \forall x \in \Lambda \Big\}. \tag{1.6}$$

Let us denote by $|x\,\rangle$ the state obtained from the Néel state by removing an electron from the site x; similarly, we define $|x, y\rangle$. We introduce the mutually orthonormal states

$$|\psi(x)\rangle \;\equiv\; \sum_{x'} P_1^{-\frac{1}{2}}(x; x') \;\; \mathbb{P}_1\,(\lambda, t)\, |x'\rangle \;\in\; \mathcal{H}_1(\lambda, t)$$

$$|\psi(x, y)\rangle \;\equiv\; \sum_{x'y'} P_2^{-\frac{1}{2}}(x, y; x', y') \;\; \mathbb{P}_2(\lambda, t)\, |x', y'\rangle \;\in\; \mathcal{H}_2(\lambda, t) \tag{1.7}$$

where $P_1(x; y)$ and $P_2(x, x'; y, y')$ are the following matrices:

$$P_1(x; x') \;=\; \langle x|\mathbb{P}_1(\lambda, t)|x'\rangle,$$

$$P_2(x, y; x', y') \;=\; \langle x, y|\mathbb{P}_2(\lambda, t)|x', y'\rangle. \tag{1.8}$$

The fiber $\mathcal{H}_1(\lambda, t, k)$ is a one dimensional Hilbert space along the vector

$$|\psi(k)\rangle \;\equiv\; \frac{1}{|\Lambda|} \sum_{x} \; e^{ik\cdot x} \; |\psi(x)\rangle \tag{1.9}$$

while

$$\mathcal{H}_2(k) \;=\; \text{span}\; \Big\{\; |\psi(k; x)\rangle, \; x \in \Lambda \Big\} \tag{1.10}$$

where

$$|\psi(k; x)\rangle \;=\; \frac{1}{|\Lambda|} \sum_{y} \; e^{ik\cdot y} \; |\psi(y, x + y)\rangle. \tag{1.11}$$

The reduced Hamiltonian $h_1(\lambda, t)$ restricted to $\mathcal{H}_1(\lambda, t, k)$ is the operator of multiplication by the constant

$$\varepsilon_1(\lambda, t, k) \;=\; \langle \psi(k) \mid h_1(\lambda, t) \; |\psi(k)\rangle, \tag{1.12}$$

while $h_2(\lambda, t)$ restricted to the fiber $\mathcal{H}_2(\lambda, t, k) \;\approx\; l^2(\Lambda)$ is given by the one body operator with matrix elements

$$h_2(\lambda, t, k; x, y) \;=\; \langle\; \psi(k; x) \mid h_2(\lambda, t) \mid \psi(k; y)\;\rangle. \tag{1.13}$$

We have

THEOREM 2 (PAIRING). *If $|\lambda|$ and $|t|$ are smaller than a certain constant $c > 0$, then, in the two dimensional case, we have the following results:*

(i) The dispersion law for one isolated hole is given by

$$\varepsilon_1(0,t,k) = \Big(-\frac{4}{3}t^2 + O(t^4)\Big) + \frac{158}{3^4 \cdot 5^3} t^6 \sum_{|y|_2=\sqrt{2}} (\cos k \cdot y - 1), \tag{1.16}$$

and

$$\varepsilon_1(\lambda,t,k) - \varepsilon_1(0,t,k) = O(\lambda t^2). \tag{1.17}$$

(ii) The reduced Hamiltonian for two holes has the form

$$\begin{aligned} h_2(0,t,k;x,y) = &-\frac{8}{3}t^2\delta_{xy} - t^2\delta_{xy}\Big[\Big(\frac{1}{3} - \frac{2V' + 3V'^2}{(2+V')^2}\Big)\delta(\|x\|-1) \\ &+ \Big(V' - \frac{8V'}{3(3-V')^2}\Big)\delta(\|x\|-2) + \frac{V'^2}{3(3+V')^2}\delta(\|x\|-3)\Big] \\ &+ \frac{1}{2+V'}t^2(1-\delta_{xy})\delta(\|x\|-1)\delta(\|y\|-1)\Big[1 - e^{ik\cdot x} - e^{-ik\cdot y}\Big] \end{aligned} \tag{1.18}$$

and

$$h_2(\lambda,t,k;x,y) - h_2(0,t,k;x,y) = O\big((t^2\lambda)^{\|x-y\|} + t^{3\|x-y\|_1 \cup 6}\big) \tag{1.19}$$

where $a \cup b = \max(a,b)$.

(iii) For all k small enough, the ground state of the operator $h_2(\lambda,t,k)$ is a symmetric (i.e. "s-wave") bound state of energy $E_{\text{binding}} = \frac{-14-7V'+8V'^2}{3(2+V')^2} \cdot t^2 + O(t^4 + t^2\lambda)$. In particular, if V' is smaller then a constant ≈ 2, a bound state exists for small t and $\lambda \leq t$.

The s-wave symmetry of the lowest energy bound state is probably due to the fact that in our hand-made leading order calculation we can only consider the limit $\frac{t}{J} \ll 1$. As the ratio $\frac{t}{J}$ increases, there might be level crossing in favour of d−wave pairing, as Shraiman and Siggia [20] predict on the basis of the Brinkman-Rice approximation.

Next, we consider the valence bond solid (VBS) model described by the following Hamilto-

nian:

$$\mathcal{H}_{VBS}(t) = \sum_{\langle xy\rangle} \frac{J}{4}\Big(\vec{\sigma}_{x\langle xy\rangle} + \vec{\sigma}_{y\langle yx\rangle}\Big)^2 + \sum_{\sigma,x,\langle xy\rangle} U\, n_{\sigma,x\langle xy\rangle}\, n_{\sigma,x\langle xy\rangle}$$
$$- \sum_{x} \sum_{\langle xy\rangle\neq\langle xy'\rangle,\sigma} t\, c^{+}_{x\langle xy\rangle}(\sigma)\, c_{x\langle xy'\rangle}(\sigma) - \sum_{\langle xy\rangle}\sum_{\sigma} t'\Big(c^{+}_{x\langle xy\rangle}(\sigma)\, c_{y\langle yx\rangle}(\sigma) + \text{h.c.}\Big) \tag{1.20}$$

Here, the sum is over bonds on a large square $\Lambda \subset \mathbb{Z}^2$ to which the system is restricted, with periodic boundary conditions. For each site $x \in \Lambda$ there are $2d$ orbitals available; they are oriented in the direction of all the bonds $\langle xy\rangle$ intersecting x. Doubly occupied sites are allowed, although double occupancy is suppressed by the $U-$term. Let us denote the Hilbert space with $\mathcal{H}_{VBS}$. For this model, too, it is convenient to fix the energy units by setting $J = 1$. We also assume that the Hubbard repulsion term is large. To fix the ideas, we set $U = 1$ and $t = t'$. We are interested in the kinetic properties of an isolated hole and in the interaction of pairs of holes in the small t limit.

The ground state of $\mathbb{H}_{VBS}(0)$ at half filling has a fairly simple structure as it can be constructed by placing one electron on each available orbital and forming a spin singlet out of each pair of electrons on the same bond. In this sector, the model clearly has a spin gap of order J . If the kinetic energy term is turned on, i.e. $t > 0$, electrons can hop and quantum fluctuations arise all over the lattice. By using the dressing transformation technique described in section 3, one can extend Theorem 1 also to this model. In particular, if there are $n = 1, 2$ holes, we can define the eigenprojection $\mathbb{P}_n(t)$ on the lowest band of the reduced Hilbert space and introduce a reduced Hamiltonian

$$h_n(t) = \big(\mathbb{H}_{VBS}(t) - E_{VBS}(t)\big)\, \mathbb{P}_n(t), \tag{1.21}$$

where $E_{VBS}(t)$ is the ground state energy in the sector corresponding to half filling. The operator $h_n(t)$ is defined on the reduced Hilbert space

$$\mathcal{H}_n(t) = \mathbb{P}_n(t)\, \mathcal{H}. \tag{1.22}$$

Pairing of holes of opposite spin also occurs for the VBS Hamiltonian in (1.20), albeit for a different reason than in the anisotropic $t - J$ model considered above. In fact, in this case the

reduced Hilbert space $\mathcal{H}_2(t)$ is spanned by all wavefunctions of the form

$$\sum_{x'y'} P_2^{-\frac{1}{2}}(x,y;x'y')\ \mathbb{P}_2(t)\ |x'y'\rangle, \tag{1.23}$$

where $|xy\rangle$ is the state obtained by pulling both electrons out of the same bond $\langle xy\rangle$ from the $t=0$ ground state of $\mathcal{H}_{VBS}(t)$ and $\mathbb{P}_2$ is an operator defined by a matrix similar to the one in (1.8). If $t=0$, this follows from the fact that if we pull two electrons out of different bonds, then we loose twice an exchange energy of at least J. If $t>0$, the proof is based on a perturbative argument and on random walk expansions that we discuss in Sect. 4. The precise statement of the result we prove is given in the following theorem:

THEOREM 3 (VALENCE BOND SOLID). *The lowest energy band in the two holes sector of the VBS Hamiltonian, for $|t|$ small enough, is spanned by the states of the form (1.20) and the pairing energy of two holes is $E_{\text{pairing}} = -J + J\,O((\frac{t}{J})^2)$.*

Theorems 2 and 3 are proven in sections 3 and 4. Our proof makes use of dressing transformations of a type introduced by one of the authors in [16] and [17]. As we explain in Appendix A, the perturbation series of the type described in [4] are generated by applying a Rayleigh-Schroedinger expansion to a family of restricted Hamiltonians; with such techniques, one can recover, at least in principle, the same series we obtain. However, as one attempts to establish the convergence of such series and to find rigorous bounds on the error terms, one meets a serious difficulty. The problem is that the number of diagrams one has to add to obtain the $n-$th coefficient for the expectation value of a local observable grows as $n!$ and there is no compensating factor $\frac{1}{n!}$ allowing one to control the convergence of the series. The generation of a number of diagrams larger than is strictly needed also negatively affects the speed and the memory requirements of the numerical implementation of the algorithm. This is a version of the "large field problem" for Fermi systems that one also encounters in quantum field theory models. To find a remedy there are two possibilities. Either one understands the mechanism controlling the cancellations among the various diagrams originating from the Rayleigh-Schroedinger expansion, as is possible in the case of weakly interacting Fermi systems. Or one performs a

unitary dressing transformation, in the spirit of [16] and [17], on the Hamiltonian before using Rayleigh-Schroedinger expansions, in such a way that the resummed geometric series generated in this way are not affected by the large field pathology anymore. In this paper, we adopt the second technique which provides quite an elementary solution to this problem. Knowledge of the previous papers [16] and [17] is not required, as we provide the proofs in full detail and also give a streamlined version of the original arguments and constructions. In Appendix A we also review some of the salient features of the method of dressing transformations for quantum spin systems. Finally, in Appendix B we discuss the issue of phase separation.

