NEW FORMULATION OF SU(2) YANG-MILLS THEORY

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A B S T R A C T

SU(2) Yang-Mills theory is formulated in terms of gauge-covariant potentials and auxiliary (scalar) fields that transform like the elements of the gauge group. Using the gauge-covariant potentials, gauge transformation is given a geometrical interpretation as pure rotation in internal isospace. This naturally leads to the gauge-invariant formulation based on geometrical constructs. An effective theory for the gauge-invariant quantities is also derived.

1.0 Introduction

Gauge theories are successful in describing the fundamental forces of nature[1]. Their success is due to the gauge symmetry which render them renormalizable[2]. Unfortunately, it is also this gauge symmetry, the presence of excess degrees of freedom that pose a problem in quantization. This problem is dealt with in two ways. One is by fixing the gauge, the process of choosing a representative field configuration of each gauge orbit in calculating physical quantities. However, since physical quantities are gauge-invariant and are calculated by choosing a particular gauge, the gauge-invariance of the results must be verified. This may be done by calculating the physical quantities using different gauge and show that the results are the same. Alternatively, the gauge-invariance of the result can also be guaranteed by establishing the Ward-Takahashi identities.

Another way of dealing with the excess degrees of freedom is by dealing directly with the effective theories of gauge-invariant quantities. Unfortunately, a successful derivation of the effective dynamics of physical quantities from the fundamental Lagrangian does not exist as of this time.

In this paper, we will present a different method of dealing with the excess degrees of freedom of SU(2) Yang-Mills theory. First, we will decompose the Yang-Mills potential in terms of a gauge-covariant vector field and auxiliary, scalar fields that transform like the elements of the gauge group. Using the gauge-covariant potential, gauge transformation is given a geometrical interpretation as pure rotations. This leads to a gauge-invariant formulation based on geometrical constructs such as lengths and angles.

2.0 The Isovector Potentials and The Auxiliary Fields

The SU(2) Yang-Mills Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}\, F^{\mu\nu}\right) \tag{1a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu,\, A_\nu\right]. \tag{1b}$$

The Lagrangian is invariant under

$$A_\mu \longrightarrow A'_\mu = \Omega A_\mu \Omega^{-1} - i(\partial_\mu \Omega)\Omega^{-1}, \tag{2}$$

where $\Omega \in SU(2)$.

Let us now introduce the new fields B_μ and K via the following:

$$A_\mu = B_\mu - i(\partial_\mu K)K^{-1} \tag{3}$$

In (3), we are effectively replacing the twelve A^a_μ's by twelve B^a_μ's and the three elements of K given by $e^{i\Lambda^a T^a}$. We therefore expect a bigger symmetry group if the action is expressed in terms of B_μ and K. And this is what exactly happens as we show below.

The field strength tensor expressed in terms of B_μ and K is given by

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - i\left[B_\mu,\ B_\nu\right] - \left[B_\mu, (\partial_\nu K)K^{-1}\right] + \left[B_\nu, (\partial_\mu K)K^{-1}\right]. \tag{4}$$

If we impose that K transforms like an element of the gauge group, i.e.,

$$K' = \Omega K\ , \tag{5a}$$

then we find that

$$B'_\mu = \Omega B_\mu \Omega^{-1}\ , \tag{5b}$$

and (4) transforms covariantly under (5;a,b,). On the other hand, if we say that K is invariant, then $F_{\mu\nu}$ transforms covariantly if we impose that

$$B'_\mu = \Omega B_\mu \Omega^{-1} - i(\partial_\mu \Omega)\Omega^{-1} + i\left[(\partial_\mu K)K^{-1} - i\Omega(\partial_\mu K)K^{-1}\Omega^{-1}\right] \tag{6}$$

Naively, it seems that we can get as many symmetries as we want since we have two fields B_μ and K to fit the transformation of A_μ. However, if we carry out the Dirac constraint formalism,[3] we find the following first-class constraints (see also the algebra given by Equation (17) in the paper)

$$\Pi_0^a = 0 \tag{7a}$$

$$P^a + \left[B_i, \frac{\delta(\partial_0 K)K^{-1}}{\delta(\partial_0 \Lambda^a)}\right]^b \Pi_i^b = 0, \tag{7b}$$

$$\left(D_i^{ab}(B) - \left[(\partial_i K)K^{-1}, T^a\right]^b\right)\Pi_i^b = 0 \tag{7c}$$

It can be easily shown that (7b) generates the infinitesimal transformation given by (5;a,b) while (7c) generates the infinitesimal version of (6). And since there are nine first-class constraints, the independent degrees of freedom is fifteen less nine giving six, the same number had we used the A_μ^a's.

