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Stable Quasi-Particle Picture in Thermal Quantum Field Physics

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It is well known that physical particles are thermally dissipative at finite temperature. In this paper we formulate both the equilibrium and non-equilibrium thermal field theories in terms of stable quasi-particles. All operators, including the dissipative physical particle operators, are realized in a Fock space defined by the stable quasi-particles. The propagators of the physical particles are expressed in terms of the operators of such stable quasi-particles which is a simple diagonal matrix with the diagonal elements being the temporal step functions, same as the propagators in the usual quantum field theory without thermal degree of freedom. The proper self-energies are also expressed in terms of these stable quasi-particle propagators. This formalism inherits the definition of on-shell self-energy in the usual quantum field theory. With this definition, a self-consistent renormalization is formulated which leads to quantum Boltzmann equation and the entropy law. With the aid of a doublet vector algebra we have an extremely simple recipe for computing Feynman diagrams. We apply this recipe to several examples of equilibrium and non-equilibrium two-point functions, and to the kinetic equation for the particle numbers. The response function is also discussed.

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I. INTRODUCTION

In the past decade there has been a remarkable progress in the physics of thermal quantum field theories [1,2]. Although there have been many confusions, they are mostly cleared in recent several years and we now have a reasonably good understanding of the subject. Therefore, it may now be an appropriate time to reconstruct these theories in a compact and calculable form. It is thus the goal of this paper to reformulate the thermo field dynamics (TFD) in such a form. TFD will be shown to be very simple for computations of Feynman diagrams with the introduction of a doublet vector algebra.

Most thermal quantum field theories have indicated the need for doubling the degrees of freedom. The super-operator formalism [3,4] suggested a twin formalism [5] in which left- and right-sided actions on the density operator were introduced. These two actions form a commutative double sets of operations [7–9]. The closed path formalism (CPF) can be formulated by a kind of doublet formalism when the field on the line above the real axis is defined as the first component and the one below the second [6]. However, these two components do not commute with each other. In equilibrium framework the most solid basis for doubling the degrees of freedom in the thermal quantum field theory was given by the C^*-algebra. An explicit canonical operator formalism for the commutative doublet formalism was realized by the thermo field dynamics (see, for example, [10]). In TFD the thermal physics is formulated in terms of pure states in which thermal effects arise through a kind of vacuum correlation similar to what occurs in two mode squeezed states. Since the need for doubling the degrees of freedom is widely recognized, we shall take it as one of the postulates of finite temperature quantum field theory.

In the development of thermal quantum field theories a serious confusion was caused by the thermal instability of observed particles; all the observed particles are thermally dissipative. This trouble was first pointed out in [11]. To understand this problem, it is helpful to recall that in the usual quantum field theory without thermal degrees of freedom all quasi-particles are stable and observable, and are described by the asymptotic fields. The thermal instability forces us to sacrifice one of the two features: observability or stability. Landsman proposed to preserve observability, sacrificing stability. This leads us to the generalized field theory [12,13]. However, it proves difficult to develop a practical and useful computational method in the framework of this formulation. In [14] it was suggested that a simpler formalism can be obtained by preserving stability, thus sacrificing observability. We shall reformulate the thermal field dynamics following this approach. In this formalism all the operators are realized in a Fock space formed by a set of stable free particles described by the annihilation operators $(\xi_k, \tilde{\xi}_k)$ and the creation operators $(\xi_k^\dagger, \tilde{\xi}_k^\dagger)$. These quasi-particles will be simply called ξ-particles and the operators called ξ-operators. All other operators are defined in terms of these stable free particle operators $\hat{A} = A(\xi, \tilde{\xi}, \xi^\dagger, \tilde{\xi}^\dagger)$ (the dynamical map). We define the ξ-particles as the quasi-particles (which are different from the observable particles). In other words the operators in terms of which dynamical maps of all the operators are expressed are called the quasi-particle operators. Then, in the dynamical map of the observable particle operators, the linear coefficients of the ξ-operators damp in time. Thus the thermal instability of observed particles is caused, not by the instability of quasi-particles, but by the temporal diminishing of the weight probability of stable quasi-particles in the observed particles. The purpose of this paper is to put this formalism of TFD in a

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compact form.