2. The Ising Limit.

The proof of the first two results in section 1 is particularly simple in the Ising limit $\lambda = 0$. In this case, the eigenprojection $\mathbb{P}_n(0,t)$ can be expanded in a geometric series as follows:

$$\mathbb{P}_n(0,t) = \oint_C \frac{dz}{2\pi i} \frac{1}{z - \mathbb{H}_{t,J}(0,t)} = \sum_{j=0}^{\infty} (-t)^j \oint_C \frac{dz}{2\pi i} \frac{1}{z - \mathbb{S}} \left(\mathbb{T} \frac{1}{z - \mathbb{S}}\right)^j \tag{2.1}$$

where

$$C = \left\{ z \in \mathbb{C} : |z| = \frac{1}{2} \right\}, \tag{2.2}$$

$$\mathbb{S} = \sum_{\substack{x,y \in \Lambda : \\ \|x-y\|_1 = 1}} \frac{1}{2} n_x n_y \left(\sigma_x^{(3)} \sigma_y^{(3)} + 1\right) + \sum_{\|x-y\|_1 = 2} V' n_x n_y \tag{2.3}$$

and

$$\mathbb{T} = \sum_{\substack{x,y \in \Lambda \\ \|x-y\|_1 = 1}} \sum_{\sigma} c_{x\sigma}^{+} c_{y\sigma}. \tag{2.4}$$

Here, we fix the energy units by setting $J = 1$. Since the relative bound of the operator $t \cdot \mathbb{T}$ with respect to $\mathbb{S}$ is $\approx c \cdot t$, the constant being uniform in $|\Lambda|$ as usual, the random walk expansion (2.1) is assured to converge if t is small enough. If $\lambda > 0$ this is not true anymore because of the extra terms that arise in this case, and, to generate the random walk expansion, one has to perform a dressing transformation first. Refering to the next section for these developments, we shall first focus our attention on the $\lambda = 0$ case.

NOTATION: For $x, x' \in \mathbb{Z}^2$, let $\omega \equiv (\omega_0 = x,\ \omega_1, \ldots,\ \omega_{|\omega|} = x') : x \to x'$ be a path in $\mathbb{Z}^2$ from x to x' such that $\|\omega_i - \omega_{i-1}\|_1 = 1$. An operator T_ω is then defined by

$$\mathbb{T}_\omega = \prod_{i=1,\ldots|\omega|} \Big(\sum_\sigma (c^+_{\omega_i\sigma}\, c_{\omega_{i-1}\sigma} + \text{h.c.}) \Big). \tag{2.5}$$

The situation with only one hole was first discussed by Brinkman and Rice [13] in the context of semiconductor physics. In this case we have that

$$\mathbb{P}_1(0,t)\,|x\rangle = \sum_{x'} \sum_{\omega: x \to x'} \sum_{j=|\omega|}^{\infty} (-t)^j \cdot \oint_C \frac{dz}{2\pi i} \Big(\prod_{i=1,\ldots,|\omega|} (z - S_1(\pi_i\omega))^{-1} \Big)\, \mathbb{T}_\omega\,|x\rangle \tag{2.6}$$

where we set

$$\pi_i\omega = (x,\ \omega_1, \ldots, \omega_i) \tag{2.7}$$

for any path $\omega = (x, \omega_1, \ldots \omega_{|\omega|})$ starting at x and $S_1(\pi_i\omega)$ is the eigenvalue of $\mathbb{S}$ corresponding to the eigenvector $T_{\pi_i\omega}|x\rangle$, i.e.

$$\mathbb{S}\,\mathbb{T}_{\pi_i\omega}\,|x\rangle = S_1(\pi_i\omega)\,\mathbb{T}_{\pi_i\omega}\,|x\rangle. \tag{2.8}$$

The subscript 1 stands for "one hole". As the hole moves under the action of the operator $\mathbb{T}$, it leaves behind a string of frustrated spins which can be eliminated only if the path intersects itself in a suitable way. A possibility is that the path is self retracing, i.e., that the hole passes through each site an even number of times, thus ending up at the same site it started from. The length of such paths is ≥ 2. The paths ω of lowest order in t for which $\mathbb{T}_\omega|x\rangle = |y\rangle$ with $x \neq y$ are those for which the hole performs one and a half turns around a plaquette. Such paths have length 6.We retain only terms yelding contributions of order at most $O(t^2)$ to the diagonal matrix elements of the reduced Hamiltonian h_1 and terms corresponding to contributions of order not higher than $O(t^6)$ to the off diagonal terms. In our formulas, the order in t of the remainder terms we neglect varies with the nature of the terms involved. i.e. it depends on whether the term is diagonal or off diagonal or whether it involves a frustrated spin. To have a

unified notation, we use suspension dots to denote all these higher order terms.

$$\begin{aligned}\mathbb{P}_1(0,t)\,|x_0\rangle \;=\; c_0(t)|x_0\rangle \;+\; \sum_{|y_1-x_0|=1} \Big\{ & c_1(t)\sigma^{(1)}_{x_0}\,|y_1\rangle \\ + \sum_{\substack{|y_2-y_1|=1\\ |y_2-x_0|=\sqrt{2}}} & \Big[c_2(t)\sigma^{(1)}_{x_0}\sigma^{(1)}_{y_1}\,|y_2\rangle \;+\; c_4(t)\sigma^{(1)}_{y_2}\sigma^{(1)}_{y_1}\,|x_0\rangle \;+\; c_5(t)\sigma^{(1)}_{y_2}|y_1\rangle \\ & + c_6(t)\,|y_2\rangle \;+\; \sum_{\substack{|y_3-y_2|=1\\ |y_3-x_0|=1}} c_3(t)\sigma^{(1)}_{x_0}\sigma^{(1)}_{y_1}\sigma^{(1)}_{y_2}\,|y_3\rangle\Big]\Big\} \\ & + \ldots\ldots,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned} c_0(t) &= 1 \;+\; 4(-t)^2 \oint \frac{dz}{2\pi i}\,\frac{1}{z^2}\,\frac{1}{z-3} \;=\; 1-\frac{4}{9}t^2 \;+\; O(t^4), \\ c_1(t) &= (-t)\oint \frac{dz}{2\pi i}\,\frac{1}{z}\,\frac{1}{z-3} \;=\; \frac{t}{3} \;+\; O(t^3), \end{aligned}$$

$$\begin{aligned} c_2(t) &= (-t)^2 \oint \frac{dz}{2\pi i}\,\frac{1}{z(z-3)z-5)} \;=\; \frac{t^2}{15} + O(t^4), \\ c_3(t) &= (-t)^3 \oint \frac{dz}{2\pi i}\,\frac{1}{z(z-3)(z-5)(z-6)} \;=\; \frac{t^3}{90} \;+\; O(t^5), \\ c_4(t) &= (-t)^4 \oint \frac{dz}{2\pi i}\,\frac{1}{z(z-3)(z-5)(z-6)} \;=\; \frac{1}{2\cdot 3^2\cdot 5^2}\,t^4 \;+\; O(t^6), \\ c_5(t) &= (-t)^5 \oint \frac{dz}{2\pi i}\,\frac{1}{z(z-3)(z-5)^2(z-6)} \;=\; \frac{1}{2\cdot 3^3\cdot 5}\,t^5 \;+\; O(t^7), \\ c_6(t) &= (-t)^6 \oint \frac{dz}{2\pi i}\,\frac{1}{z^2(z-3)^2(z-5)^2(z-6)} \;=\; -\frac{37}{2^2\cdot 3^4\cdot 5^3}\,(-t)^6 \;+\; O(t^8). \end{aligned} \tag{2.10}$$

We have

$$P_1(x;y) \;=\; \langle x|\mathbb{P}(0,t)|y\rangle \;=\; (1-\frac{4}{9}t^2 + O(t^4))\;\delta(x-y) + O(t^{6\cup 3\|x-y\|})\;(1-\delta(x-y)) \tag{2.11}$$

and

$$P_1^{-\frac{1}{2}}(x,y) \;=\; (1+\frac{2}{9}t^2 + O(t^4))\;\delta(x-y) \;+\; O(t^{6\cup 3\|x-y\|})(1-\delta(x-y)). \tag{2.12}$$

Here, $a\cup b \;=\; \max(a,b)$. The Hilbert space is spanned by the othonormal basis $\{|\psi(x)\rangle\;\; x\in\Lambda\}$ defined in (1.7). The vector $\{\psi(x)\}$ has an expansion of the form (2.9) with coefficients

$$c_0'(t) \;=\; 1-\frac{2}{9}t^2 \;+\; O(t^4), \tag{2.13}$$

while $c_i'(t) = (1 + O(t^2))\, c_i(t)$, for $i \geq 0$. The diagonal terms of the reduced Hamiltonian are easy to compute. We obtain

$$\begin{aligned}\langle\psi(x)|h_1|\psi(x)\rangle &= \langle\psi(x)|\mathbb{S} - t\mathbb{T}|\psi(x)\rangle \\ &= 4\cdot 3\cdot\frac{1}{9}t^2 + 2\cdot 4\cdot(-t)\frac{1}{3}t + O(t^4) = -\frac{4}{3}t^2 + O(t^4).\end{aligned} \tag{2.14}$$

The off-diagonal terms fulfill the bound

$$h_1(x_0, y_0) = O(t^{6\cup 3\|x_0 - y_0\|}) \tag{2.15}$$

and the leading term describes next neighbour hopping between sites with $|x_0 - y_0|_2 = \sqrt{2}$. In this case, we find that

$$\langle\psi(y_0)|\mathbb{S}|\psi(x_0)\rangle = 4\cdot c_6(t) + 4\cdot 3c_1(t)c_5(t) + 4\cdot 5c_2(t)c_4(t) + 2\cdot 6c_3(t)^2 + O(t^8), \tag{2.16}$$

and

$$\langle\psi(y_0)|(-t\mathbb{T})|\psi(x_0)\rangle = -4\cdot tc_5(t) - 4tc_1(t)c_4(t) - 4\cdot tc_2(t)c_3(t). \tag{2.17}$$

Hence

$$\langle\psi(y_0)|h_1|\psi(x_0)\rangle = \frac{158}{3^4\cdot 5^3}t^6 + O(t^8). \tag{2.18}$$

The dispersion law is thus given by

$$\begin{aligned}\varepsilon_1(k;t,0) &= \sum_y h_1(y,0)\, e^{ik\cdot y} = -\frac{4}{3}t^2 - \frac{158}{3^4\cdot 5^3}\sum_{|y|_2=\sqrt{2}} t^6 e^{ik\cdot y} \\ &= \Big(-\frac{4}{3}t^2 + O(t^4)\Big) + \frac{158}{3^4\cdot 5^3}t^6 \sum_{|y|_2=\sqrt{2}} (\cos\, k\cdot y - 1) + \ldots..\end{aligned} \tag{2.19}$$

where the suspension points stand for k-independent terms of order $O(t^4)$ and k-dependent terms of order $O(t^8)$.