The problem with maintaining the symmetry generated by (7c) is that we do not have a purely covariant B_μ which is given by (5b). The purely covariant nature of B_μ is important for it naturally leads to gauge-invariant quantities which are geometric in nature.

The question now is how to have a purely covariant B_μ. Effectively, we want to break the extra symmetry generated by (7c) which is a consequence of the fact that we have too many degrees of freedom, three more than the A_μ^a's. To get the same number of degrees of freedom, we can impose the condition that the B_i^a's satisfy

$$n_i B_i^a(\vec{x}, t) = 0, \tag{8}$$

where the $n_i (n_i^2 = 1)$ represent arbitrary directions in real space which we will eventually average out. Note that conditions (8) are invariant under (5b) but not under (6). Now, the degrees of freedom exactly tally, there are six independent components in B_i^a's plus three Λ^a's in K for a total of nine. However, there are three first-class constraints given by (7b) yielding a total of six independent degrees of freedom which we can take to be the six gauge-invariant geometrical constructs based on the covariant nature of B_i.

Before we proceed, one word of caution is in order. Equation (8) is not a gauge-fixing condition as the theory is still gauge-invariant under (5;a,b). Equation (8) should be interpreted as part of the equation (3) that defined the new degrees of freedom B_i and K so that the number of fields are the same.

3.0 "Spherical" Decomposition of Fields

The spherical decomposition of the fields is most appropriate because of the gauge transformations given by (5a,b). The physical degrees of freedom are clearly visible in this decomposition.

Since K transforms like an element of the gauge group, we will parametrize both K and Ω in the same manner, in terms of the Euler angles (α, β, γ) and (ψ, θ, φ) respectively, i.e.,

$$K = \begin{bmatrix} e^{i/2(\alpha+\psi)}\cos\beta/2 & ie^{i/2(\alpha-\gamma)}\sin\beta/2 \\ ie^{-i/2(\alpha-\gamma)}\sin\beta/2 & e^{-i/2(\alpha+\gamma)}\cos\beta/2 \end{bmatrix} \qquad (9a)$$

$$\Omega = \begin{pmatrix} e^{i/2(\psi+\phi)}\cos\theta/2 & ie^{i/2(\psi-\phi)}\sin\theta/2 \\ ie^{-i/2(\psi-\phi)}\sin\theta/2 & e^{-i/2(\psi+\phi)}\cos\theta/2 \end{pmatrix} \qquad (9b)$$

Equation (5a) translates into the following:

$$\beta' = 2\sin^{-1}\left\{\cos^2\frac{\theta}{2}\sin^2\frac{\beta}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{\beta}{2} + \frac{1}{2}\sin\theta\sin\beta\cos(\phi+\alpha)\right\}^{1/2} \qquad (10a)$$

$$\alpha' = \sin^{-1}\left\{\frac{1}{\sin\beta'}\left[\sin\theta\sin\psi\cos\beta + \sin\beta\,(\sin\psi\cos\theta\cos(\phi+\alpha) + \cos\psi\sin(\phi+\alpha)\right]\right\}, \qquad (10b)$$

$$\alpha' = \cos^{-1}\left\{\frac{1}{\sin\beta'}\left[\cos\theta\cos\gamma\sin\beta + \sin\theta\left(\cos\gamma\cos\beta\cos(\phi+\alpha) - \sin\gamma\sin(\phi+\alpha)\right)\right]\right\}. \qquad (10c)$$

