FPRINT-93-54

D=3 GENERAL RELATIVITY IS NOT D=3 GRAVITY

13

Y. Verbin*
Physics Group
The Open University of Israel
Tel Aviv 61392, Israel

Three-dimensional General Relativity has many intriguing properties. Among them is a vanishing Newtonian potential which disqualifies this theory to serve as a theory for three-dimensional gravity. The proper description of three-dimensional gravity is obtained by dimensional reduction of four-dimensional General Relativity, and shown to be a scalar-tensor theory. This theory admits a Newtonian limit and can further incorporate all the non-Newtonian objects like gauge strings and global strings as well. Coupling with electromagnetic field in three dimensions is also obtained by dimensional reduction and the result is found to be essentially different from the D=3 Maxwell-Einstein theory.

PACS numbers: 04.20.Cr, 04.40.+c, 98.80.Cq

---

* e-mail: verbin@openu.ac.il

1. INTRODUCTION

Lower dimensional field theories are widely used as "toy models" for studying various aspects which are difficult to handle in their four-dimensional counterparts. Three-dimensional General Relativity [1,2,3] (D=3 GR) is no exception. Although careful authors avoid the explicit statement, there is an impression that D=3 GR is the appropriate theory for describing D=3 gravity. This impression stems partly from the fact that (straight) cosmic strings [4] can be described in both ways, either as linear matter distributions (along, say, the x^3 axis) with $T^3{}_3=T^0{}_0$ in four dimensional spacetime [5,6,7], or as point sources in three dimensions [8,9,10,11]. There exists however, a very serious difficulty, namely, the absence of Newtonian limit in D=3 GR. This phenomenon may occur also in D=4 as demonstrated very clearly by cosmic strings. These are known to generate around them a conical space with a mass dependent deficit angle. Consequently, cosmic strings exert no forces on non-relativistic matter. In D=3, however, the Newtonian limit is always absent and gravitation has no "action at a distance". This is, of course, a manifestation of the fact that in D=3 GR source-free regions are flat. This issue has been discussed from the three-dimensional point of view to some extent by several authors [2,3] who accepted the absence of the Newtonian limit as an inevitable consequence of the general-relativistic approach to three-dimensional gravity.

In this paper I take a more conservative approach and claim that due to the fact that three-dimensional General Relativity fails to provide a Newtonian limit, it cannot be considered as a proper description of D=3 gravity. Naturally, this leaves us with the question: what is then the correct theory? Our starting point for answering this question will be D=4 GR and our aim will be to obtain from this theory an effective three-dimensional one which will be able to give a proper description of the physics of parallel linear sources and other x^3-independent matter distributions. This is our definition of the term "three-dimensional gravity". The straightforward way for this purpose is to perform a dimensional reduction (see e.g. [12] and references therein) from four into three dimensions. As a result, we will find that the effective lower dimensional theory is a scalar-tensor theory (with the scalar being the 33-component of the metric tensor), rather than pure three-dimensional Einsteinian General Relativity. This of course settles the issue of the absence of "action at a distance" in D=3, since the scalar field now serves as a source to the D=3 metric tensor thus surpassing the above-mentioned conclusion that matter-free regions are flat.

2. D=3 GRAVITY FROM D=4 GENERAL RELATIVITY

As we have just mentioned we wish to obtain from a D=4 point of view an effective theory which will determine the dynamics of matter distributions with one-dimensional translational symmetry, and could be considered as a proper theory of D=3 gravity. For concreteness we may think of these matter distributions as strings. We take them to be parallel to the x^3 axis and let them move only perpendicular to x^3, i.e. $\dot{x}^3=0$. In order to find the D=3 effective theory we first write the line element in the form:

$$ds^2 = \gamma_{\alpha\beta}\, dx^\alpha dx^\beta + 2B_\alpha dx^\alpha dx^3 - \Phi^2 (dx^3)^2 \qquad (1)$$

where $\gamma_{\alpha\beta}$ is the D=3 metric tensor with α,β ranging from 0 to 2, i.e. the $\alpha\beta$ components of the D=4 metric, $g_{\mu\nu}$. B_α and Φ are related to the other components of $g_{\mu\nu}$ in an obvious way*. All these quantities are naturally independent of x^3.

Note that this parametrization of the metric tensor is somewhat unconventional since we do not have any freedom in the interpretation of the lower-dimensional metric tensor. In other words, it is $g_{\alpha\beta}$ which should be identified as the real lower-dimensional metric tensor, not any other combination of $g_{\mu\nu}$, as is usually done in Kaluza-Klein theories.