Let us pass to the case of two holes. Let us fix a bond $\langle x_0\, y_0\rangle$ and let $|x_0\, y_0\rangle$ be the state with two holes on such a bond. Expanding in a geometric series, we find

$$\begin{aligned}&\oint \frac{dz}{2\pi i}\frac{1}{z - \mathbb{S} + t\mathbb{T}}\,|x_0 y_0\rangle = |x_0 y_0\rangle \\ &\quad - t\oint\frac{dz}{2\pi i}\frac{1}{z(z-2-V')}\Big(\sum_{\substack{\|x - x_0\| = 1\\ x\neq y_0}}\sigma^{(1)}_{x_0}|x\, y_0\rangle + \sum_{\substack{\|y - y_0\| = 1\\ y\neq x_0}}\sigma^{(1)}_{y_0}|x_0\, y\rangle\Big) \\ &\quad + t^2\oint\frac{dz}{2\pi i}\frac{1}{z^2(z-2-V')}\Big(6|x_0\, y_0\rangle + \sum_{\langle xy\rangle\cap\langle x_0 y_0\rangle\neq\emptyset}|xy\rangle\Big) + \ldots..\end{aligned} \tag{2.20}$$

where the remainder contains terms of order $O(t^2)$ which are orthogonal to all states $|x\ y\rangle$ together with terms of order $O(t^3)$. Hence

$$\mathbb{P}_2(t)\,|x_0y_0\rangle = (1-\frac{6}{(2+V')^2}t^2)|x_0y_0\rangle$$
$$+\frac{t}{(2+V')}\Big(\sum_{\substack{\|x-x_0\|=1\\ x\neq y_0}}\sigma^{(1)}_{x_0}|xy_0\rangle+\sum_{\substack{|y-y_0|=1\\ y\neq x_0}}\sigma^{(1)}_{y_0}|x_0\,y\rangle\Big)$$
$$-\frac{1}{2(2+V')}t^2\sum_{\substack{\langle xy\rangle\cap\langle x_0y_0\rangle\neq\emptyset\\ \langle xy\rangle\neq\langle x_0y_0\rangle}}|xy\rangle+\ldots\ldots \tag{2.21}$$

We have

$$P_2(x_0,y_0;x_0,y_0) = 1-\frac{6}{(2+V')^2}t^2+O(t^4). \tag{2.22}$$

Moreover, if $\langle xy\rangle\cap\langle x_0y_0\rangle\neq\emptyset$ and $\langle xy\rangle\neq\langle x_0y_0\rangle$, we have

$$P_2(x_0,y_0;x,y) = -\frac{1}{2(2+V')}t^2+O(t^4). \tag{2.23}$$

By expanding in a binomial series, we find

$$P_2^{-\frac{1}{2}}(x_0,y_0;x_0,y_0) = 1+\frac{3}{(2+V')^2}t^2+O(t^4), \tag{2.24}$$

$$P_2^{-\frac{1}{2}}(x_0,y_0;x,y) = \frac{1}{2(2+V')^2}t^2+O(t^4). \tag{2.25}$$

Hence, if $|x_0-y_0|=1$, we have

$$\sum_{xy}P_2^{-\frac{1}{2}}(x_0,y_0;x,y)\,\mathbb{P}_2(t)\,|xy\rangle = \Big(1-\frac{3}{(2+V')^2}t^2\Big)\,|x_0y_0\rangle$$
$$+\frac{t}{(2+V')}\Big(\sum_{\|x-x0\|=1,x\neq y_0}\sigma^{(1)}_{x_0}\,|xy_0\rangle+\sum_{\substack{\|y-y_0\|=1\\ y\neq x_0}}\sigma^{(1)}_{y_0}\,|x_0\,y\rangle\Big)$$
$$-\frac{t^2}{2(2+V')^2}\sum_{\substack{\langle xy\rangle\cap\langle x_0y_0\rangle\neq\emptyset\\ \langle xy\rangle\neq\langle x_0y_0\rangle}}|xy\rangle+\ldots\ldots \tag{2.26}$$

Hence, the leading order terms of the effective Hamiltonian $h_1(x_0,y_0;x,y)$ for $\|x_0-y_0\|=1$ are non zero only if the sites x,y are on neighbouring bond and in this case we have

$$h_2(x_0,y_0;x,y) = \begin{cases} -\frac{12t^2}{(2+V')^2} & \text{if } x=x_0,\ y=y_0\\ -\frac{t^2}{2+V'} & \text{if } \begin{cases}\langle xy\rangle\neq\langle x_0\,y_0\rangle\\ \langle x,y\rangle\cap\langle x_0\,y_0\rangle\neq\emptyset\end{cases}\end{cases} \tag{2.27}$$

Next we consider a state of two holes with $\|x_0 - y_0\|_1 = 2$. We have that

$$\mathbb{P}_2|x_0, y_0\rangle = \Big(1 - \frac{8}{(3-V')^2}t^2\Big)|x_0y_0\rangle + \frac{t}{3-V'}\Big(\sum_{\|y-y_0\|=1} \sigma^{(1)}_{y_0}|y\rangle + \sum_{\|x-x_0\|=1} \sigma^{(1)}_{x_0}|x\rangle\Big) + \ldots\ldots, \tag{2.28}$$

$$P_2(x_0, y_0; x_0, y_0) = 1 - \frac{8}{(3-V')^2}t^2 + O(t^4). \tag{2.29}$$

$$|\psi(x_0, y_0)\rangle = \Big(1 - \frac{4}{(3-V')^2}t^2\Big)|x_0y_0\rangle + \frac{t}{(3-V')}\Big(\sum_{\|y-y_0\|=1} \sigma^{(1)}_{y_0}|x_0y\rangle + \sum_{\|x-x_0\|=1} \sigma^{(1)}_{x_0}|xy_0\rangle\Big) + \ldots\ldots, \tag{2.30}$$

so that

$$h_2(x_0, y_0; x_0, y_0) = V' - \frac{8(3-2V')}{(3-V')^2}t^2. \tag{2.31}$$

Finally, if $\|x_0 - y_0\|_1 \geq 3$ we have

$$h_2(x_0, y_0; x_0, y_0) = \begin{cases} -\Big[2 + \frac{3+2V'}{(3+V')^2}\Big]t^2 & \text{if } \|x_0 - y_0\|_1 = 3 \\ -\frac{8}{3}t^2 & \text{if } \|x_0 - y_0\,{}_1 \geq 4 \end{cases} \tag{2.32}$$

Passing to the center of mass frame, we find

$$\begin{aligned} h_2(k; x, y) &= \sum_z e^{ik\cdot z}\, h_2(0, x; z, y+z) \\ &= -\frac{8}{3}\,t^2\delta_{xy} - t^2\delta_{xy}\Big[\Big(\frac{1}{3} - \frac{2V' + 3V'^2}{(2+V')^2}\Big)\delta(\|x\| - 1) \\ &\quad + \Big(V' - \frac{8V'}{3(3-V')^2}\delta(\|x\| - 2) + \frac{V'^2}{3(3+V')^2}\delta(\|x\| - 3)\Big] \\ &\quad + \frac{1}{2+V'}t^2\delta(\|x\| - 1)\delta(\|y\| - 1)(1 - \delta(x-y))\Big[1 - e^{ik\cdot x} - e^{-ik\cdot y}\Big] \end{aligned} \tag{2.33}$$

which proves (1.18) in case $\lambda = 0$.

Due to the Perron-Frobenius theorem and to the negative sign of the third term at the bottom of the band, i.e. for $k = 0$, and at $\lambda = 0$, the ground state of the reduced two-hole Hamiltonian is a stricly positive function and has s-symmetry. As we show in the next section, for λ small the situation does not change, while, for λ near one and larger values of t, there might be a crossover to d-wave pairing, as one would expect on the basis of the calculation in [20].

3. Dressing Transformation for the tJ Model.

The Heisenberg Hamiltonian is invariant with respect to global spin flips. To deal with this degeneracy there are two alternatives: Either we choose to express the dressing transformation through spin flip operators involving only the matrices $\sigma^{(1)}$, as was done in [16], and, in this case, we respect the symmetry but give up unitarity; or we promote the global spin flip symmetry to a local gauge symmetry by extending the Hamiltonian to a bigger Hilbert space $\bar{\mathcal{H}}$ and then use techniques of [17] of which we give a new, streamlined presentation in this paper, i.e. we follow the second alternative which is mathematically simpler. However, the reader should be warned that in order to optimize the algorithm for numerical calculations, the artificial increase in the number of degrees of freedom resulting from the second strategy should be avoided, and alternative setups will be more convenient. The point made in this paper is that local gauge invariance is useful in order to control large local deviations from the Ising ground state consisting of large bubbles surrounded by Peierls contours. Such large deviations show up only at orders of perturbation theory as large as $\approx c \cdot \lambda^{-1}$.

On the extended Hilbert space, a state is described by a configuration of happy and frustrated bonds, and it turns out that unitarity of the dressing transformation can be enforced in such a way that the local symmetry is preserved. We set

$$\bar{\mathcal{H}} = \otimes_{b=\langle xy\rangle \subset \mathbb{Z}^d} (\mathbb{C}_b^2 \oplus \mathbb{C}_b^5), \tag{3.1}$$

where the fiber corresponding to a bond b is the direct sum of two spaces, $\mathbb{C}_b^2$ and $\mathbb{C}_b^5$. A basis for $\mathbb{C}_b^2$ consists of two vectors, $|h\rangle$ and $|f\rangle$, where h stands for "happy" and f for "frustrated". The basis for $\mathbb{C}_b^5$ is $\{|e\uparrow\rangle,\ |e\downarrow\rangle\ ,|\uparrow e\rangle, |\downarrow e\rangle,\ |ee\rangle\}$, where the first argument refers to the left site for horizontal bonds and to the top site for vertical bonds. There is a natural injection $\mathcal{H}_{tJ} \rightarrow \bar{\mathcal{H}}_{tJ}$, and there is a canonical extension of the tJ Hamiltonian $\mathbb{H}_{tJ}(\lambda, t)$ to the larger Hilbert space $\bar{\mathcal{H}}_{tJ}$ of the form

$$\mathbb{H}_{tJ}(\lambda, t) = \sum_b \left(\hat{\varepsilon}_b + \lambda\, \hat{k}_b + t\, \hat{\delta}_b\right). \tag{3.2}$$

It is sufficient to restrict our attention to a subspace $\hat{\mathcal{H}}_{tJ}$ of $\bar{\mathcal{H}}_{tJ}$ containing the physical space $\mathcal{H}_{tJ}$ and invariant under the action of the extended Hamiltonian. Such an invariant space must

share with the physical subspace the property that, for any given site, the states of the bonds meeting in that site agree on whether the site is empty or occupied. Thus only doubly occupied bonds can be chosen in a nonphysical way, that is, in such a way that the Peierls contours are not all closed; this provides the extra degree of freedom required to construct the kind of unitary dressing transformation we shall make use of. The diagonal operator $\hat{\varepsilon}_b$ is defined as follows: $\hat{\varepsilon}_b$ has eigenvalues 0 and 1; it annihilates states over a bond b for which the bond b is either happy or not doubly occupied, and its eigenspace of eigenvalue one is spanned by those states for which the bond b is frustrated, thus penalizing such configurations. The action of the operators $\hat{k}_b$ and $\hat{\delta}_b$ is illustrated in Fig.1, below. If the bond b is either frustrated or partially empty, then $\hat{k}_b$ annihilates the state. Otherwise, if b is happy it leaves it happy and changes the state of the neighbouring bonds $c \cap b \neq \emptyset$ in such a way that if c was happy it becomes frustrated and viceversa; moreover, if c was occupied by just one spin on the bond b, then this spin gets flipped. Finally, $\hat{\delta}_b$ acts nontrivially only on states for which one and only one site of b is empty while the other site is occupied; if this is the case $\hat{\delta}_b$ only changes the state of the bonds b and c if $b \cap c \neq \emptyset$, and its action is uniquely determined by the requirement that the Hamiltonian in (3.2) is an extension of the tJ Hamiltonian in (1.1). See Fig.1 for a graphic description of the action of $\hat{k}_b$ and $\hat{\delta}_b$.

