FPRINT-93-58

GENERAL SOLUTIONS FOR SCALAR-TENSOR COSMOLOGIES

John D Barrow

Astronomy Centre, University of Sussex, Brighton BN1 9QH, UK

(Received) Sussex U.

Abstract

A simple representation of scalar-tensor gravity theories is presented which permits the general solution of the vacuum and radiation-dominated Friedman universes after the gravity theory is specified. A range of theories are solved explicitly.

PACS numbers: 98.80.Hw, 12.10.Gq, 04.50.+h, 04.60.+n

There are now many situations in which it is advantageous to have simple exact cosmological models in scalar-tensor theories of gravitation. They permit rigorous quantification of the allowed deviations from the general theory of relativity through confrontation with standard solar system tests [1], primordial nucleosynthesis [2] and astronomical observations of the binary pulsar [3]. They arise in studies of dimensional reduction in the early universe [4], in studies of varying 3-dimensional constants induced by the variation in mean size of extra dimensions [5], in the study of extended inflationary universe models [6],[7], in some proposed resolutions of the dark matter problem [8] and possible modifications to primordial black hole formation and evaporation [9]. Studies of 'oscillating physics' [10] and its possible connection with quasi-periodicity of large-scale structure in redshift space [11] also need exact cosmological models which permit variation of Newton's gravitational 'constant' in order to to compute other observational consequences of this idea. Information is also potentially available about the scalar-tensor nature of gravitation in the very early Universe from the super-adiabatic production of gravitons [12].

Until recently very few cosmological solutions to scalar-tensor theories of gravity have been found. Attention has focused upon the simplest case of the Brans-Dicke theory [13].

FPRINT-93-58

Here we present a simple procedure which gives the general vacuum and radiation-dominated Friedman cosmological solutions to scalar-tensor gravity theories in a manner that facilitates understanding of the range of behaviour that are possible at small and large times.

Scalar-tensor gravity theories have lagrangians of the form

$$L_\varphi = -\varphi R + \varphi^{-1}\omega(\varphi)\partial_a\varphi\partial^a\varphi + 16\pi L_m$$

Fermilab Library
0 1160 0017882 6

where φ is a scalar field coupled to the space-time 4-curvature, R, by $\omega(\varphi)$ and L_m is the lagrangian of the remaining matter fields. The theory of Brans and Dicke is the special case ω = constant. The generalised Einstein equations and the wave equation for φ are

$$R_{ab} - \tfrac{1}{2}g_{ab}R = -8\pi\varphi^{-1}T_{ab} - \omega(\varphi)\varphi^{-2}\{\varphi_a\varphi_b - \tfrac{1}{2}g_{ab}\varphi_i\varphi^i\} - \varphi^{-1}\{\varphi_{a;b} - g_{ab}\,\Box\varphi\} \qquad (1)$$

$$\{3 + 2\omega(\varphi)\}\,\Box\varphi = 8\pi T - \omega'(\varphi)\varphi_i\varphi^i \qquad (2)$$

where T^{ab} is the covariantly conserved energy-momentum tensor of the matter content of the theory. If its trace, T, vanishes, and φ is a constant, then (1)-(2) reduce to the standard Einstein equations with gravitational constant $G = \varphi^{-1}$. Hence any exact solution of Einstein's equations with a tracefree matter source will also be a particular exact solution of the scalar-tensor theory with φ, and hence $\omega(\varphi)$, constant. In what follows we shall seek the general solutions of the isotropic and homogeneous models when neither φ nor $\omega(\varphi)$ is constant.

FERMI
JUL 22 1993
LIBRA

Consider Friedman cosmologies with metric (c = 1)

$$ds^2 = a^2(\eta)\{d\eta^2 - (1-kr^2)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2\}$$

where $a(\eta)$ is the expansion scale factor, η is the conformal time coordinate, and the

metric curvature parameter is $k = 0$ or ± 1. We shall assume that the material content of the universe is black-body radiation so the density ρ is given by $8\pi\rho = 3\Gamma a^{-4}$ where $\Gamma \geq 0$ is a constant. The case $\Gamma = 0$ will define the vacuum model. Introduce a function $y_k(\eta)$ defined by

$$y_k = \varphi a^2, \tag{3}$$

where k parametrizes the curvature, and (1) and (2) become

$$\varphi' a^2 = 3^{\frac{1}{2}} A(2\omega(\varphi)+3)^{-\frac{1}{2}} \; ; \; A \text{ constant}. \tag{4}$$

$$y_k'^2 = 4\Gamma y_k + A^2 - 4k y_k^2. \tag{5}$$

Equation (5) integrates to give, in vacuum,

$$\begin{aligned} y_0(\eta) &= A(\eta + \eta_0), \\ y_{-1}(\eta) &= \tfrac{1}{2}A\sinh\{2(\eta+\eta_0)\}, \\ y_1(\eta) &= \tfrac{1}{2}A\sin\{2(\eta+\eta_0)\} \end{aligned} \tag{6}$$