In the stable quasi-particle picture, what propagate as particle waves are the quasi-particle waves. Therefore, it is tempting to write all the propagators explicitly in terms of the stable quasi-particle propagators. This will be studied in sections 3 and 4. An immediate result of this study is the definition of the on-shell self-energy. The thermal instability of the observed particles has made it rather difficult to formulate a reasonable definition of the on-shell self-energy. The instability gives us the feeling that the on-shell energy should be given by complex poles. However, this cannot be true because the complex poles lie not on the physical Riemann sheet, but on the second sheet. Since the propagators of the stable quasi-particles have poles on the real axis on the first sheet, the on-shell self-energy can be defined by these poles in the same way as the on-shell definition in the usual quantum field theory without thermal degrees of freedom.

In the stable quasi-particle formalism we present a new set of computational rules which will be explained in section 5. These rules are formulated in terms of a doublet vector algebra. They significantly simplify calculations of the Feynman diagrams. Some examples will be presented.

This paper deals with both equilibrium and nonequilibrium cases. In the latter case the Boltzman equation plays a significant role. We derive this equation in two ways. One is to use a self-consistent renormalization based on the on-shell definition mentioned above. Another is to make use of the Heisenberg equation. Agreement of these two method suggests a self-consistent nature of the on-shell definition.

To make this paper self-contained, in the next section we present a brief summary of some basic relations in TFD.

II. SOME BASICS OF THE FORMALISM

To double the degrees of freedom in TFD, to every operator A is associated its partner $\tilde{A}$ such that the non-tilde operators commute with the tilde operators. When a nontilde operator A is given, its tilde partner is obtained from it by the tilde conjugation rules which are summarized as (see, for example, [10])

$$(AB)^{\sim} = \tilde{A}\tilde{B}, \tag{1}$$

$$(c_1 A + c_2 B)^{\sim} = c_1^* \tilde{A} + c_2^* \tilde{B}, \tag{2}$$

$$(A^\dagger)^{\sim} = \tilde{A}^\dagger, \tag{3}$$

$$(\tilde{A})^{\sim} = \sigma A, \tag{4}$$

$$|0\rangle^{\sim} = |0\rangle, \tag{5}$$

$$\langle 0|^{\sim} = \langle 0|. \tag{6}$$

Here c_1 and c_2 are arbitrary c-numbers, A and B stand for arbitrary operators. It may be worthwhile to point out here that the rule of replacing a c-number by its complex conjugate together with the invariance of the vacua under tilde conjugation eliminates the trivial phase freedom in the structure of the bra- and ket-vacuum.

In order for the thermal vacuum to be invariant under the tilde-conjugation at any time (as shown by Eqs. (5) and (6)), the Hamiltonian $\hat{H}$ should change sign under the tilde conjugation; i.e. $\hat{H}$ must be anti-tildian. It is further required that the dynamical equations are not modified by the thermal degrees of freedom, implying that *$\hat{H}$ does not contain any terms which mix tilde and non-tilde operators of the basic fields.* This requirement indicates a sharp distinction between the dynamical effects and the thermal effects. This distinction plays an important role in the present formulation. Let H denote the usual dynamical energy operator, then the above requirement uniquely specifies the Hamiltonian:

$$\hat{H} = H - \tilde{H}. \tag{7}$$

As was pointed out in the introduction, we formulate TFD in a Fock space associated with a set of free quasi-particles called ξ-particles. Their annihilation operators are $(\xi_k, \tilde{\xi}_k)$, and the creation operators are $(\xi_k^\dagger, \tilde{\xi}_k^\dagger)$. Thus, denoting the vacuum of the Fock space by $|0\rangle$ and $\langle 0|$ (these are the thermal vacua), we have

$$\xi_k|0\rangle = \tilde{\xi}_k|0\rangle = 0, \tag{8}$$

$$\langle 0|\xi_k^\dagger = \langle 0|\tilde{\xi}_k^\dagger = 0. \tag{9}$$

We make use of the thermal doublet notations:

$$\xi_k^1 = \xi_k,\ \xi_k^2 = \tilde{\xi}_k^\dagger, \tag{10}$$

$$\tilde{\xi}_k^1 = \xi_k^\dagger,\ \tilde{\xi}_k^2 = -\sigma\tilde{\xi}_k, \tag{11}$$

where σ is $+1(-1)$ for boson (fermion).