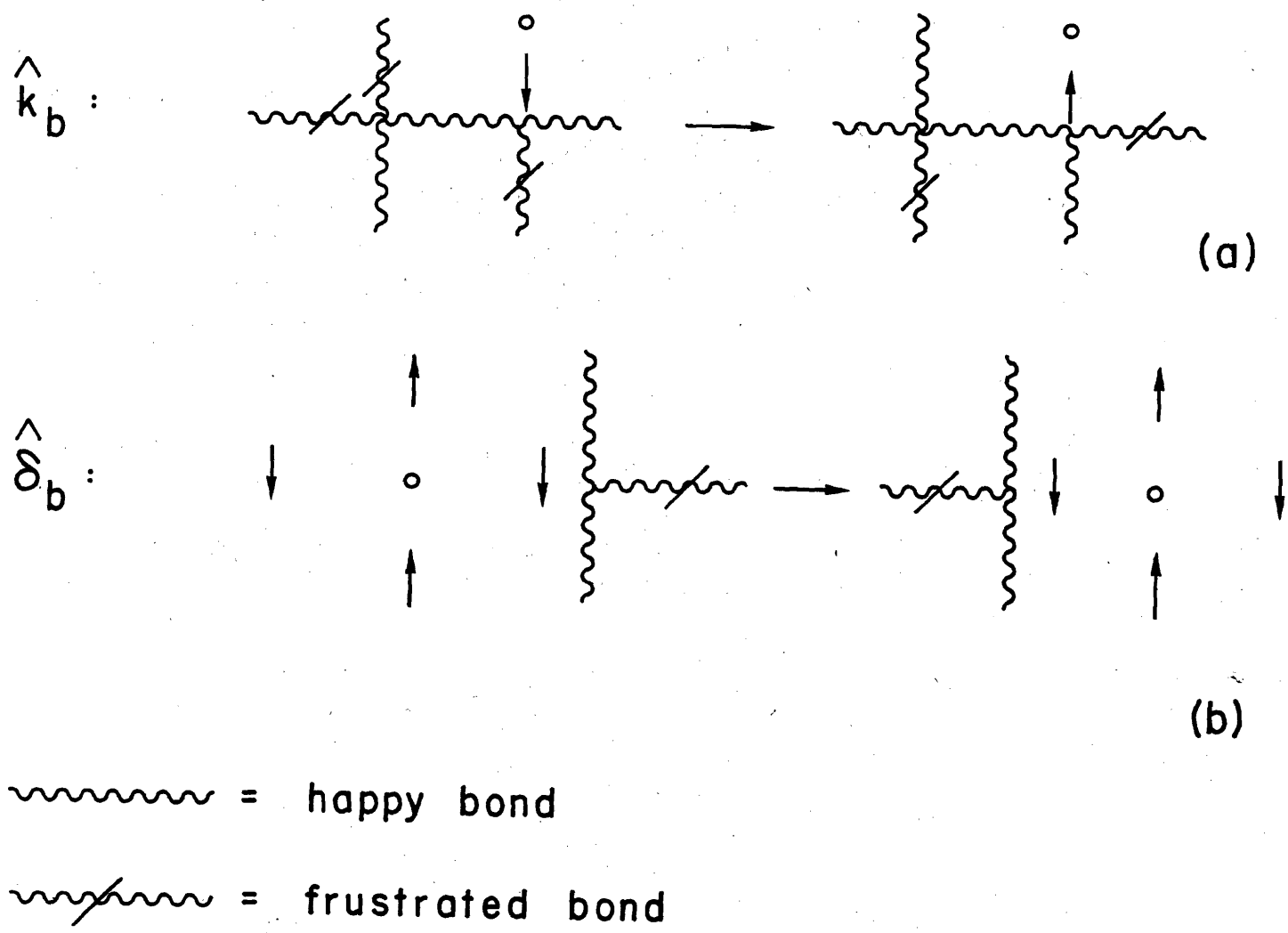


Fig. 1

To define the dressing transformation, let us first consider the model with a half filled band, i.e. without any empty sites. In the half filled sector, an excitation is described by a function $\gamma : \bar{\Lambda} \to \{0,1\}$, where $\bar{\Lambda}$ is the set of bonds of Λ. Let $\sigma_1, i\sigma_2$ be the matrices

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{O}_5, \qquad i\sigma^{(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \mathbb{O}_5. \tag{3.3}$$

on the vector space $\mathbb{C}^2 \oplus \mathbb{C}^5$, where $\mathbb{O}_5$ is the zero operator on the subspace $\mathbb{C}^5$ of the fiber over b. Here, we identify $|h\rangle$ with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and f with $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The state $|h\rangle$ carries the label 0, the state $|f\rangle$ the label 1. Let $\sigma_b^{(1)}$, $i\sigma_b^{(2)}$ be the corresponding operators on the fiber $\mathbb{C}_b^2 \oplus \mathbb{C}_b^5$. We introduce the real skewsymmetric operator

$$\tau_\gamma = \frac{1}{|\gamma|} \sum_{b \in \gamma} i\sigma_b^{(2)} \prod_{\substack{c \neq b \\ c \in \gamma}} \sigma_c^{(1)}, \tag{3.4}$$

where $|\gamma|$ is the volume

$$|\gamma| = \sum_b \gamma(b). \tag{3.5}$$

Since no confusion can arise with spin $\frac{1}{2}$ particles, we also denote by γ the support of the excitation γ. Let $d(\gamma)$ be the number of bonds of the smallest connected set containing γ. The vectors

$$|\gamma> = \tau_\gamma |0> \tag{3.6}$$

define an excitation basis on $\tilde{\mathcal{H}}$ in which we can expand all wave functions as follows:

$$|\psi> = \sum_\gamma \psi_\gamma |\gamma> . \tag{3.7}$$

We define the l^1-norm of $|\psi\rangle$ by

$$\|\psi\|_1 = \sum_\gamma |\psi_\gamma|. \tag{3.8}$$

We also denote by $\|\cdot\|_1$ the corresponding operator norm.

Our goal is to construct a unitary operator $\mathbb{U}(\lambda)$ on $\hat{\mathcal{H}}$ analytic in a disc $\{|\lambda| \leq \lambda_0\}$, $\lambda_0 > 0$, such that the ground state of $\mathbb{H}_{tJ}(\lambda, t)$ in the half filled sector is $\mathbb{U}(\lambda)|0>$ and that $\mathbb{U}(\lambda)|0\rangle$

belongs to the subspace of $\hat{\mathcal{H}}$ identified with $\mathcal{H}$. To determine $\mathbb{U}(\lambda)$ in a unique way, we restrict its form as follows:

$$\mathbb{U}(\lambda) = \lim_{v\to\infty} \exp(\mathbb{R}^1(\lambda))\ldots\exp(\mathbb{R}^v(\lambda)) \tag{3.9}$$

where

$$\mathbb{R}^v(\lambda) = \sum_{n=1}^{\infty} \lambda^n \, \mathbb{R}_n^v \tag{3.10}$$

and

$$\mathbb{R}_n^v = \sum_{|\gamma|=v} r_{n\gamma} \, \tau_\gamma. \tag{3.11}$$

This prescription gives rise to a cluster expansion for the coefficients $r_{n\gamma}$: As one can easily see, we have that $r_{n\gamma} = 0$ in case $d(\gamma) > 6n$.

One can show that Theorem 1 of Sect. 1 is a straightforward consequence of the following result:

LEMMA 3.1 (DRESSING TRANSFORMATION). *There is a constant $\lambda_0 > 0$ such that a unique operator $\mathbb{U}(\lambda)$ of the form (3.9) exists, is analytic in the disc $\{|\lambda| \leq \lambda_0\}$ and solves the conjugagy problem*

$$\mathbb{U}(\lambda)^{-1}\mathbb{H}_{t,J}(\lambda)\mathbb{U}(\lambda) = \mathbb{S} + \mathbb{V}(\lambda) + E_0(\lambda), \tag{3.12}$$

with

$$\mathbb{S} = \sum_b \hat{\varepsilon}_b \tag{3.13}$$

and

$$\mathbb{V}(\lambda)|0> = 0. \tag{3.14}$$

The ground state of $\mathbb{H}_{t,J}(\lambda, t)$ in the half filled sector is $\mathbb{U}(\lambda)|0>$ and belongs to $\mathcal{H}$, its energy being the constant $E_0(\lambda)$ in (3.12). The operator $\mathbb{V}(\lambda)$ is relatively form bounded with respect to $\mathbb{S}$ in the following sense:

$$\langle\Psi|\mathbb{V}(\lambda)|\Psi\rangle \leq c \cdot |\lambda| \, \langle\Psi|\mathbb{S}|\Psi\rangle, \tag{3.15}$$

for all $\Psi \in \mathcal{H}$. Finally, we have that $r_n(\gamma) = 0$ if $d(\gamma) > 6n$, and that, for some positive constant c,

$$r_n^* \equiv \sup_b \sum_{\gamma: b\in\gamma} |r_{n\gamma}| \leq c^n. \tag{3.16}$$

PROOF: Let us consider the conjugacy problem

$$\mathbb{U}(\lambda)^{-1}\,\mathbb{H}(\lambda)\,\mathbb{U}(\lambda)\,|0> = E_0(\lambda)\,|0>, \tag{3.17}$$

where $E_0(\lambda)$ is a constant and $\mathbb{U}(\lambda)$ is a unitary operator on $\hat{\mathcal{H}}$ of the form (3.10). Expanding both sides of (3.12) in powers of λ, we find the following recurrence relations for the operators $\mathbb{R}_n^v$:

$$\begin{aligned}
&\sum_v \mathbb{R}_n^v|0> \\
&= -\sum_b \Big\{ \sum_{k\geq 2} \sum_{\substack{|\gamma_1|\leq\ldots\leq|\gamma_k| \\ i_1+\ldots+i_k=n}} \frac{1}{n(|\gamma_1|,\ldots|\gamma_k|)} \mathbb{S}^{-1}\mathbb{P}_\perp \left[\ldots\left[\hat{\epsilon}_b, r_{i_1\gamma_1}\tau_{\gamma_1}\right],\ldots r_{i_k\gamma_k}\tau_{\gamma_k}\right]|0\rangle \\
&+ \sum_{k\geq 0} \sum_{\substack{|\gamma_1|\leq\ldots\leq|\gamma_k| \\ i_1+\ldots+i_k=n-1}} \frac{1}{n(|\gamma_1|,..|\gamma_k|)} \mathbb{S}^{-1}\mathbb{P}_\perp \left[\ldots\left[\hat{k}_b, r_{i_1\gamma_1}\tau_{\gamma_1}\right],\ldots. r_{i_k\gamma_k}\tau_{\gamma_k}\right]|0>\Big\}
\end{aligned} \tag{3.18}$$

where $\mathbb{S}$ is defined in (3.12), $\mathbb{P}_\perp = \mathbb{I} - |0\rangle\langle 0|$ and we set

$$n(|\gamma_1|,..|\gamma_k|) = \prod_{v=0}^{\infty} (\#\{i : |\gamma_i| = v\}!). \tag{3.19}$$

Equation (3.18) generates a cluster expansion. When $n = 1$ then $r_{1\gamma} = 0$, unless $|\gamma| = 1$, while, at the n-th order of perturbation theory, only clusters of at most $n+1$ bonds are present.