Since B_μ transforms like an isovector, we will parametrize its spatial components by

$$B_i = \frac{1}{2}R_i\begin{pmatrix} \cos\theta_i & \sin\theta_i\, e^{-i\phi_i} \\ \sin\theta_i\, e^{i\phi_i} & -\cos\theta_i \end{pmatrix}, \qquad (11)$$

a "spherical polar" coordinate system in the internal space. The time components B_0^a will be conveniently left in terms of the Cartesian components because they are not dynamical degrees of freedom (conjugate momentum is zero). The gauge transformation (5b) changes the "spherical" components by

$$R'_i = R_i, \tag{12a}$$

$$\theta'_i = \cos^{-1}\left\{\cos\theta_i \cos\theta - \sin\theta_i \sin\theta \sin(\varphi_i - \phi)\right\} \tag{12b}$$

$$\varphi'_i = \tan^{-1}\left[\frac{m_i}{n_i}\right], \tag{12c}$$

where $m_i = \cos\psi\sin\theta\cos\theta_i - \sin\theta_i\sin\psi\cos(\varphi_i - \phi) + \sin\theta_i\cos\psi\cos\theta \quad \sin(\varphi_i - \phi)$; $n_i = \sin\theta\cos\theta_i\sin\psi + \sin\theta_i\cos\psi\cos(\varphi_i - \phi) + \sin\theta_i\sin\psi \cos\theta \quad \sin(\varphi_i - \phi)$. Note that the "lengths" R_i are invariant which make them suitable candidates for physical degrees of freedom.

We would like to note that the parametrization (11) is not appropriate for the A_i because of the inhomogeneous term in the gauge transformation which will result in a non-invariant length.

The field strength components can be expressed in terms of the fields defined above by using equation (6). For example:

$$\begin{aligned}
F^1_{oi} = {} & (\partial_o R_i)\sin\theta_i \cos\varphi_i + R_i \cos\theta_i \cos\varphi_i(\partial_o\theta_i) \\
& - R_i \sin\theta_i \sin\varphi_i(\partial_o\varphi_i) - \partial_i B^1_o - \tfrac{1}{2}B^3_o R_i \sin\theta_i \sin\varphi_i \\
& + \tfrac{1}{2}B^2_o R_i \cos\theta_i + B^3_o \sin\alpha(\partial_i\beta) + B^2_o(\partial_i\alpha) + B^2_o \cos\beta(\partial_i\gamma) \\
& - B^2_o \sin\beta \cos\alpha(\partial_i\gamma) - R_i \cos\theta_i \sin\alpha(\partial_o\beta) \\
& - R_i \sin\theta_i \sin\varphi_i(\partial_o\alpha) - R_i \sin\theta_i \sin\varphi_i \cos\beta(\partial_o\gamma) \\
& + R_i \cos\theta_i \sin\beta \cos\alpha(\partial_o\gamma);
\end{aligned} \tag{13a}$$

$$\begin{aligned}
F^1_{ij} = {} & \partial_i R_j \sin\theta_j \cos\varphi_j + R_j \cos\theta_j \cos\varphi_j(\partial_i\theta_j) \\
& - R_j \sin\theta_j \sin\varphi_j(\partial_i\varphi_j) - \tfrac{1}{2}R_i R_j \cos\theta_i \sin\theta_j \sin\varphi_j
\end{aligned}$$

$$+ R_i \cos\theta_i \sin\alpha(\partial_i\beta) + R_i \sin\theta_i \sin\varphi_i(\partial_j\alpha)$$

$$+ R_i \sin\theta_i \cos\beta \sin\varphi_i(\partial_j\gamma) - R_i \cos\theta_i \sin\beta \cos\alpha(\partial_j\gamma)$$

$$- (\text{same terms but with } i \leftrightarrow j). \qquad (13b)$$

There are similar terms for the others but we will not write them down as the derivation is rather straightforward.

4.0 The Constraint Structure and the Symmetries

The Yang-Mills action in terms of B_o^a, $(R_i, \theta_i, \varphi_i)$ and (α, β, γ) is still first-order. Following the Dirac constraint formalism, we get the following primary constraints:

$$\Pi_o^a = 0 \qquad (14a)$$

$$\chi_1 = P_\alpha - \sum_{i=1}^{3} \Pi_{\varphi_i} = 0, \qquad (14b)$$

$$\chi_2 = P_\beta + \sum_i \left\{ \sin(\varphi_i+\alpha)\Pi_{\theta_i} + \cot\theta_i \cos(\varphi_i+\alpha)\Pi_{\varphi_i} \right\} = 0, \qquad (14c)$$

$$\chi_3 = P_\gamma - \sum_i \left\{ \Pi_{\theta_i} \sin\beta \cos(\varphi_i+a) + \Pi_{\varphi_i} [\cos\beta - \cot\theta_i \sin\beta \sin(\varphi_i + \alpha)] \right\} = 0. \qquad (14d)$$