---

* We use the conventions: $R = g^{\mu\nu} R^{\kappa}{}_{\mu\kappa\nu}$, $R^{\kappa}{}_{\lambda\mu\nu} = \partial_\nu \Gamma_\mu{}^{\kappa}{}_\lambda - \ldots$ and signature $(+,-,-,-)$. All the geometric quantities will be covariant in the D=3 sense; e.g. ∇_α is the covariant derivative computed from $\gamma_{\alpha\beta}$, and not the α component of ∇_μ.

Let us consider first the motion of a point particle in the D=3 space-time. It's motion is obviously determined by the D=4 geodesic equations. The x^3 independence of $g_{\mu\nu}$ is not a sufficient condition to ensure that no motion in the x^3 direction will occur for obtaining a consistent D=3 description. We should further exclude forces in the x^3 direction, or, to put it differently, $\dot{x}^3=0$ should be a solution of the D=4 geodesic equations of motion. This means that $\Gamma_\alpha{}^3{}_\beta$ must vanish. Direct computation of $\Gamma_\alpha{}^3{}_\beta$ in terms of the D=3 fields in (1) yields the condition:

$$\Gamma_\alpha{}^3{}_\beta = -\frac{1}{2}(\Phi^2 + B^\alpha B_\alpha)^{-1}(\nabla_\alpha B_\beta + \nabla_\beta B_\alpha) = 0. \tag{2}$$

It thus seems that in order to fulfill $\Gamma_\alpha{}^3{}_\beta=0$ the D=3 metric tensor must admit Killing vectors, which is evidently too restrictive. Moreover, the Killing vectors should be identified with the vector degrees of freedom of the dimensionally reduced theory, thus rendering them non-dynamical. The way to surpass this constraint is to set $B_\alpha=0$, which is consistent with (2) and imposes no restrictions on $\gamma_{\alpha\beta}$.

We found therefore that D=3 gravity should be described in terms of the metric tensor $\gamma_{\alpha\beta}$ and the scalar field, Φ. It turns out, however, that Φ does not have a direct influence on the motion of point particles since the motion is geodesic also from the D=3 point of view. The reason is that the D=3 components of the D=4 connection $\Gamma_\lambda{}^\mu{}_\nu$ are identical with those of the D=3 connection, $\gamma_\alpha{}^\beta{}_\varepsilon$. The scalar field does have as we will see a crucial affect on the D=3 metric thus still having an indirect influence on particles' motion.

The next step is writing Einstein's equations in terms of the D=3 quantities. One may find after some straightforward algebra:

$$G_{\alpha\beta} + \nabla_\alpha\nabla_\beta \ln\Phi + \partial_\alpha \ln\Phi \, \partial_\beta \ln\Phi - \gamma_{\alpha\beta}(\nabla^\gamma\nabla_\gamma \ln\Phi + \nabla^\gamma \ln\Phi \, \nabla_\gamma \ln\Phi) = -8\pi G \, T_{\alpha\beta} \tag{3a}$$

$$\frac{1}{2}R = 8\pi G \, T^3{}_3 \tag{3b}$$

In order to clarify the situation we take the trace of (3a) and using also (3b) we find a simple equation for the scalar field:

$$\frac{1}{\Phi}\nabla^\alpha\nabla_\alpha\Phi = 4\pi G(T^\alpha{}_\alpha - T^3{}_3) \tag{4a}$$

While for the gravitational field we get:

$$G_{\alpha\beta} = \frac{1}{\Phi}(\gamma_{\alpha\beta} \nabla^\gamma\nabla_\gamma \Phi - \nabla_\alpha\nabla_\beta \Phi) - 8\pi G \, T_{\alpha\beta} \tag{4b}$$

We arrived therefore in a new theory of D=3 gravity, which is a slight modification of the Brans-Dicke theory (see e.g. [13]). The geometry of the D=3 space-time is determined by the matter distribution as well as by the scalar field. The combination $T^\alpha{}_\alpha - T^3{}_3$ plays the role of the source for the scalar field itself. $T^3{}_3$ thus has to be interpreted from a D=3 point of view as a new kind of scalar density. The differences with respect to the Brans-Dicke theory are this extra $T^3{}_3$ density and the coupling of gravity with matter which in the Brans-Dicke theory amounts to the replacement $G \to 1/\Phi$ in (4).