For $n = 1$, we have

$$r_1^* = \sup_b \|\sum_{c\cap b\neq\emptyset} \mathbb{S}^{-1}\hat{k}_c|0\rangle\|_1 = 1. \tag{3.20}$$

Next, suppose that $n \geq 2$. Let us call (connected) k-polymer a family of clusters $p = \{b, \gamma_1, \ldots\gamma_k\}$ such that $b \in \bar{\Lambda}$, $\gamma_1 \cap b \neq \emptyset$, $\gamma_k \cap (b \cup \ldots \cup \gamma_{k-1}) \neq \emptyset$. We also introduce a space $\mathcal{P}$ of k-polymers given by $[p] = \{\mathcal{T}_x p; x \in \Lambda, p \in \mathcal{P}\}$ where $\mathcal{T}_x$ is the operator of translation by x. There is a state $|\bar{\gamma}(p)\rangle$ such that

$$\left[..\left[\hat{\epsilon}_b,\ r_{i_1\gamma_1}\tau_{\gamma_1}\right],\ldots r_{i_k}\tau_{\gamma_k}\right]|0\rangle = f_{i_1\ldots i_k}(p)\,|\bar{\gamma}(p)\rangle, \tag{3.21}$$

for some coefficient $f_{i_1\ldots i_k}(p)$. This function is obviously translation invariant, because the $t-J$ Hamiltonian is. (In case $\mathbb{H}_{tJ}$ had a similar form but lacked this symmetry, we would consider

$\sup_x |f(\mathcal{T}_x p)|$ and all the arguments below would apply as well). The contribution to r_n^* coming from all connected k-polymers $[p_0] \in \mathcal{P}$ is bounded from above by

$$\sup_{b_0} \sum_{\substack{p \in [p_0] \\ \bar{\gamma}(p) \ni b_0}} |f_{i_1 \ldots i_k}([p_0])\, \|\mathbb{S}^{-1}|\bar{\gamma}([p_0])\rangle\|_1$$
$$= |\bar{\gamma}(p)|\, |f_{i_1 \ldots i_k}([p_0])|\, \|\mathbb{S}^{-1}\, |\bar{\gamma}(p(\pi))\rangle\|_1 \;=\; |f_{i_1 \ldots i_k}([p_0])|. \tag{3.22}$$

where $p(\pi)$ is a representative of the equivalence class π. Hence, the contribution to r_n^* coming from the first sum in (3.18) is

$$\leq \sum_{k \geq 2} \sum_{i_1 + \ldots + i_k = n} \sum_{[p_0]} \frac{1}{n(p(\pi))}\, |f_{i_1 \ldots i_k}(p(\pi))|$$
$$\leq \sup_b \sum_{k \geq 2} \sum_{\substack{|\gamma_1| \leq \ldots |\gamma_k| \\ i_1 + .. + i_k = n}} \frac{1}{n(|\gamma_1|, .. |\gamma_k|)}\, |f_{i_1 .. i_k}(b, \gamma_1, \ldots \gamma_k)|. \tag{3.23}$$

where if $p_0 = \{b, \gamma_1, \ldots \gamma_k\}$ we set $n([p_0]) = n(\gamma_1, \ldots \gamma_k)$. We have that

$$\sum_{\substack{\gamma_k \,:\, \gamma_k \cap (b \cup \ldots \cup \gamma_{k-1}) \neq \emptyset \\ |\gamma_k| = v_k}} |f_{i_1 .. i_k}(b, \gamma_1, .. \gamma_k)|$$
$$\leq 2 \left[k + (|\gamma_1| + .. |\gamma_{k-1}| - k + 1)\, v_k^{-1} \right] |f_{i_1 .. i_{k-1}}(b, \gamma_1, \ldots \gamma_{k-1})|$$
$$\cdot \sup_{c \in \bar{\Lambda}} \sum_{\substack{\gamma_k \ni c \\ |\gamma_k| = v_k}} |r_{i_k \gamma_k}|. \tag{3.24}$$

The first term, k in the factor within square brakets, originates from the k centers of noncommutativity of $b \cup \gamma_1 \ldots \cup \gamma_{k-1}$, while the second term is due to the center of noncommutativity in τ_{γ_k}. The factor v_k^{-1} is inherited from (3.4). Since the volumes are ordered, we have

$$(|\gamma_1| + .. + |\gamma_{k-1}| - k + 1)\, v_k^{-1} \;\leq\; k. \tag{3.25}$$

Iterating this bound, we find

$$\sup_b \sum_{\substack{|\gamma_1| = v_1, \ldots |\gamma_k| = v_k \\ i_1 + \ldots + i_k = n \\ k \geq 2}} \frac{1}{n(v_1, ..v_k)} |f_{i_1 ..i_k}(b, \gamma_1, \ldots \gamma_k)|$$

$$\leq (4 \cdot k) \cdot (4 \cdot (k-1)) \ldots 4 \cdot \prod_{j=1,\ldots k} \sup_b \sum_{\substack{|\gamma_j| = v_j \\ \gamma_j \ni b}} |r_{i_j \gamma_j}|$$

$$\leq 2^{2k}\, k! \prod_{j+1,\ldots k} r^*_{i_j v_j}, \tag{3.26}$$

where

$$r^*_{i_j v_j} = \sup_b \sum_{\substack{\gamma \ni b \\ |\gamma_j| = v_j}} |r_{i_j \gamma_j}|. \tag{3.27}$$

Similarly, we find that the second term in (3.18) gives a contribution to r^*_n which is

$$\leq 6 \cdot 2^{2k} \cdot k! \prod_{j=1,\ldots k} r^*_{i_j v_j}. \tag{3.28}$$

The extra factor 6 is due to the fact that the operator $\hat{k}_b$ has support on 6 bonds rather than on just one as in the previous case.

For all $n \geq 2$, we thus have

$$r^*_n \leq \Big(\sum_{k \geq 2} \sum_{\substack{v_1 \leq \ldots \leq v_k \\ i_1 + \ldots + i_k = n}} + 6 \cdot \sum_{k \geq 1} \sum_{\substack{v_1 \leq \ldots \leq v_k \\ i_1 + \ldots + i_k = n-1}} \Big) \frac{2^{2k} \cdot k!}{n(v_1, ..v_k)} \left(\prod_{j=1}^{k} r^*_{i_j v_j} \right)$$

$$\leq \Big(\sum_{k \geq 2} \sum_{i_1 + .. + i_k = n} + 6 \cdot \sum_{k \geq 1} \sum_{i_1 + \ldots + i_k = n-1} \Big) 2^{2k} \prod_{j=1}^{k} r^*_{i_j}. \tag{3.29}$$

Introducing the formal power series

$$r^*(\lambda) \approx \sum_{n=1}^{\infty} \lambda^n r^*_n, \tag{3.30}$$

we find

$$r^*(\lambda) \leq \lambda + 6 \cdot \lambda \Big[(1 - 4r^*(\lambda))^{-1} - 1 \Big] + \Big[(1 - 4r^*(\lambda))^{-1} - 1 - 4r^*(\lambda) \Big]. \tag{3.31}$$

Since the equation

$$a^*(\lambda) = \lambda + 8\,(1 - 4a^*(\lambda))^{-1}\left[3 \cdot \lambda a^*(\lambda) + 2a^*(\lambda)^2\right] \tag{3.32}$$

has a solution $a^*(\lambda)$ analytic for $|\lambda| \leq \lambda_0$ for some constant $\lambda_0 > 0$ and since $a^*(\lambda)$ majorizes the series (3.30), we conclude that also $r^*(\lambda)$ converges, and hence, $\mathbb{U}(\lambda)$ is analytic in λ, for $|\lambda| < \lambda_0$. It is important to remark that the constant λ_0 above depends on d but is uniform in the volume. A straightforward calculation gives $\lambda_0 = \frac{3-\sqrt{8}}{20}$ which is much smaller than the expected radius of convergence, namely $\lambda_0 = 1$. The quality of our bound on λ_0 can however be improved by explicitly computing the dressing operator up to somewhat higher orders of perturbation theory, not stopping at the first order as we have done here.

Next we pass to the proof that $\mathbf{V}(\lambda)$ is relatively bounded with respect to $\mathbb{S}$. We have the following expression for $\mathbf{V}(\lambda)$:

$$\begin{aligned}\mathbf{V}(\lambda) = \sum_{j=1}^{\infty} \lambda^j \Big\{ \sum_b \sum_{k\geq 1} \sum_{\substack{v_1 \leq \ldots \leq v_k \\ i_1 + \ldots + i_k = j}} \frac{1}{n(v_1, \ldots v_k)} \Big[\ldots \Big[\hat{\varepsilon}_b, \mathbb{R}_{i_1}^{v_1}\Big], \ldots \mathbb{R}_{i_k}^{v_k}\Big] \\ + \sum_{k\geq 0} \sum_{\substack{v_1 \leq \ldots \leq v_k \\ i_1 + \ldots + i_k = j-1}} \frac{1}{n(v_1, \ldots v_k)} \Big[\ldots \Big[\hat{k}_b, \mathbb{R}_{i_1}^{v_1}\Big], \ldots \mathbb{R}_{i_k}^{v_k}\Big]\Big\} - E_0(\lambda)\end{aligned} \tag{3.33}$$

Expanding $\mathbf{V}(\lambda)$ in series of operators $v(\gamma)$ of support $\gamma \subset \Lambda$ and using the fact that $\mathbf{V}(\lambda)|0\rangle = 0$, we see that

$$\mathbf{V}(\lambda) = \sum_{\gamma \subset \Lambda} v(\gamma) = \sum_{\gamma \subset \Lambda} \mathrm{ad}v(\gamma), \tag{3.34}$$

where $\mathrm{ad}v(\gamma)$ is the operator such that

$$\mathrm{ad}v(\gamma)\,|\gamma'\rangle = \mathrm{ad}v(\gamma)\,\tau_{\gamma'}\,|0\rangle = \left[v(\gamma), \tau_{\gamma'}\right]|0\rangle. \tag{3.35}$$

Let us fix a vector $|u\rangle \in \mathcal{H}$ and let us expand it in the excitation basis

$$|u\rangle = \sum_{\gamma} u_\gamma |\gamma\rangle. \tag{3.36}$$

One then derives from (3.33) and (3.34) that

$$
\begin{aligned}
|\langle u|\mathbf{V}(\lambda)|u\rangle| &\leq \sum_{\gamma} |u_\gamma|^2 \, |\langle\gamma|\mathbf{V}(\lambda)|\gamma\rangle| + 2 \sum_{\substack{\gamma' : |u_{\gamma'}| \leq |u_\gamma| \\ \gamma' \neq \gamma}} |u_\gamma u_{\gamma'}| \, |\langle\gamma'|\mathbf{V}(\lambda)|\gamma\rangle| \\
&\leq 2 \sum_{\gamma} |u_\gamma|^2 \sum_{\gamma'} |\langle\gamma'|\mathbf{V}(\lambda)|\gamma\rangle| \\
&\leq 2 \sum_{\gamma} |u_\gamma|^2 \sum_{b\in\gamma} \sum_{\gamma' : b\in\gamma'} \|\mathrm{ad}v(\gamma')\|_1 . \qquad (3.37)
\end{aligned}
$$

The relative boundedness estimate in (3.15) follows from the inequality

$$
\sup_{b} \sum_{\gamma' : b\in\gamma'} \|\mathrm{ad}v(\gamma')\|_1 \leq c \cdot |\lambda| \qquad (3.38).
$$

which is a direct consequence of the bounds on the operators $\mathbb{R}_n^v$ in the first part of the proof. Q.E.D..

The proof of theorem 2 for $\lambda \neq 0$ follows from a combination of the expansion in section 2 and of the relative boundedness estimate for $\mathbf{V}(\lambda)$ we just derived. In fact, the dressed Hamiltonian in the one and two holes sectors acquires an extra term $t\mathbb{T}(\lambda)$ such that

$$
\mathbb{U}^{-1}\mathbb{H}_{t,J}\mathbb{U} = \mathbb{S} + \mathbf{V}(\lambda) + t\mathbb{T}(\lambda) + E_0(\lambda). \qquad (3.39)
$$

As long as there is a finite number of holes and t is small enough, $t\mathbb{T}(\lambda)$ is relatively bounded with respect to $\mathbb{S}$. Working in the dressed representation, a geometric series expansion in $t\mathbb{T}(\lambda)$ analogous to the one in section 2 converges for λ small, and the conclusions concerning binding are preserved, as long as $|\lambda|$ is small enough.