The extended Hamiltonian is given by

$$\mathcal{H}_E = \frac{1}{2} \sum_i \left\{ \Pi_{R_i}^2 + \left(\frac{1}{R_i^2}\right) \Pi_{\theta_i}^2 + \left(\frac{1}{R_i^2 \sin^2\theta_i}\right) \Pi_{\varphi_i}^2 \right\}$$

$$+ \frac{1}{4} \sum_{i,j,a} F_{ij}^{a^2}(R_i, \theta_i, \varphi_i; \alpha, \beta, \gamma \text{ and derivatives})$$

$$+ \sum_{a=1} \left[B_o^a \; G^a + U_a \; \Pi_o^a + V_a \chi_a \right], \tag{15}$$

where u^a and v_a are Lagrange multipliers. The G^a's are the Gauss' operators and they are given below:

$$G^1 = \sum_{i=1}^{3} \Big\{ \partial_i \Big[\sin\theta_i \; \cos\varphi_i \; \Pi_{R_i} + \left(\frac{1}{R_i}\right) \cos\theta_i \; \cos\varphi_i \; \Pi_{\theta_i}$$
$$- \left(\frac{1}{R_i}\right) \frac{\sin\varphi_i}{\sin\theta_i} \Pi_{\varphi_i} \Big] - \Pi_{R_i} (11)_i - \frac{1}{2}\Pi_{\theta_i} \; \sin\varphi_i$$
$$- \left(\frac{1}{R_i}\right) \Pi_{\theta_i} (21)_i - \frac{1}{2}\Pi_{\varphi_i} \; \cot\theta_i \; \cos\varphi_i - \left(\frac{1}{R_i}\right) \Pi_{\varphi_i} (31)_i \Big\}, \tag{16a}$$

$$G^2 = \sum_i \Big\{ \partial_i \Big[\sin\theta_i \; \cos\varphi_i \; \Pi_{R_i} + \left(\frac{1}{R_i}\right) \sin\varphi_i \; \cos\theta_i \; \Pi_{\theta_i}$$
$$+ \left(\frac{1}{R_i}\right) \frac{\cos\varphi_i}{\sin\theta_i} \Pi_{\varphi_i} \Big] - \Pi_{R_i} (12)_i + \frac{1}{2}\Pi_{\theta_i} \; \cos\varphi_i$$
$$- \left(\frac{1}{R_i}\right) \Pi_{\theta_i} (22)_i - \frac{1}{2} \cot\theta_i \; \sin\varphi_i \; \Pi_{\varphi_i} - \left(\frac{1}{R_i}\right) \Pi_{\varphi_i} (32)_i \Big\}, \tag{16b}$$

$$G^3 = \sum_i \Big\{ \partial_i \Big[\cos\theta_i \; \Pi_{R_i} - \left(\frac{1}{R_i}\right) \sin\theta_i \; \Pi_{\theta_i} \Big]$$
$$- \Pi_{R_i} (13)_i - \left(\frac{1}{R_i}\right) \Pi_{\varphi_i} (23)_i + \frac{1}{2} \Pi_{\varphi_i} - \left(\frac{1}{R_i}\right) \Pi_{\varphi_i} (33)_i \Big\}. \tag{16c}$$

The $(jk)_i$ terms with $j,k = 1,2,3$ are the jth row and kth column of the matrix given below:

$$\begin{pmatrix} \sin\theta_i \ \sin\varphi_i a_i + \cos\theta_i b_i & -\sin\theta_i \ \cos\varphi_i a_i + \cos\theta_i c_i & -\sin\theta_i \ \cos\varphi_i a_i - \sin\theta_i \sin\varphi_i c_i \\ \cos\theta_i \ \sin\varphi_i a_i - \sin\theta_i b_i & -\cos\theta_i \ \cos\varphi_i a_i - \sin\theta_i c_i & -\cos\theta_i \ \cos\varphi_i b_i - \cos\theta_i \sin\varphi_i c_i \\ \dfrac{\cos\varphi_i}{\sin\theta_i} a_i & \dfrac{\sin\varphi_i}{\sin\theta_i} a_i & \dfrac{\sin\varphi_i}{\sin\theta_i} b_i^{\bullet} - \dfrac{\cos\varphi_i}{\sin\theta_i} c_i \end{pmatrix}$$

where $a_i = \partial_i\alpha + \cos\beta(\partial_i\gamma)$; $b_i = \sin\alpha(\partial_i\beta) - \sin\beta\ \cos\alpha(\partial_i\gamma)$ and $c_i = \cos\alpha(\partial_i\beta) + \sin\beta\ \sin\alpha(\partial_i\gamma)$.