The general definition of the thermal doublets are

$$A^1 = A,\ A^2 = \tilde{A}^\dagger, \tag{12}$$

$$\tilde{A}^1 = A^\dagger,\ \tilde{A}^2 = -\sigma\tilde{A}, \tag{13}$$

where A stands for any operator consisting of non-tilde operators only.

For sake of simplicity, we consider only the fields of type 1, those without anti-particles. Extension of the formalism to those of type 2 is straightforward, as will be briefly explained toward the end of this paper. Thus our basic Heisenberg field has the Fourier transform

$$\psi_H(x)^\mu = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{i\vec{x}\cdot\vec{k}} a_{Hk}(t)^\mu, \tag{14}$$

$$\bar{\psi}_H(x)^\mu = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{-i\vec{x}\cdot\vec{k}} \bar{a}_{Hk}(t)^\mu. \tag{15}$$

Here the suffix H refers to the Heisenberg operators. We have the equal-time commutation relation:

$$[\psi_H(x)^\mu, \bar{\psi}_H(y)^\nu]_\sigma \delta(t_x - t_y) = \delta_{\mu\nu}\delta(x-y), \tag{16}$$

or

$$[a^{\mu}_{Hk}(t), \bar{a}^{\nu}_{Hl}(t)]_{\sigma} = \delta_{\mu\nu}\delta(\vec{k}-\vec{l}). \tag{17}$$

Here too, use was made of the thermal doublet notations:

$$\psi^1_H = \psi_H,\ \psi^2_H = \tilde{\psi_H}^{\dagger}, \tag{18}$$
$$\bar{\psi}^1_H = \psi_H{}^{\dagger},\ \bar{\psi}^2_H = -\sigma\tilde{\psi}_H, \tag{19}$$

and

$$a^1_{Hk} = a_{Hk},\ a^2_{Hk} = \tilde{a}^{\dagger}_{Hk}, \tag{20}$$
$$\bar{a}^1_{Hk} = a^{\dagger}_{Hk},\ \bar{a}^2_{Hk} = -\sigma\tilde{a}_{Hk}. \tag{21}$$

Since we formulate our calculation in the interaction representation, we use the free field operators for the observed particles:

$$\phi(x)^{\mu} = \frac{1}{(2\pi)^{3/2}}\int d^3k\ e^{i[\vec{x}\cdot\vec{k}-\omega_k t]}a^{\mu}_k, \tag{22}$$
$$\bar{\phi}(x)^{\mu} = \frac{1}{(2\pi)^{3/2}}\int d^3k\ e^{-i[\vec{x}\cdot\vec{k}-\omega_k t]}\bar{a}^{\mu}_k. \tag{23}$$

with the commutation relation:

$$[a^{\mu}_k, \bar{a}^{\nu}_l]_{\sigma} = \delta_{\mu\nu}\delta(\vec{k}-\vec{l}). \tag{24}$$

The free field Hamiltonian is

$$\hat{H}_0 = H_0 - \tilde{H}_0 \tag{25}$$

with

$$H_0 = \int d^3k\ \omega_k\ a^{\dagger}_k a_k. \tag{26}$$

When a thermal situation changes in time, the observed energy ω_k may depend on time. However, calculations are much simpler if we use a time-independent ω_k such as the energy at the final state.