4. The VBS Model and the Sign Problem.

Dressing transformations are a tool flexible enough to accomodate also the VBS Hamiltonian in (1.19). This toy model is interesting from both the physical and the methodological points of view. Physically, the VBS Hamiltonian shows that the attractive interactions due to pure

exchange which we suppress in our variation of the $t - J$ model, are able, in other contexts, to give rise to a pairing mechanism free of phase separation pathologies. Methodologically, the VBS Hamiltonian is interesting because the sign problem obstructs the development of low and zero temperature polymer expansions solely on the basis of the functional integral formalism. Also the $t - J$ model has a sign problem. However, since we perturb around the half filled case and the Heisenberg Hamiltonian is equivalent to a model of hard core bosons, this problem is rather mild. In contrast, the VBS model shows a full fledged "bulk" sign problem. Some of the techniques which are necessary to readapt our methods to fermion models with sign problem difficulty are discussed in this paper in the context of the VBS model. In a subsequent article [18] we discuss how to extend the analysis to models having not only a sign problem but also a spontaneously broken discrete symmetry and to low, not necessarily zero temperature.

The Hilbert space of the VBS Hamiltonian in (1.19) can be identified with the tensor product space

$$\mathcal{H}_{VBS} \approx \otimes_{\langle xy\rangle \in \Lambda} \mathbb{C}^{16}_{\langle xy\rangle} \tag{4.1}$$

once an ordering for the bonds of the lattice Λ and for the basis vectors of the spaces $\mathbb{C}^{16}_{\langle xy\rangle}$ is fixed. The basis of $\mathbb{C}^{16}_{\langle xy\rangle}$ we choose consists of the following vectors:

$$|0\rangle_{\langle xy\rangle} = \frac{1}{\sqrt{2}}\Big(|\uparrow, \downarrow\rangle_{\langle xy\rangle} - |\downarrow, \uparrow\rangle_{\langle xy\rangle}\Big), \qquad |1\rangle_{\langle xy\rangle} = \frac{1}{\sqrt{2}}\Big(|\uparrow, \downarrow\rangle_{\langle xy\rangle} + |\downarrow, \uparrow\rangle_{\langle xy\rangle}\Big),$$

$$|2\rangle_{\langle xy\rangle} = |\uparrow, \uparrow\rangle_{\langle xy\rangle}, \qquad |3\rangle_{\langle xy\rangle} = |\downarrow, \downarrow\rangle_{\langle xy\rangle}, \qquad |4\rangle_{\langle xy\rangle} = |0, 0\rangle_{\langle xy\rangle}$$

$$|5\rangle_{\langle xy\rangle} = |0, \uparrow\rangle_{\langle xy\rangle}, \qquad |6\rangle_{\langle xy\rangle} = |\uparrow, 0\rangle_{\langle xy\rangle}, \qquad |7\rangle_{\langle xy\rangle} = |0, \downarrow\rangle_{\langle xy\rangle},$$

$$|8\rangle_{\langle xy\rangle} = |\downarrow, 0\rangle_{\langle xy\rangle}, \qquad |9\rangle_{\langle xy\rangle} = |\uparrow\downarrow, \uparrow\rangle_{\langle xy\rangle}, \qquad |10\rangle_{\langle xy\rangle} = |\uparrow, \uparrow\downarrow\rangle_{\langle xy\rangle}$$

$$|11\rangle_{\langle xy\rangle} = |\uparrow\downarrow, \downarrow\rangle_{\langle xy\rangle}, \qquad |12\rangle_{\langle xy\rangle} = |\downarrow, \uparrow\downarrow\rangle_{\langle xy\rangle}, \qquad |13\rangle_{\langle xy\rangle} = |0, \uparrow\downarrow\rangle_{\langle xy\rangle}$$

$$|14\rangle_{\langle xy\rangle} = |\uparrow\downarrow, 0\rangle_{\langle xy\rangle}, \qquad |15\rangle_{\langle xy\rangle} = |\uparrow\downarrow, \uparrow\downarrow\rangle_{\langle xy\rangle}. \tag{4.2}$$

These are eigenstates of the operator

$$s_{\langle xy\rangle} \equiv \frac{1}{4}(\vec{\sigma}_{x\langle xy\rangle} + \vec{\sigma}_{y\langle xy\rangle})^2. \tag{4.3}$$

Let us introduce operators $t^{\pm}_{z\langle xy\rangle,\sigma}$ where $z = x, y$ and $\sigma = \uparrow, \downarrow$, by setting

$$t^{\pm}_{z\langle xy\rangle,\sigma} = c^{+}_{z\langle xy\rangle,\sigma} \pm c_{z\langle xy\rangle,\sigma}. \tag{4.4}$$

Let $\mathbb{T}^{\pm}_{\langle xy\rangle,i}$, $i = 1, ...15$, be the linear combinations of products of the operators above such that

$$\mathbb{T}^{\pm}_{\langle xy\rangle,i}|0\rangle = |i\rangle_{\langle xy\rangle}. \tag{4.5}$$

An excitation is given by a map $\gamma : \{\langle xy\rangle \in \Lambda\} \to \{0, ...15\}$ and is associated to the skewsymmetric operator

$$\tau_\gamma = \frac{1}{|\gamma|} \sum_{\langle xy\rangle\in\gamma} \mathbb{T}^{-}_{\langle xy\rangle\gamma(\langle xy\rangle)} \prod_{\langle x'y'\rangle\notin\gamma}{}^{\prec} \mathbb{T}^{+}_{\langle xy\rangle\gamma(\langle xy\rangle)}. \tag{4.6}$$

As usual, we set $\gamma = \text{supp}\ \gamma$. Let us call an excitation γ admissible if the state $|\gamma\rangle$ is not orthogonal to the cyclic subspace

$$\mathcal{H}_0 = \text{span}\{\mathbb{H}^n_{VBS}|0\rangle, n = 0, 1, 2....\}. \tag{4.7}$$

We notice that, in the perturbative regime (i.e. for $|t|$ small), the ground state of $\mathbb{H}_{VBS}$ is not orthogonal to $\mathcal{H}_0$. If γ is admissible then the operator τ_γ contains an even number of Fermi creation and annihilation operators. Hence, if γ and γ' are two admissible excitations of disjoint support, we have

$$[\tau_\gamma, \tau_{\gamma'}] = 0. \tag{4.8}$$

This property is crucial to achieve clustering. However, in case the particles obey Fermi statistics, one has to give up the kind of weak commutativity we had in section 3. Hence, in order to enforce this property, the proof of convergence of the cluster expansion must be modified. As we show in the following, it turns out that, as long as there is no spontaneous symmetry breaking and as long as the volumes are ordered, it suffices to use a stronger norm for the generators of the dressing transformation in order to establish convergence. Since the estimates are recursive, it is not so surprising that the optimal norm can be neither too weak nor too strong. The best norm turns out to be not the l^1 norm that suffices in section 3, but a weighted l^1 norm instead, the weight of an excitation being its zeroth order energy.

The unitary dressing transformation is given, also in this case, by an operator of the form

$$\mathbb{U} = \lim_{v\to\infty} e^{\mathbb{R}^1(t)}....e^{\mathbb{R}^v(t)} \tag{4.10}$$

where

$$\mathbb{R}^v(t) = \sum_{n=v+1}^{\infty} t^n \, \mathbb{R}^v_n \tag{4.11}$$

and

$$\mathbb{R}^v_n = \sum_{|\gamma|=v} r_{n\gamma} \, \tau_\gamma . \tag{4.12}$$

where γ ranges over admissible excitations. Of course, in the VBS case the operators τ_γ are the ones defined above in (4.6). The recurrence relations determining the coefficients $r_{n\gamma}$ are similar to those in section 3 but will be written in a slightly different way, i.e.

$$\begin{aligned}\sum_v \mathbb{S}\mathbb{R}^v_n \, |0\rangle = -\Big\{ &\sum_{k\geq 2} \sum_{\substack{v_1 \leq \ldots \leq v_k \\ i_1 + \ldots + i_k = n}} \frac{1}{n(v_1, \ldots v_k)} \, \mathbb{P}_\perp \Big[\ldots \Big[\mathbb{S}, \mathbb{R}^{v_1}_{i_1} \Big], \ldots . \mathbb{R}^{v_k}_{i_k} \Big] \, |0\rangle \\ &+ \sum_{k\geq 0} \sum_{\substack{v_1 \leq \ldots \leq v_k \\ i_1 + \ldots + i_k = n-1}} \frac{1}{n(v_1, \ldots v_k)} \, \mathbb{P}_\perp \Big[\ldots \Big[\mathbb{K}, \mathbb{R}^{v_1}_{i_1} \Big], \ldots . \mathbb{R}^{v_k}_{i_k} \Big] \, |0\rangle \Big\} \\ &+ E_{0n} |0\rangle ,\end{aligned} \tag{4.13}$$

where

$$\mathbb{K} = \mathbb{H}_{VBS} - \mathbb{S}. \tag{4.14}$$

Let us rewrite equation (4.13) as follows:

$$\sum_v \mathbb{P}_z c_0^n \mathbb{S} \mathbb{R}_n^v |0\rangle = -\sum_{k\geq 2} \sum_{\substack{v_1 \leq .. \leq v_k \\ i_1 + ... + i_k = n}} \frac{1}{n(v_1, ...v_k)} \mathbb{P}_z c_0^n \mathbb{P}_\perp \Big[.. \Big[\mathbb{S}, \mathbb{R}_{i_1}^{v_1}\Big], ...\mathbb{R}_{i_k}^{v_k}\Big] |0\rangle$$
$$+ \sum_{k\geq 0} \sum_{\substack{v_1 \leq ... \leq v_k \\ i_1 + ... + i_k = n-1}} \frac{1}{n(v_1, ...v_k)} \mathbb{P}_z c_0^n \mathbb{P}_\perp \Big[... \Big[\mathbb{K}, \mathbb{R}_{i_1}^{v_1}\Big], ...\mathbb{R}_{i_k} v_k\Big] |0\rangle \Big\} \tag{4.15}$$

where $\mathbb{P}_z$ is the orthogonal projector onto the subspace spanned by excitations $|\gamma\rangle$ with $z \in \gamma$, $\mathbb{P}_\perp$ is the projector on the subspace ortogonal to the unperturbed ground state $|0\rangle$ and

$$c_0 = \sup \Big\{ \|T_{xi}^+\|, \|T_{x_i}^-\|, x \in \Lambda, i = 1, ...15 \Big\} \tag{4.16}$$

As a norm for the generators $\mathbb{R}_n^v$ we choose the following sequence:

$$r_n^* = \sup_{z\in\Lambda} \sum_{\gamma \ni z} c_0^n |\gamma| |r_{n\gamma}| \leq \sup_{z\in\Lambda} \|\mathbb{P}_z c_0^n \mathbb{S} \sum_v \mathbb{R}_n^v |0\rangle\|_1. \tag{4.17}$$