The letter G is used to represent the constraints given by (18) because they appear like Gauss' operators. However, they are not the generators of the gauge transformations given by (5;a,b) but the $x_{a's}$. However, the $G^{a's}$ generate the symmetry given by equation (6).

Carrying out the Dirac consistency iteration for the constraint $\Pi_o^a = 0$ yields $G^a = 0$. The constraints given by (14a, b, c, d) and by $G^a = 0$ satisfy the following algebra:

$$[\chi_1(\vec{x},t),\ G^1(\vec{x},t)] = -G^2\delta^3(\vec{x}-\vec{x}'), \tag{17a}$$

$$[\chi_1(\vec{x},t),\ G^2(\vec{x}',t)] = G^1\delta^3(\vec{x}-\vec{x}'), \tag{17b}$$

$$[\chi_1(\vec{x},t),\ G^3(\vec{x},t)] = 0, \tag{17c}$$

$$[\chi_2(\vec{x},t),\ G^1(\vec{x}',t)] = -\sin\alpha\ G^3\delta^3(\vec{x}-\vec{x}'), \tag{17d}$$

$$[\chi_2(\vec{x},t),\ G^2(\vec{x}',t)] = -\cos\alpha\ G^3\delta^3(\vec{x}-\vec{x}') \tag{17e}$$

$$[\chi_2(\vec{x},t),\ G^3(\vec{x},t)] = (\sin\alpha\ G^1 + \cos\alpha\ G^2)\delta^3(\vec{x}-\vec{x}'), \tag{17f}$$

$$[\chi_3(\vec{x},t),\ G^1(\vec{x}',t)] = (\sin\beta\ \cos\alpha\ G^3 - \cos\beta\ G^2)\delta^3(\vec{x}-\vec{x}'), \tag{17g}$$

$$[\chi_3(\vec{x},t),\ G^2(\vec{x}',t)] = (-\sin\beta\ \sin\alpha\ G^3 + \cos\beta\ G^1)\delta^3(\vec{x}-\vec{x}') \tag{17h}$$

$$[\chi_3(\vec{x},t),\ G^3(\vec{x}',t)] = (-\sin\beta\ \cos\alpha\ G^1+\sin\beta\ \sin\alpha\ G^2)\delta^3(\vec{x}-\vec{x}') \tag{17i}$$

$$[G^a(\vec{x}',t),\ G^b(\vec{x}',t)] = f^{abc}\ G^c\ \delta^3(\vec{x}-\vec{x}') \tag{17j}$$

with all the others giving zero. Furthermore, all the constraints have vanishing brackets with the Hamiltonian density

$$\mathscr{H} = \frac{1}{2}\sum_i \left\{\Pi^2_{R_i} + \left(\frac{1}{R_i^2}\right)\Pi^2_{\theta_i} + \left(\frac{1}{R_i^2\ \sin\theta_i^2}\right)\Pi^2_{\varphi_i}\right\} + \frac{1}{4}\sum_{a,i,j} F^{a^2}_{ij}. \tag{18}$$

The derivation of the above results are rather lengthy but straightforward. Operator ordering was not taken into account as we are dealing with classical Poisson brackets. However, it is easy to see that the algebra is also valid as a quantum relation if the symmetrization rule $q^n p^m \longrightarrow \frac{1}{2}[\hat{q}^n\hat{p}^m + \hat{p}^m\hat{q}^n]$ is used.

The algebra shows that the constraints $x_a = 0$ and $G^a=0$ would have been all first-class if the $B_{i's}$ are not restricted (and this is due to the excess degrees of freedom). However, we imposed a condition on the B_i which is consistent with its isovector character and is given by equation (8). In spherical coordinates, these conditions read:

$$\rho_1 = \sum_i n_i R_i\ \cos\theta_i = 0, \tag{19a}$$

$$\rho_2 = \sum n_i R_i\ \sin\theta_i\ \sin\varphi_i = 0, \tag{19b}$$

$$\rho_3 = \sum n_i R_i\ \sin\theta_i\ \cos\varphi_i = 0. \tag{19c}$$

It is easy to verify that $[\chi_a, \rho_b] \approx \rho$ and that $[G^a, \rho_b] \neq 0$, thus the only surviving first class constraints are the $\chi_{a's}$.