The fact that a system can be in a variety of thermal situations can be expressed by the statement that the operators of the observed particles $(a_k, \tilde{a}_k)$ are not the same as those of the ξ-particles, but are related to the ξ-operators through linear relations of the form:

$$\xi^{\mu}_k = B_k(\theta)^{\mu\nu}a^{\nu}_k, \tag{27}$$
$$\bar{\xi}^{\mu}_k = \bar{a}^{\nu}_k B^{-1}_k(\theta)^{\nu\mu}. \tag{28}$$

The general structure of the matrix B_k will be discussed soon later. Here we point out only that θ stands for a set of continuous parameters. The dependence of $B_k(\theta)$ on the parameters θ signifies the fact that the realization of the physical operators $(a^{\mu}_k, \bar{a}^{\mu}_k)$ depends on the thermal situations which are classified by these parameters.

In order to consider the possibility of time-dependent thermal situation we need to further elaborate on the notations. When a thermal situation changes in time, the parameters in the B_k-matrix also depend on time. Thus we should write the matrix as $B_k(\theta_t)$ where θ_t denotes a time-dependent θ. Then the unperturbed physical particle operators $(a^{\mu}_k, \bar{a}^{\mu}_k)$ depend on time through the B_k-matrix. They will be denoted by $(a^{\mu}_{tk}, \bar{a}^{\mu}_{tk})$. Thus, Eq.(28) reads

$$\xi^{\mu}_k = B_k(\theta_t)^{\mu\nu}a^{\nu}_{tk}, \tag{29}$$
$$\bar{\xi}^{\mu}_k = \bar{a}^{\nu}_{tk}B^{-1}_k(\theta_t)^{\nu\mu}, \tag{30}$$

It is convenient to introduce the notations

$$a_k(t)^{\mu} \equiv e^{-i\omega_k t}\ a^{\mu}_{tk}, \quad \bar{a}_k(t)^{\mu} \equiv e^{i\omega_k t}\ \bar{a}_{tk}. \tag{31}$$

Then the free fields in (22) and (23) become

$$\phi(x)^{\mu} = \frac{1}{(2\pi)^{3/2}}\int d^3k\ e^{i\vec{x}\cdot\vec{k}}a^{\mu}_k(t), \tag{32}$$
$$\bar{\phi}(x)^{\mu} = \frac{1}{(2\pi)^{3/2}}\int d^3k\ e^{-i\vec{x}\cdot\vec{k}}\bar{a}^{\mu}_k(t). \tag{33}$$

These are the unperturbed physical fields in the perturbational calculation formulated in the interaction representation.

The field equations for the quasi-particle operators ($\xi(t)$-operators) are the usual free field equations:

$$[\frac{d}{dt} + i\omega_k(t)]\xi_k(t)^{\mu} = 0, \tag{34}$$
$$\bar{\xi}_k(t)^{\mu}[\frac{\overleftarrow{d}}{dt} - i\omega_k(t)] = 0. \tag{35}$$

Here we used the notations

$$\xi_k(t)^{\mu} \equiv e^{-i\omega_k t}\xi^{\mu}_k, \quad \bar{\xi}^{\mu}_k \equiv e^{i\omega_k t}\bar{\xi}^{\mu}_k. \tag{36}$$

The Hamiltonian for these equations is

$$\hat{H}^{\xi}_0 = \int d^3k\ \omega_k\ [\xi^{\dagger}_k\xi_k - \tilde{\xi}^{\dagger}_k\tilde{\xi}_k]. \tag{37}$$

Since this should be the same as the free Hamiltonian in (25), the condition for the B_k-matrix is that it keep the free Hamiltonian (25) invariant. In other words, the thermal degree of freedom is the freedom in choosing the operations under which the form of the free Hamiltonian is invariant. There are three generators which commute with $\hat{H}_0$ (25), providing us with a linear mapping with three parameters. The general form of the B_k-matrix depends on the three parameters (α, s_k, n_k) through [15,10]:

$$B_k(\theta)^{\mu\nu} = (1+\sigma n_k)^{1/2}e^{s_k\tau_3}\begin{bmatrix} 1 & -f_k^{\alpha_k} \\ -\sigma f_k^{1-\alpha_k} & 1 \end{bmatrix}, \tag{38}$$

where

$$f_k = \frac{n_k}{1+\sigma n_k}, \tag{39}$$

which gives $n_k = f_k/[1-\sigma f_k]$. We choose α_k to be independent of $\vec{k}$ and denote it by α. The symbol θ stands

for these three parameters (n_k, α, s_k). Use of Eqs. (9) and (8) gives

$$n_k \delta(\vec{k} - \vec{l}) = \langle 0 | a_k^\dagger a_l | 0 \rangle. \tag{40}$$

n_k is called the number parameter.