These differences are reflected by the fact that the D=3 energy-momentum tensor of the matter alone is not covariantly conserved, since energy can be interchanged not only between matter and the metric tensor, but also between matter and the scalar field. The energy-momentum conservation equation thus reads:

$$\nabla_\alpha T^\alpha{}_\beta + \partial_\alpha \ln\Phi \; T^\alpha{}_\beta - \partial_\beta \ln\Phi \; T^3{}_3 = 0 \tag{5}$$

and it can be shown easily that (5) which follows from the D=4 conservation law is equivalent to the covariant conservation equation of the total D=3 energy-momentum tensor which appears as the right-hand-side of (4b).

Finally we note that the field equations (3) or (4) correspond to a very simple but peculiar gravitational action functional with the combination ΦR as a Lagrangian density. Although the action does not contain explicitly a kinetic term for the scalar field, such a term appears in the field equations.

3. SIMPLE SOLUTIONS AND THE NEWTONIAN LIMIT

In order to get more insight on the nature of the D=3 gravity described by (3) or (4) and its Newtonian limit, we consider few simple cases.

The first and the simplest is the local cosmic string. It may be described by $T^3{}_3 = T^0{}_0$ which vanish everywhere except on the x^3-axis. To find the D=4 metric tensor around such a string we may use the coordinates in which the line element takes the form:

$$ds^2 = Adt^2 - dr^2 - Bd\phi^2 - \Phi^2 dz^2 \qquad (6)$$

which is one of the several equivalent forms of the general static cylindrically symmetric metrics [14]. The solution to Einstein equations is a conical space [5,6,7]

$$A = 1 \ , \qquad B = (1-4G\mu)^2 r^2 \ , \qquad \Phi = 1 \qquad (7)$$

with a deficit angle of $8\pi G\mu$, μ being the linear mass density of the string.

We must stress here the difference between this mass and the total **gravitational** (or Tolman) mass (per unit length), M. μ represents the "inertial mass" (i.e. that part which is included in $T^0{}_0$), while M includes the energy of the gravitational field as well [15]. For a local string we expect M=0 due to its vanishing gravitational potential, and this is indeed the case as we will find in eq. (9).

The three-dimensional point of view is provided by equations (4). Equation (4a) has an identically vanishing source term outside as well as inside the string. Thus, the only static solution to (4a) is a constant Φ. Consequently, (4b) reduces to the D=3 Einstein equations with a D=3 energy-momentum tensor, which is totally unaware of the existence of an extra dimension. The only non-vanishing component of $T^{\alpha}{}_{\beta}$ is $T^{0}{}_{0}$ which corresponds to a D=3 point source, and the D=3 metric tensor is well-known [1,2,3] to be identical with (7). It seems therefore that the origin of the equivalence between the two descriptions of the local strings is merely the accidental cancellation between $T^{\alpha}{}_{\alpha}$ and $T^{3}{}_{3}$. Any other matter distribution which has this same property can be also described in terms of pure D=3 GR. Matter distributions with non vanishing $T^{\alpha}{}_{\alpha}-T^{3}{}_{3}$ will produce gravitational fields which inevitably require the additional scalar field for their proper D=3 description.

The simplest systems where a scalar field is required for a consistent D=3 description are different kinds of D=3 point-like sources with non-vanishing $T^{\alpha}{}_{\alpha}-T^{3}{}_{3}$. These correspond in the D=4 description to other types of strings (e.g. global strings [4]) or other linear matter distributions. We may include in the discussion also extended (but finite) sources with cylindrical symmetry.

The metric tensor outside all these sources is known to be of the Kasner form [14] which translates in our D=3 language to:

$$A = (kr)^{2a}\ , \qquad B = (\beta/k)^{2}(kr)^{2b}\ , \qquad \Phi = (kr)^{c} \tag{8a}$$

where k sets the length scale and the following conditions hold:

$$a + b + c = a^2 + b^2 + c^2 = 1 \tag{8b}$$

The dimensionless constants a, b, c and β are determined by the internal properties of the source. For example, the total gravitational mass per unit length, M, is known [15] to be:

$$M = \beta a/2G \tag{9}$$

The existence of this family of solutions demonstrates the richness of the scalar-tensor theory with respect to the very constrained D=3 GR. One aspect of this richness is represented by eq. (8) which evidently describe a D=3 curved space-time which is source-free. The corresponding gravitational potential is attractive for positive mass and in small enough regions around $r=1/k$ it can even be approximated by a logarithmic function:

$$A \sim 1 + 2a \ln(kr) \tag{10}$$

We have therefore an explicit resolution of the puzzle of the missing Newtonian limit in D=3. It reappears due to the presence of the scalar field which is naturally included in this theory of D=3 gravity.