We now show that the $\chi_{a's}$ generate the gauge transformations (5;a,b). The infinitesimal limit of (10;a,b,c) and (12;a,b,c) are:

$$\delta\alpha = \delta\psi + \delta\phi - \sin\alpha \cot\beta \, \delta\theta, \tag{20a}$$

$$\delta\beta = \cos\alpha \, \delta\theta, \tag{20b}$$

$$\delta\gamma = \frac{1}{\sin\beta} \sin\alpha \, \delta\theta, \tag{20c}$$

$$\delta R_i = 0, \tag{20d}$$

$$\delta\theta_i = \sin\varphi_i \, \delta\theta, \tag{20e}$$

$$\delta\varphi_i = -(\delta\phi + \delta\psi) + \cot\theta_i \cos\varphi_i \, \delta\theta. \tag{20f}$$

The conserved change is given by

$$Q = \int d^3x \left\{ \chi_1(\delta\psi + \delta\phi) + [\cos\alpha \, \chi_2 + \frac{\sin\alpha}{\sin\beta} \chi_3 - \sin\alpha \cot\beta \, \chi_1]\delta\theta \right\}, \tag{21}$$

expressed only in terms of the $\chi_{a's}$, thus proving that they generate the gauge transformation.

5.0 Effective Dynamics of the Gauge-Invariants

We have already identified three gauge-invariant quantities, the "lengths" $R_i(\vec{x},t)$ of the isovector. The other three are the "angles" between the isovectors $B_i(\vec{x},t)$ defined by

$$\cos\theta_{ij}(\vec{x},t) = \frac{2\,\mathrm{tr}(B_i B_j)}{R_i R_j} = \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \cos(\varphi_i - \varphi_j) \tag{22}$$

Note that these invariant quantities are geometric objects in the internal isospace at each space-time point. In this section, we will derive the effective action that governs the dynamics of these quantities.

For simplicity, consider the $A_o^a = 0$ gauge of Yang-Mills which is still invariant under time-independent gauge transformations. The effective dynamics for the gauge-invariants will be derived from:

$$\int (dA_i^a)\, e^{iS_{YM}} = N \int (dR_i)(d\theta_{ij})\, e^{iS_{eff}.(R_i,\ \theta_{ij})} . \tag{23}$$

To arrive at the right hand side of (23), we need to do the following steps:

(a) Express the measure (dA_i^a) in terms of B_i^a and K

$$(dA_i^a) = J_1 (dB_i^a)(d\Lambda^a)\ \delta(\hat{n}\circ\vec{B}^a) \tag{24}$$

The Jacobian of this transformation is

$$J_1 = \mathrm{Det}\begin{bmatrix} \delta^{ab} & 0 & \delta^{ab}\partial_1 + \frac{1}{2}f^{abc}\ \partial_1\Lambda^c + O(\Lambda^2) + \ldots \\ 0 & \delta^{ab} & \delta^{ab}\partial_2 + \frac{1}{2}f^{abc}\ \partial_2\Lambda^c + O(\Lambda^2) + \ldots \\ -\frac{n_1}{n_3}\delta^{ab} & -\frac{n_2}{n_3}\delta^{ab} & \delta^{ab}\partial_3 + \frac{1}{2}f^{abc}\ \partial_3\Lambda^c + O(\Lambda^2) + \ldots \end{bmatrix}\delta^3(\vec{x}-\vec{x}'),$$

and Det refers to the functional determinant.

(b) We will gauge-fix by imposing $K = 1\!\!1$ (or $\Lambda^a = 0$) to simplify the field strengths and J_1. Note that this gauge condition is always realizable and unique. The Fadeev-Popov determinant for this gauge-fixing is

$$\Delta_F = \frac{1}{\int (d\Omega)},$$

the reciprocal of the divergent gauge volume term.