The inverse of the above matrix is

$$B_k^{-1}(\theta)^{\mu\nu} = (1 + \sigma n_k)^{1/2} \begin{bmatrix} 1 & f_k^\alpha \\ \sigma f_k^{1-\alpha} & 1 \end{bmatrix} e^{-s_k \tau_3}, \tag{41}$$

Among the three parameters the one which physically classifies the thermal situations is the number parameter n_k. Choices of the other two parameters, (α, s_k) are rather a matter of convention and preference, as will be explained later.

On the other hand, the unperturbed operators of physical particles ($a(t)$-operators) satisfy the equations [15]

$$[\frac{d}{dt} + i\omega_k(t) - iP_k(t)] a_k(t) = 0, \tag{42}$$

$$\bar{a}_k [\frac{\overleftarrow{d}}{dt} - i\omega_k(t) + iP_k(t)] = 0, \tag{43}$$

which can be obtained from Eqs. (34,35) for the quasiparticles, where

$$P_k(t) \equiv iB_k^{-1}(t) \frac{d}{dt} B_k(t). \tag{44}$$

It is due to this definition of $P_k(t)$ that $[d/dt - iP_k(t)]$ is called the covariant time derivative.

The Hamiltonian of the unperturbed physical fields (the $a(t)$-operators) is

$$\hat{H}_Q(t) = \hat{H}_0(t) - \hat{Q}(t), \tag{45}$$

where

$$\hat{Q}(t) = \int d^3k \, \bar{a}_k(t) P_k(t) a_k(t), \tag{46}$$

Note that *no time-dependent thermal processes start without the* $\hat{Q}$-term.

Let us separate the dynamical energy operator H into the free and interaction parts:

$$H = H_0 + H_{int}. \tag{47}$$

Then the total Hamiltonian has the structure

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \tag{48}$$

with

$$\hat{H}_{int} = H_{int} - \tilde{H}_{int}. \tag{49}$$

Since our unperturbed Hamiltonian is not $\hat{H}_0$ but $\hat{H}_Q(t)$, *the interaction Hamiltonian in this perturbation calculation contains the counter term* $\hat{Q}$ as

$$\hat{H}_I(t) = \hat{H}_{int} + \hat{Q}(t), \tag{50}$$

because the total Hamiltonian $\hat{H}$ should not be modified.

Since we consider the interaction representation, the time-change of the state vectors $|t\rangle$ is generated by this interaction Hamiltonian:

$$|t\rangle = \hat{u}(t)|t_0\rangle \tag{51}$$

with

$$\frac{\partial}{\partial t} \hat{u}(t) = -i\hat{H}_I(t)\hat{u}(t). \tag{52}$$

Here we denote the coincidence time of the Heisenberg and interaction representation by t_0. It has been shown [16,10] that, when the form of the number parameter n_k is not restricted to the equilibrium form, the Feynman formalism for the perturbation calculation can be obtained only for two choices of the parameters; (a) $\alpha = 1$ for $t_0 = -\infty$, or (b) $\alpha = 0$ for $t = \infty$. We use $\alpha = 1$ with $t_0 = -\infty$ in the following consideration.