and, for radiation,

$$\begin{aligned} y_0(\eta) &= \Gamma(\eta + \eta_0)^2 - A^2/4\Gamma, \\ y_{-1}(\eta) &= -\tfrac{1}{2}\Gamma + \tfrac{1}{2}(A^2 - \Gamma^2)^{\frac{1}{2}} \sinh\{2(\eta+\eta_0)\} \text{ if } A^2 > \Gamma^2 \\ y_{-1}(\eta) &= -\tfrac{1}{2}\Gamma + \tfrac{1}{2}\exp[2(\eta+\eta_0)] \text{ if } A^2 = \Gamma^2 \\ y_{-1}(\eta) &= -\tfrac{1}{2}\Gamma + \tfrac{1}{2}(\Gamma^2 - A^2)^{\frac{1}{2}} \cosh\{2(\eta+\eta_0)\} \text{ if } \Gamma^2 > A^2 \\ y_1(\eta) &= \tfrac{1}{2}\Gamma + \tfrac{1}{2}(\Gamma^2 + A^2)\sin\{2(\eta+\eta_0)\} \end{aligned} \tag{7}$$

If the scalar-tensor theory is defined by the specification of a scalar function $f(\varphi)$ via

$$2\omega(\varphi) + 3 = 3\varphi^2 f'^2/f^2$$

then (4) integrates to give, in vacuum,

$$f(\varphi)/f_0 = \begin{cases} \eta + \eta_0 & \text{for } k = 0 \\ \tanh(\eta + \eta_0) & \text{for } k = -1 \\ \tan(\eta + \eta_0) & \text{for } k = +1 \end{cases} \tag{8}$$

with f_0 constant; the corresponding radiation solutions are,

$$f(\varphi)/f_0 = \begin{cases} \left[\dfrac{2\Gamma\eta + 2\Gamma\eta_0 - A}{2\Gamma\eta + 2\Gamma\eta_0 + A}\right] & k = 0 \\ \left[\dfrac{\Gamma\tan(\eta+\eta_0) + (\Gamma^2+A^2)^{\frac{1}{2}} - A}{\Gamma\tan(\eta+\eta_0) + (\Gamma^2+A^2)^{\frac{1}{2}} + A}\right] & k = +1 \\ \left[\dfrac{\Gamma\tanh(\eta+\eta_0) - A + (A^2-\Gamma^2)^{\frac{1}{2}}}{A + \Gamma\tanh(\eta+\eta_0) + (A^2-\Gamma^2)^{\frac{1}{2}}}\right] & k = -1, \; A^2 > \Gamma^2 \\ \left[\dfrac{A\tanh(\eta+\eta_0) - \Gamma + (\Gamma^2-A^2)^{\frac{1}{2}}}{-\Gamma - A\tanh(\eta+\eta_0) + (\Gamma^2-A^2)^{\frac{1}{2}}}\right] & k = -1, \; \Gamma^2 > A^2 \\ 1 - \Gamma\exp(-2(\eta+\eta_0)) & k = -1, \; A^2 = \Gamma^2 \end{cases} \tag{9}$$

The scale factor $a(\eta)$ is obtained from (3), (6) or (7) and (8) or (9) in the vacuum and radiation cases respectively, since,

$$a^2(\eta) = \frac{y_k(\eta)}{f^{-1}[\eta+\eta_0]}$$

where f^{-1} is the inverse function of f obtained by inverting (8) or (9) in the vacuum or radiation cases respectively. In the $k \leq 0$ radiation-dominated case $f(\varphi)$ approaches a constant as $\eta \to \infty$ and the general relativistic solutions are approached. In the vacuum case $f(\varphi)$ approaches a constant in the $k = -1$ universes as $\eta \to \infty$ because solutions asymptote to the Milne vacuum model of general relativity ($a \propto \exp(\eta)$). When $k \geq 0$ there are no vacuum Friedman models to approach and the asymptotic behaviour is sensitive to the functional form of $f(\varphi)$. Five different forms for $f(\varphi)$ are given below to indicate the range of behaviours for $a(\eta)$ and $\varphi(\eta)$. We give only the $k = 0$ solutions but the algebraically cumbersome $k \neq 0$ models are easily obtained if required. We have

defined the zero of η time by setting $\eta_0 = 0$. The vacuum models are as follows:

$f(\varphi)/f_0$	$\varphi(\eta)$	$a^2(\eta)$
φ^p	$\eta^{1/p}$	$A\eta^{1-1/p}$
$\exp(\lambda\varphi^n)$	$\lambda^{-1/n}\,\ell n^{1/n}(\eta)$	$A\lambda^{1/n}\,\eta\,\ell n^{1/n}(\eta)$
$\ell n^m(\varphi)$	$\exp[\eta^{1/m}]$	$A\eta\,\exp[-\eta^{1/m}]$
$\tanh^{-1}(\lambda\varphi)$	$\lambda^{-1}\tanh(\eta)$	$A\lambda\eta\,\coth(\eta)$
$\tanh(\lambda\varphi)$	$\lambda^{-1}\tanh^{-1}(\eta)$	$A\lambda\eta\,(\tanh^{-1}(\eta))^{-1}$