(c) We will exponentiate the restricting conditions on the B_i^a, i.e.,

$$\begin{aligned}\delta(\hat{n}\circ\vec{B}^a) &= \exp\left\{-\frac{i}{2\xi}\int d^4x\ \mathrm{tr}\ (\vec{n}\circ\vec{B})^2\right\} \\ &= \exp.\left\{-\frac{i}{2\xi}\int d^4x\left\{\sum_i R_i^2 + \sum_{i\neq j} n_i n_j R_i R_j \cos\theta_{ij}\right]\right\}.\end{aligned}$$

The value $\xi = 0$ is the "unitary" limit.

(d) The measure (dB_i^a) will be written in terms of the spherical coordinates $(R_i, \theta_i, \varphi_i)$

$$(dB_i^a) = (\prod_i \mathrm{Det}(R_i^2 \sin\theta_i))(dR_i)(d\theta_i)(d\varphi_i)$$

(e) Next, we express the action (with $K = \mathbb{1}$) in terms of the gauge-invariants. To illustrate how this will be done, consider the kinetic term:

$$\mathrm{tr}(\partial_o B_i)^2 = \frac{1}{2}\sum_i \left[(\partial_o R_i)^2 + R_i^2(\partial_o \theta_i)^2 + R_i^2(\sin\theta_i)^2(\partial_o\varphi_i)^2\right].$$

To get the gauge-invariant components, take B_3 along the z-axis. This means $\theta_1 = \theta_{13}$, $\theta_2 = \theta_{23}$ and from $\cos\theta_{12} = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$, then $\varphi_1 - \varphi_2 = \varphi$ is gauge-invariant. The kinetic term then becomes

$$\frac{1}{2}\Big[(\partial_o R_1)^2 + (\partial_o R_2)^2 + (\partial_o R_3)^2 + R_1^2(\partial_o\theta_{13})^2 + R_2^2(\partial_o\theta_{13})^2$$
$$+ R_1^2 \sin^2\theta_{13}(\partial_o\varphi_1)^2 + R_2^2\sin^2\theta_{23}(\partial_o\varphi_2)^2\Big]$$

We then decompose $\varphi_1 = \frac{1}{2}\varphi + \varphi_T$; $\varphi_2 = \varphi_T - \frac{1}{2}\varphi$ where φ_T is non-invariant. The non-invariant angles are then φ_T and the orientation of the B_3 vector in the internal isospace given by θ_3 and φ_3. Using the relation for $\cos\theta_{12}$, we will get an expression for $(\partial_o\varphi)$ and use this to extract the gauge-invariant (the $(\partial_o\varphi_T)$ independent terms) components of the kinetic term.

But the result we will get from above will not be symmetric with respect to the i = 1,2,3 indices. To make the result symmetric, we do the same calculations with B_1 along z and also with B_2 along z and average the three results to give:

$$(\text{Kinetic term})_{\text{invariant}} = \frac{1}{2}\sum_{i} (\partial_o R_i)^2 + \frac{1}{6} \sum_{i\neq j} R_i^2 (\partial_o \theta_{ij})^2$$

$$+ \frac{1}{24} \sum_{(i,j,k)}{}' R_i^2 \left[\sin^2\theta_{ik} (\partial_o \varphi)^2_{(i,jk)} + \sin^2\theta_{ij} (\partial_o \varphi)^2_{(k,i,j)} \right],$$

where

$$(\partial_\mu \varphi)_{(i,j,k)} = \Big\{ \sin\theta_{ij} \sin\theta_{ik} \sin\theta_{jk} (\partial_\mu \theta_{ij})$$

$$+ \sin\theta_{jk} (\cos\theta_{ij} \cos\theta_{ik} - \cos\theta_{jk}) \partial_\mu \theta_{ik} + \sin\theta_{ik} (\cos\theta_{ij} \cos\theta_{jk}$$

$$- \cos\theta_{ik})(\partial_\mu \theta_{jk}) \Big\} \Big/ \Big\{ \left[\sin\theta_{ij} \sin\theta_{jk} \right] \times \Big[-2 + \sin^2\theta_{ij} + \sin^2\theta_{ik}$$

$$+ \sin^2\theta_{jk} + 2 \cos\theta_{ij} \cos\theta_{ik} \cos\theta_{jk} \Big]^{1/2} \Big\}.$$

The prime in the third summation denotes that (i,j,k) is a cyclic permutation of (1,2,3).