4. THE MAXWELL-EINSTEIN SYSTEM

The D=3 Maxwell-Einstein system has been discussed already by several authors [16,17,18]. It is however clear from our analysis that this system cannot be considered as a proper description of gravity coupled to electromagnetism in D=3. To obtain a consistent D=3 theory one should start as before from the D=4 point of view, assuming a gauge potential A_μ which is x^3-independent and satisfies $A_3=0$. This yields the following energy-momentum tensor:

$$T^{\alpha}{}_{\beta} = F^{\alpha\gamma}F_{\gamma\beta} + \frac{1}{4}\,\gamma^{\alpha}{}_{\beta}\,F^{\gamma\delta}F_{\gamma\delta} \tag{11a}$$

$$T^{3}{}_{3} = \frac{1}{4}\,F^{\gamma\delta}F_{\gamma\delta} = -T^{\alpha}{}_{\alpha} \tag{11b}$$

The gravitational field equations will be therefore equations (4) with an energy-momentum tensor expressed by (11). The electromagnetic field will satisfy the inhomogeneous D=3 Maxwell equations which include a coupling between the electromagnetic field and the scalar:

$$\nabla_{\alpha}\,F^{\alpha\beta} + \partial_{\alpha}\ln\Phi\,F^{\alpha\beta} = 0 \tag{12}$$

Note that the field equations in this case can be interpreted as a pure D=3 theory without any reference to its extra dimensional origin, which is represented in (4) by $T^{3}{}_{3}$. The reason is equation (11b) where $T^{3}{}_{3}$ is given in terms of D=3 quantities only. consequently, pure electromagnetism in a given D=3 curved background is incomplete without specifying the form of the scalar Φ. In other words, the electromagnetic field equations depend on the embedding D=4 spacetime. Only when Φ is constant, does it decouple from the electromagnetic field and the conventional Maxwell equations are obtained.

As we have seen in section 3, the D=3 metric tensor outside an uncharged source is a special case of the D=4 solution (eq. (8)) with $\Phi=1$. This is not the case for a charged source, or any other matter distribution which produces an external electromagnetic field. A possible way to see it is to compare the specific D=3 charged solution (see e.g. [18]) with the D=4 static Maxwell-Einstein solution with cylindrical symmetry and a radial electric field [14]. One may be easily convinced that there is no coordinate transformation that might connect these two solutions. Similarly, the D=3 and D=4 solutions of "constant" magnetic field are also inequivalent.

The reason for this inequivalence is more clearly seen in the field equation (4a). This equation allows a constant Φ solution only if $T^3{}_3=T^\alpha{}_\alpha$, which is clearly not satisfied by an electromagnetic system. The D=3 Maxwell-Einstein system is therefore completely different from the D=3 theory obtained here. Moreover, it is not even contained in it as a special case.

Acknowledgments: I am grateful to A. Davidson, J. Fredsted, A.L. Larsen and especially N.K. Nielsen for useful discussions. It is also a pleasure to thank for warm hospitality the Fysisk Institut, Odense Universitet, where part of this work has been completed.

REFERENCES

[1] S. Deser, R. Jackiw and G. 't Hooft, Ann. Phys. (N.Y.) 152 (1984) 220

[2] J.R. Gott and M. Alpert, Gen. Rel. Grav. 16 (1984) 243

[3] S. Giddings, J. Abbott and K. Kuchar, Gen. Rel. Grav. 16 (1984) 751

[4] A. Vilenkin, Phys. Rep. 121 (1985) 263

[5] A. Vilenkin, Phys. Rev. D 23 (1981) 852

[6] J.R. Gott, Ap. J. 288 (1985) 422

[7] W.A. Hiscock, Phys. Rev. D 31 (1985) 3288

[8] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) 192 (1989) 352

[9] J.R. Gott, Phys. Rev. Lett. 66 (1991) 1126

[10] S.M. Carroll, E. Farhi and A.H. Guth, Phys. Rev. Lett. 68 (1992) 263

[11] S. Deser, R. Jackiw and G. 't Hooft, Phys. Rev. Lett. 68 (1992) 267

[12] M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Rep. 130 (1986) 1

[13] S. Weinberg, Gravitation and Cosmology (John Wiley & Sons, Inc. 1972)

[14] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, Exact Solutions of Einstein's Field Equations (Cambridge Univ. Press, 1980), Ch. 20

[15] V.P. Frolov, W. Israel and W.G. Unruh, Phys. Rev. D 39 (1989) 1084

[16] S. Deser and P.O. Mazur, Class. Quantum Grav. 2 (1985) L51

[17] M.A. Melvin, Class. Quantum Grav. 3 (1986) 337

[18] J.R. Gott, J.Z. Simon and M. Alpert, Gen. Rel. Grav. 18 (1986) 1019