The $f = \varphi^p$ case includes the Brans–Dicke theory since $\omega(\varphi) = \frac{1}{2}(p^2 - 1)$ = constant. The $k = 0$ radiation models, are given as follows:

$\underline{f(\varphi) = f_0\varphi^p}$

$$\varphi(\eta) = \left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]^{1/p}$$

$$a^2(\eta) = \frac{\frac{1}{4}\Gamma^{-1}[2\Gamma\eta - A]^{(p+1)/p}}{[2\Gamma\eta - A]^{(p-1)/p}}$$

$\underline{f = f_0\exp(\lambda\varphi^n)}$

$$\varphi(\eta) = \lambda^{-1/n}\,\ell n^{1/n}\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]$$

$$a^2(\eta) = \frac{\lambda^{1/n}[\Gamma\eta^2 - \frac{1}{4}A^2\Gamma^{-1}]}{\ell n^{1/n}\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]}$$

$\underline{f(\varphi) = f_0\ell n^m(\varphi)}$

$$\varphi(\eta) = \exp\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]^{1/m}$$

$$a^2(\eta) = \exp\left[-\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]^{1/m}\right] \times [\Gamma\eta^2 - \tfrac{1}{4}A^2\Gamma^{-1}]$$

$\underline{f = f_0\tanh^{-1}(\lambda\varphi)}$

$$\varphi(\eta) = \lambda^{-1}\tanh\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]$$

$$a^2(\eta) = \lambda^{-1}\coth\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right] \times [\Gamma\eta^2 - \tfrac{1}{4}A^2\Gamma^{-1}]$$

$\underline{f(\varphi) = f_0\tanh(\lambda\varphi)}$

$$\varphi(\eta) = \lambda^{-1}\tanh^{-1}\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]$$

$$a^2(\eta) = \frac{\lambda[\Gamma\eta^2 - \frac{1}{4}A^2\Gamma^{-1}]}{\tanh\left[\frac{2\Gamma\eta - A}{2\Gamma\eta + A}\right]}$$

The method presented is not restricted to the forms of $f(\varphi)$ displayed here. It enables us to evaluate the physical consequences of scalar–tensor cosmological models for the radiation–dominated phase of the early universe. Baryosynthesis, nucleosynthesis and graviton production can be studied in detail. This will enable observational bounds to be placed upon the allowed deviations from general relativity and then can be compared with those derived from local solar–system tests.

Acknowledgements

The author acknowledges a Senior Research Fellowship from the Royal Society and the Leverhulme Trust.

References

[1] C. Will, *Theory and experiment in gravitation physics*, Cambridge UP., (1981). T. Damour, Gravitation and Experiment, in *Gravitation and Quantizations*, ed. B. Julia and J. Zinn-Justin, North-Holland, Amsterdam 1993.

[2] J.D. Barrow, Mon. Not. Roy. astr. Soc. 184, 677 (1978). J. Garcia-Bellido and M. Quiros, Phys. Lett. B 243, 45 (1990).

[3] C. Will, Int. J. Mod. Phys. D 1, 13 (1992).

[4] P.G.O. Freund, Nucl. Phys. B 209, 146 (1982).

[5] J.D. Barrow, Phys. Rev. D 35, 1805 (1987).

[6] D. La and P.J. Steinhardt, Phys. Rev. Lett. 62, 376 (1989). J.D Barrow and K. Maeda, Nucl. Phys. B 341, 294 (1990).

[7] P.J. Steinhardt and F.S. Accetta, Phys. Rev. Lett. 64, 2740 (1990).

[8] T. Damour, G. Gibbons and C. Gundlach, Phys. Rev. Lett. 64, 123 (1990).

[9] J.D. Barrow, Phys. Rev. D 46, R3227 (1992).

[10] M. Morikawa, Astrophys. J., 362, L37. C.T. Hill, P.J. Steinhardt and M.S. Turner, Phys. Lett. B 252, 343 (1990). R.G. Crittenden and P.J. Steinhardt, Astrophysical J. 395, 360 (1992).

[11] T.J. Broadhurst, R.S. Ellis, D.C. Koo and A.S. Szalay, Nature 343, 726 (1990).

[12] J. D. Barrow, J.P. Mimoso and M.R. de Garcia Maia, Phys. Rev D (submitted).

[13] C. Brans and R.H. Dicke, Phys. Rev. D 124, 925 (1961). H. Nariai, Prog. Theo. Phys. 40, 49 (1968), L.E. Gurevich, A.M. Finkelstein and V.A. Ruban, Astrophys. Space Sci. 22, 231 (1973), D. Lorenz-Petzold, Astrophys. Space Sci. 96, 451 (1983), J. O'Hanlon and B.O.J. Tupper, Nuovo Cimento 137, 305 (1970). J.D. Barrow, Phys. Rev. D 47, 000 (1993)