The potential part, $\frac{1}{4} \sum_{a,i,j} (F^a_{ij})^2$, involves a lot more work. Fortunately, there are some terms that are obviously invariant and they lessen the calculation a little bit. The other terms would have to be worked out in the same manner as the kinetic term and after a lengthy calculation yields:

$$(\text{Potential})_{\text{inv.}} = \frac{1}{2} \sum_{i\neq j}\left[(\partial_i R_j)^2+\frac{1}{4}R_i^2\ R_j^2\ \sin^2\theta_{ij}\right]+\frac{1}{6} \sum_{i\neq j\,;i\neq k} R_i^2(\partial_j\theta_{ik})^2$$

$$+ \frac{1}{24} \sum_{(i,j,k)}{}' R_i^2\left[\sin^2\theta_{ik}(\partial_\ell\varphi)^2_{(i,j,k)} + \sin^2\theta_{ij}(\partial_\ell\varphi)^2_{(k,i,j)}\right]$$

$$+ \sum_{(i,j,k)}{}' R_i R_j\ D_{(i,j,k)} + \sum_{(i,j,k)}{}' \left[R_i\partial_i R_j E^{(1)}_{(i,j,k)} + R_j\partial_j R_i E^{(2)}_{(i,j,k)}\right]$$

$$+ \sum_{(i,j,k)}{}' \left[R_i^2\ R_j\ G^{(1)}_{(i,j,k)} + R_j^2\ R_i\ G^{(2)}_{(i,j,k)}\right].$$

The coefficients D, E and G are functions of the $\theta_{ij's}$. We will not write down their explicit forms because of their length and their derivation is rather straightforward anyway.

(f) Penultimately, we express the measure (dB_i^a) in terms of the gauge-invariants. Following the same procedure for the kinetic and potential terms, we find

$$(dB_i^a) = \prod_i (\text{Det } R_i^2) \prod_{(i,j,k)}{}' \text{Det}^{1/3}\left(\frac{\sin\theta_{ij}}{(\sin\varphi)_{(i,j,k)}}\right)$$
$$\times\ (dR_i)(d\theta_{ij})\left[d(\text{non-inv.})\right],$$

where

$$(\cos\varphi)_{(i,j,k)} = \frac{\cos\theta_{ij} - \cos\theta_{ik}\ \cos\theta_{ik}}{\sin\theta_{ij}\ \sin\theta_{ik}}\ .$$

The non-invariant measure corresponds to the angles of the vector we put on the z-axis and the azimuthal angle φ_T defined from the other two. They correspond to the freedom of orienting the z-axis and rotating the other two vectors by the same azimuth about the z-axis.

(g) Finally, we average over the directions n_i to restore the real space symmetry. This procedure, will primarily affect the $n_i n_j R_i R_j \cos\theta_{ij}$ term from the $\delta(\vec{n}\circ\vec{B})$.

Taking everything into account, the effective Lagrangian for the gauge-invariant, is

$$S_{eff} = \int d^4x \Big\{ \text{Kinetic} + \text{Potential} - \frac{1}{2\xi} \sum_i R_i^2 + 2 \sum_i \ln R_i + \frac{1}{3} \sum_{(i,j,k)}{}' \ln\left[\frac{\sin\theta_{ij}}{(\sin\varphi)_{(i,j,k)}}\right] \Big\} + \ln\left[1 + \frac{1}{2}\left(\frac{-i}{2\xi}\right)^2 \sum_{i\neq j} \int d^4x d^4x' \left(R_i R_j \cos\theta_{ij}\right)_X \left(R_i R_j \cos\theta_{ij}\right)_{X'} + \ldots\right]$$

This is the final result of this paper.

6.0 Summary

We have reformulated Yang-Mills theory in terms of gauge-covariant potentials and auxiliary scalar fields. From the gauge-covariant potentials, we identified the gauge-invariant variables from geometrical constructs. We also derived the effective action for these invariants. Physical consequences from this effective action is presently under investigation.

References:

(1) The Standard Model is now a standard material in textbooks. For example, see K. Huang, "Quarks, Leptons and Gauge Fields", World Scientific, Singapore 1982.

(2) G.'t Hooft, Nuclear Physics B33 (1971) 173. Also see any of the current textbooks in Field Theory.

(3) P.A.M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University, New York 1964.