Let $n \geq 2$ and let us fix a sequence i_j, v_j, γ_j, , $j = 1, ...k$, such that $i_1 + ... + i_k = n$, $|\gamma_j| = v_j$, $v_1 \leq ... \leq v_k$. Let $\bar{\gamma}_k$ be the excitation such that

$$c_0^n \Big[... \Big[s_{\langle xy\rangle}, r_{i_1\gamma_1} \tau_{\gamma_1} \Big], ...r_{i_k\gamma_k} \tau_{\gamma_k} \Big] |0\rangle = f_{i_1...i_k}(\langle xy\rangle, \gamma_1, ...\gamma_k) |\bar{\gamma}_k\rangle. \tag{4.18}$$

Thanks to the translation invariance of $\mathbb{H}_{VBS}$, we have

$$\sup_{z\in\Lambda} \sum_{\langle xy\rangle} \sum_{\substack{|\gamma_1| = v_1 ... |\gamma_k| = v_k \\ z \in \bar{\gamma}_k}} c_0^n |f_{i_1...i_k}(\langle xy\rangle, \gamma_1, ...\gamma_k)|$$
$$\leq \sup_{\langle xy\rangle} \sum_{|\gamma_1| = v_1 ... |\gamma_k| = v_k} c_0^n |\bar{\gamma}_k| |f_{i_1...i_k}(\langle xy\rangle, \gamma_1, ...\gamma_{k-1})|$$
$$\cdot 2c_0 |\bar{\gamma}_{k-1}| \sup_z \sum_{\substack{\gamma_k \ni z \\ |\gamma_k| = v_k}} |r_{i_k v_k}|$$
$$\leq \sup_{\langle xy\rangle} \sum_{|\gamma_1| = v_1 ... |\gamma_{k-1}| = v_{k-1}} c_0^{n_k - 1} |\bar{\gamma}_{k-1}| |f_{i_1...i_{k-1}}(\langle xy\rangle, \gamma_1, ...\gamma_{k-1})|$$
$$\cdot 2c_0\, k \sup_z \sum_{\substack{\gamma_k \ni z \\ |\gamma_k| = v_k}} v_k |r_{i_k\gamma_k}|, \tag{4.19}$$

where

$$\bar{\gamma}_{k-1} \equiv \langle xy \rangle \cup \gamma_1 \cup \ldots \cup \gamma_{k-1}. \tag{4.20}$$

and

$$n_{k-1} = i_1 + \ldots + i_{k-1}. \tag{4.21}$$

Here, we use the fact that volumes are ordered, so that

$$|\bar{\gamma}_k| \leq (2 + |\gamma_1| + \ldots + |\gamma_k| + 1 - k) \leq k\,|\gamma_k|. \tag{4.22}$$

To estimate the commutator with τ_{γ_k}, we also use the fact that γ_k intersects $\bar{\gamma}_{k-1}$ in at most $i_k + 1$ points. Iterating this estimate, we arrive at the following final result:

$$\sup_{z \in \Lambda} \sum_{\langle xy \rangle} \sum_{\substack{|\gamma_1| = v_1 \ldots |\gamma_k| = v_k \\ z \in \bar{\gamma}_k}} |f_{i_1 \ldots i_k}(\langle xy \rangle, \gamma_1 \ldots \gamma_k)| \leq (2c_0)^k\, k! \prod_{j=1}^{k} v_j\, |r^*_{i_j v_j}| \tag{4.23}$$

where

$$r^*_{i_k v_k} = \sup_{x \in \Lambda} \sum_{\substack{\gamma \ni x \\ |\gamma| = v_k}} |r_{i_k \gamma}|. \tag{4.24}$$

The proof can now be completed along the lines followed in the previous section.

Appendix A. Numerical Implementations.

In our treatment of the anisotropic Heisenberg antiferromagnet in Section 3, we make use of a dressing transformation to transform the Hamilton operator to a form that can be used in Rayleigh-Schroedinger expansions for the spectral projections. The reason why we proceed in this way is to avoid the large fields problem for Fermi systems. In this section, we show how this pathology presents itself in the context of the cluster expansion algorithm first used by Gelfand, Singh and Huse [4] to compute zero temperature cluster expansions numerically.

Let us consider the Hamiltonian

$$\mathbb{H} = \sum_{\langle xy \rangle} (\sigma_x^{(3)} \sigma_y^{(3)} - 1) + \lambda_{\langle xy \rangle} (\sigma_x^{(1)} \sigma_y^{(1)} + \sigma_x^{(2)} \sigma_y^{(2)}) \tag{A.1}$$

where all the $\lambda_{\langle xy \rangle}$ are equal to the same small parameter λ and carry a subscript only as a bookkeeping device to distinguish the contributions to the various cluster weights coming from

the various bonds. Let us suppose that we want to compute the ground state correlation function $\langle \mathcal{O} \rangle$ of the local operator $\mathcal{O}$. Following [4], the first step of the algorithm consists in finding all connected subsets $\gamma_0 \subset \mathbb{Z}^d$ intersecting the support of $\mathcal{O}$. The calculation of the expansion coefficients is by induction in the order $n = 0, 1, \ldots$. At the n-th step we compute the weight $w(\gamma_0)$ of the clusters γ_0 with n bonds. Let $n(\gamma_0)$ denote the number of bonds of γ_0. The weights are such that

$$\langle \mathcal{O} \rangle = \sum_{\gamma_0} w(\gamma_0) \tag{A.2}$$

and

$$w(\gamma_0) = \sum_{\substack{\alpha_{\langle xy \rangle} \geq 1 \\ \langle xy \rangle \subset \gamma_0}} \prod_{\langle xy \rangle \subset \gamma_0} \lambda_{\langle xy \rangle}^{\alpha_{\langle xy \rangle}} w_\alpha(\gamma_0). \tag{A.3}$$

It is not difficult to see that if $w(\gamma_0) \neq 0$ then γ_0 must intersect $\mathcal{O}$. Supposing that all the weights of the clusters γ_0' of order $n(\gamma_0') < n(\gamma_0)$ are known, one can compute $w(\gamma_0)$ by restricting the Hamiltonian to γ_0 and evaluating the corresponding ground state $\Psi_{\gamma_0}(\lambda)$ by means of a Rayleigh-Schroedinger expansion. The weight $w(\gamma_0)$ is given by

$$w(\gamma_0) = \frac{\langle \Psi_{\gamma_0}(\lambda) | \mathcal{O} | \Psi_{\gamma_0}(\lambda) \rangle}{\langle \Psi_{\gamma_0}(\lambda) | \Psi_{\gamma_0}(\lambda) \rangle} - \sum_{\gamma_0' \subset \gamma_0} w(\gamma_0'). \tag{A.4}$$

To compute $\Psi_{\gamma_0}(\lambda)$, one can use the formula

$$|\Psi_{\gamma_0}(\lambda)\rangle = \oint_C \frac{dz}{2\pi i} \frac{1}{z - \mathbb{S}_{\gamma_0} - \lambda \mathbb{K}_{\gamma_0}} |\Psi_{\gamma_0}(0)\rangle, \tag{A.5}$$

where $C = \{|z| = \frac{1}{2}\}$, and $\mathbb{S}_{\gamma_0}$, $\mathbb{K}_{\gamma_0}$ are the restricions to γ_0 of the Z part $\mathbb{S}$ and the XY part $\mathbb{K}$ of the Heisenberg Hamiltonian.

The idea of Gelfand, Singh and Huse of computing the ground state wavefunction of restrictions of the Hamiltonian to small clusters is interesting because it reduces a complex problem to many simpler ones. However, this technique presents some difficulties. One of the problems is that excited state properties are harder to study than ground state correlation functions; as we discuss above, with dressing transformations this is not the case. A second problem is that the value of $w(\gamma_0)$ at the leading order in λ is given by the sum of $n(\gamma_0)!$ diagrams. However, from

the analyticity result in section 3 it follows that

$$|w(\gamma_0)| \leq (c \cdot \lambda)^{n(\gamma_0)}. \tag{A.6}$$

Hence, many cancellations must be taking place among the $n(\gamma_0)!$ diagrams contributing to $w(\gamma_0)$. To set up a rigorous proof one can either attempt to understand the cancellation mechanism, as is possible for weakly interacting Fermi systems, or one can make use of dressing transformations, as we do in the previous sections.

To draw some conclusions, we would like to list some of the main properties and the general conditions of applicability of the method of dressing transformations as they emerge from this work and to add some remarks on which we elaborate more in depth in the three companion papers [12], [18], [21].

(i) The Hamiltonian is supposed to be short range and equal to the sum of a main part whose ground state wavefunction we know and of a perturbation;

(ii) Since we don't attempt a multiscale decomposition, the main part is required to possess a spectral gap which is large with respect to the size of the perturbation term, as measured by a local norm;

(iii) Unitarity is a very convenient property of dressing transformations leading to a normalized ground state wavefunction and to various semplifications in both the rigorous and numerical analysis. However, one should keep in mind that for certain applications as well as for the low temperature expansion in [18] this property is neither required nor useful.

(iv) Commutativity or weak non-commutativity, in the sense of section 3, are interesting properties, because they generate cluster expansions with the lowest number of diagrams. However, such properties are not strictly necessary to achieve convergence.

(v) If the generators of the dressing transformation are given by non-commutative operators, the volumes have to be ordered as in (3.9); otherwise, control of the convergence of the cluster expansion is jeopardized because of the high growth rate of the number of diagrams.

(vi) If a discrete symmetry is spontaneously broken or, more generally, if the main part of the Hamiltonian has a finitely degenerate ground state, then special care is required to define the

dressing transformation. In this case, only Peierls contours are associated to an energy term and can be allowed to contain centers of non-commutativity if a formal proof of convergence of the algorithm is an issue.

(vii) To control excited states, one must require the operator $V(\lambda)$ defined as in (3.33) to be relatively bounded with respect to the main part of the Hamiltonian. This property is fairly automatic if the ground state is non-degenerate but requires special attention in case it has a finite degeneracy. In the case of the anisotropic Heisenberg antiferromagnet in Sect. 3, the desire of defining a convergent unitary dressing transformation with a relatively bounded $V(\lambda)$ forced us to enlarge the Hilbert space. In [18] we show that if one is willing to give up unitarity, it is possible to respect the symmetry while remaining in the original Hilbert space.

(viii) Dressing transformations can also be used to generate time dependent cluster expansions in the adiabatic limit; a result in this direction is in [21].

(ix) If one wants to optimize dressing transformations as a numerical algorithm to compute ground state properties and reduced Hamiltonians at zero temperature, then the operators τ_γ must be chosen skewsymmetric to avoid orthonormalization problems. Such problems can be very delicate if finite precision arithmetic is used [22] because of the large dimensionality of the Hilbert space generated by elementary excitations. An algebraic solution to this problem is preferable to a brute force approach and reduces the systematic errors.

(x) If only the first few orders of a perturbation expansion are required, then the techniques used in this paper to control the large order behaviour of the series expansions in the presence of a spontaneously broken discrete symmetry are not relevant. In particular, the extension of the Hilbert space in section 3 should be avoided, because it increases, rather than lowers, the number of diagrams generated at the first few orders of perturbation theory.

(xi) The best way of using the method of dressing transformations for both numerical and rigorous calculations for the $t-J$ and similar models is to proceed in two steps: first, one solves the problem in the absence of holes, and then one conjugates the Hamiltonian in the sector with a few holes with the dressing operator found in the first step. This procedure, block diagonalizes the Hamiltonian. Each block can be seen as the effective Hamiltonian describing a number of

spinless fermions corresponding to the dressed holes in interaction with bosonic quasiparticles corresponding to spin-wave excitations. Since the unitary transformation is given by a power series, the applicability of this method is limited to the parameter region on which such series either converge or can be analytically continued by means of Pade' approximants or a similar technique. The leading block for small t and λ corresponds to the effective Hamiltonian with no spin-wave excitations. This is the only block we consider in this paper. However, as t and λ increase, there might be a level crossing among the ground state energy of this block and that of some other block. This phenomenon typically corresponds to a first order phase transition. We would like to emphasize that, as long as analyticity doesn't fail, our method can be applied also in the presence of level crossing and provides full information on the phase transition. In [18], we show this idea at work on a model of strongly interacting fermions with a Mott metal-insulator transition for which, by using particle number conservation, one can rigorously prove that analyticity of the dressing transformation is not lost on at least part of coexistence curves and one can explicetely compute both the phase boundaries and the critical behaviour of quantities like the spectral gap.

(xii) The first step of both a rigorous and a numerical calculation must be the solution of a model like the anisotropic Heisenberg model with a zeroeth order gap of size one and without quasiparticles in the ground state. To this end, it is interesting to notice that most of the clusters γ appearing at a certain order of perturbation theory have a large energy $\langle\gamma|\mathbb{S}|\gamma\rangle$ and are associated to a small amplitude $r_{n\gamma}$, the competition among the entropy and the smallness of the amplitude being won by the latter. It is thus convenient to add counterterms to the Hamiltonian in such a way to set exactly at zero the amplitude $r_{n\gamma}$ of such exotic configurations. This introduces errors but ultimately improves the precision because it saves memory and speeds up the algorithm, allowing one to push the expansion to higher orders where one can find diagrams which are more significative than the ones neglected at the previous orders. The error can be measured as in a typical exact calculation based on the Lanczos algorithm , i.e. by using the variance

$$\delta \equiv \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\|(\mathbb{H} - E_0)\Psi\|_1, \qquad (A.7)$$

where $\Psi = \mathbb{U}_{nd}|0\rangle$ and $\mathbb{U}_{n\delta}$ is the approximate n-th order transformation with error δ. Truncation methods have also been discussed in [23] and [24] in the context of models without vacuum fluctuations and turned out to be very successfull. In the presence of vacuum fluctuations, if one truncates directly the wavefunction amplitudes as in [7] the gain is not as noticeable. Our way of truncating a many-body wavefunction might reveal itself more effective because it respects the clustering properties of the ground state wavefunction and doesn't introduce spurious nonlocal effects.

The apparent advantages of our algorithm with respect to Lanczos' are that finite size corrections are absent, memory requirements for the ground state wavefunction are reduced by exploiting the clustering properties of the ground state wave-function and knowledge of the generators of the transformation is the only information which is necessary to compute both the ground state expectations and the reduced Hamiltonians for quasiparticles. The disadvantage is that we are limited by the constraints (i) and (ii) above and can't find explicetely the ground state of a system of several holes but can only generate the reduced Hamiltonians; see however the next subsection. The advantages over the cluster expansion algorithm in [4] are that fewer diagrams are generated, one doesn't need to recompute from scratch a new set of weights for each local observable, memory requirements are further reduced by neglecting contributions coming from configurations with small amplitude and, finally, excited states are not problematic. The disadvantage is that fewer exact coefficients can be computed this way. To raise the order at which one can push the cluster expansion without allowing error terms, an approach closer to the one developed by Gelfand, Singh and Huse is necessary.

(xiii) Once the quantum fluctuations of the antiferromagnetic vacuum are incorporated into a unitary dressing transformation, one can consider the case in which holes are present and use the same dressing transformation to block diagonalize the Hamiltonian, as we do in section 3. An estimate like the one in (A.7) is also valid in the sectors with a finite number of holes, as long as Ψ is reinterpreted as the ground state wavefunction of the effective Hamiltonian corresponding to the leading block. Since also the effective Hamiltonian is symmetric and the nonzero matrix elements are clustered near the diagonal, one can find such a ground state by

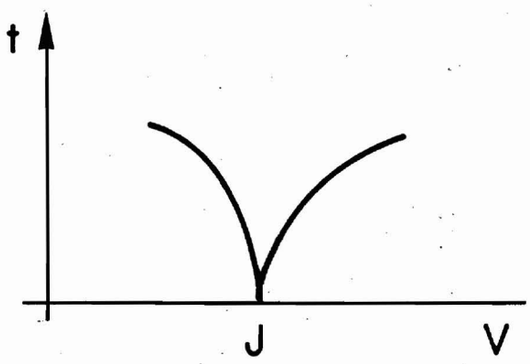
means of a Lanczos type calculation. This two steps procedure leads to a remarkable reduction of algorithmic complexity with respect to a straightforward implementation of the Lanczos algorithm directly on the bare Hamiltonian. By using translation symmetry, one can further decompose the effective Hamiltonian in about $\frac{1}{|\Lambda|}$ blocks, each one indexed by a wave vector k. On a square lattice, most values of k are 8-fold degenerate; hence one can decompose the problem into about $\frac{1}{8}|\Lambda|$ simpler problems. The dimension d of the sector of the reduced Hilbert space corresponding to a certain k in the first Brillouin zone has the following value on a 8×8 lattice with n_h holes: $d = 1$ if $n_h = 1$, $d = 32$ if $n_h = 2$, $d = 651$ if $n_h = 3$, $d = 9928$ if $n_h = 4$, $d = 119.133$ if $n_h = 5$ and $d = 1.171.147$ if $n_h = 6$. This means that, in the parameter region in which the dressing transformation converges, one can realistically plan an exact diagonalization study for the $t - J - V - V'$ model with up to 6 holes on a lattice of up to 64 sites. As a comparison, a direct application of the Lanczos algorithm to the bare Hamiltonian would involve an Hilbert space whose dimension is greater by a factor $2^{58} \approx 2.88 * 10^{17}$. With the Lanczos algorithm alone, exact diagonalizations are confined to lattices of at most 18-20 sites and the algorithmic complexity raises so steeply with the size that no forseeable progress in computer technology can improve this limit in a significative way. By implementing the algorithm we discuss in this paper, this bottleneck can probably be bypassed.

(xiv) Dressing transformations can be combined also with quantum Montecarlo algorithms. There are at least three reasons why this may be advantageous in certain situations. First, the effective Hamiltonian corresponding to the leading block of the dressed $t - J$ Hamiltonian describes spinless fermions; hence there are no difficulties related to the enforcement of the constraint of no double occupancy. Second, as is explained in greater detailed in [18], dressing transformations make the sign problem become milder and, in some cases, one can prove that optimal lower bounds on the expectation of the sign operator are satisfied. Third, dressing transformations are useful because they tipically renormalize bare strong coupling interactions into weak coupling interactions among dressed quasiparticles.

Appendix B. Phase Separation: a Heuristic Discussion.

Phase separation of holes into bubbles is a physical phenomenon that has been observed

in several copper oxides near the superconducting phase transition [8]. Numerical studies of the $t - J$ model also confirm the theoretical analysis of Kivelson, Emery and Lin [3] who predict phase separation in the small t limit. The reason why phase separation is stable is that in slightly doped antiferromagnetic insulators it is hard to strike a balance between the tendency of the holes to delocalize in order to lower the kinetic energy and the tendency of the magnetic vacuum to relax to the antiferromagnetic ground state. As a consequence, if the interaction among holes is of the static type, holes are expelled from the insulator and concentrate in bubbles. Emery and Kivelson suggest that high T_c superconductivity is a phenomenon related to frustrated phase separation and arises thanks to the long range Coulomb forces which destabilize such bubbles. Recent numerical simulations by Troyer *et al.* showed that phase separation in the $t - J - V - V'$ model is actually suppressed by long range repulsive forces and that pairing susceptibility is enhanced, but that nonetheless a charge density wave instability appears to suppress superconductivity. Although the problem of long range forces requires more work to be settled, it is also worth exploring other scenarios which can possibly lead to superconductivity through short range interactions. The mechanism we propose in this paper involves an effective interaction whose leading term is velocity dependent and is such that the effective attraction among two holes is strengthened by the delocalization of the pair. The mechanism provided by dynamic binding, if available, would be a conceptually simpler way of producing superconductivity than a mechanism based on long range interactions. In fact, in the first case melting of the phase separated bubbles is not due to the fact that the kinetic energy terms or long range forces are so large that they dominate the effective attractive interaction of magnetic origin, but to the fact that an interplay between magnetic forces and the hopping term favours the binding of two holes and the delocalization of pairs of two holes. From this point of view, high temperature superconductivity is also due to a kind of frustrated phase separation, but the competition in our case is between different kinds of order inside the bubbles which makes an otherwise subleading interaction dominate in the crossover region.



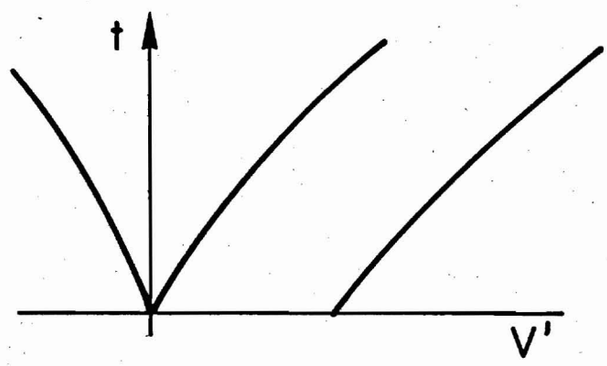


Fig. 2

The physically relevant parameter region for the $t - J - V - V'$ model has $t \approx J$ and $\lambda \approx J$ and can be reached only with numerical methods. However, it is conceivable that by carefully tuning the interaction parameters one can stabilize a phase at small t and λ in which the relevant pairing mechanism which is responsible for high temperature superconductivity dominates. The question for the numerical studies would then be to find out whether this phase becomes more stable or, instead, it disappears as we move to larger values of t and λ. Setting $V = J$ in (1.1), we neutralize the exchange interaction among holes, so that it is neither attractive nor repulsive in the limit $t = \lambda = V' = 0$. The occurrence of phase separation for $\lambda << t << V$ depends on the value of the next neighbour coupling V'. If $V' < 0$, then at $\lambda = t = 0$ holes arrange themselves in a checkerboard configuration. If t and λ are switched on, this phase is still stable for $V' < -(c_1 t^2 + c_2 \lambda^2)$, for some constants $c_1, c_2 > 0$ of order $O(1)$. If $V' = 0$ and $t > \lambda$, the most stable configuration appears to be a double chain of holes forming a domain wall. In fact, virtual hoppings of the holes in and out of the double chain are not suppressed by any frustration effect. This results in a resonance among all configurations in which one or more holes which are not neighbours of each other make one step out of the chain. As a result, the contribution to the ground state energy coming from the holes is proportional to $(-t)$ times the number of holes. The dynamic binding mechanism in section 2 is already at work in this parameter region, but since it gives rise to a ground state energy $\approx -t^2$ per hole, it is suppressed. Double chains are much less stable then crystals and as V' increases they undergo an instability for $V' \approx t$. There are two constants of order $O(1)$ such that, if $c_3 t < V' < c_4 J$, then double chains are unstable and the leading interaction mechanism is the dynamical binding in section 2. Hence in this parameter region we expect the stable phase to be a superfluid of tightly bound hole pairs. Beyond the threshold $V' = c_4 J$, also dynamic binding becomes unstable and static interactions dominate. We thus expect some kind of phase separation to take over again.

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