The above choice of the parameters is based on the conventional reason that we wish to make use of the Feynman formalism; if we do not mind a complicated computation rule without the Feynman method, the choice of α is not limited. A remarkable aspect of the perturbation calculation in TFD is that *it does not need the Gell-Mann-Low trick* [16,10]. Finally the choice of α is not limited when the number parameter has the equilibrium form. It may be worth recalling that the equilibrium number parameter seems to follow from the stability condition for the thermal vacuum which states $S|0\rangle = |0\rangle$ and $\langle 0|S = \langle 0|$, where $S \equiv \hat{u}(t = \infty)$. We used the word "seems" because the complete proof has been given only for the fact that the stability condition is sufficient for the number parameter to have the equilibrium distribution; it has not been proven that this condition is necessary, though perturbation calculations suggest that it indeed seems to be so. The emergence of the temperature seems to be based on the stability of thermal vacuum. In this paper we derive the concept of temperature from a different condition. We show that the temperature emerges from the condition that a thermal situation is stationary (that is, time-independent) when there are no interactions with external systems.

Though we have chosen $\alpha = 1$, we are still left with the parameter s_k besides the number parameter. We shall choose

$$s_k = \ln [1 + \sigma n_k]^{1/2}, \tag{53}$$

because this makes the matrix B_k linear in n_k and simplifies the structure of the heat term $\hat{Q}$ as will be shown bellow though a careful investigation shows that the choice of s_k has no physical significance in the heat term. Since the parameters (α, s_k) are specified, B_k depends only on the number parameter n_k. We will therefore denote it by $B_k[n]$ in the following. With this choice of α and s_k we obtain

$$B_k[n] = \begin{bmatrix} 1+\sigma n_k & -n_k \\ -\sigma & 1 \end{bmatrix}, \tag{54}$$

the inverse of which is

$$B_k^{-1}[n] = \begin{bmatrix} 1 & n_k \\ \sigma & 1+\sigma n_k \end{bmatrix}. \tag{55}$$

With this $B_k[n]$ the operator $\hat{Q}$ takes a simple form; Eqs.(44) and (46) give

$$P_k(t) = i\sigma \dot{n}_k(t) T_0 \tag{56}$$

and

$$\hat{Q}(t) = i\sigma \int d^3k\, \dot{n}_k(t) \bar{a}_k(t) T_0 a_k(t), \tag{57}$$

where

$$T_0 \equiv \begin{bmatrix} 1 & -\sigma \\ \sigma & -1 \end{bmatrix}, \tag{58}$$

It may be worthwhile to point out some significant differences between the closed path method and TFD [17]. In TFD we have the freedom in choosing the parameters (α, s_k). We chose these parameters according to our convenience. The parameter s_k does not show up in the unperturbed propagator in a stationary situation, but it does explicitly appear in the propagator in a time-dependent situation, as will be shown in the next section. Another difference between TFD and CPF seems to be more fundamental. In TFD, the free Green's function is unambiguously defined, the time dependence of the number parameters is explicit. In CPF however, the number parameter in the free Green's function is rather vague. As a matter of fact n_k is the initial number parameter defined at time $t_0 = -\infty$ and does not have the time dependence. Therefore the derivation of the Boltzmann equation in CPF, particularly the collision term, does not withstand close scrutiny, i.e., we cannot have a consistent integro-differential equation for either the number parameter n_k or the observable number density $n_H(t) =< a^\dagger(t) a(t) >$.

With this preparation we are now ready to enter an essential part of the construction of TFD in this paper.

III. DIAGONAL PROPAGATOR OF THE QUASI-PARTICLE

We now come to an issue of essential significance. In the *stable* quasi-particle picture, the particles are defined by the quasi-particles so that what propagates as particle waves is the propagator of the quasi-particles, which is given by the diagonal matrix

$$\langle 0|T[\xi_k(t)^\mu\, \bar{\xi}_l(t')^\nu]|0\rangle = \Theta(\omega_k : t-t')^{\mu\nu} \delta(\vec{k}-\vec{l}) \tag{59}$$

with

$$\Theta(\nu : t-t') \equiv e^{-i\nu(t-t')} \begin{pmatrix} \theta(t-t') & 0 \\ 0 & -\theta(t'-t) \end{pmatrix}. \tag{60}$$

This motivates us to express all the propagators in terms of this quasi-particle propagator. The matter is simple in the case of unperturbed propagators of the physical particles (a-particles)

$$\Delta_k(t,t')^{\mu\nu} \delta(\vec{k}-\vec{l}) \equiv \langle 0|T[a_k(t)^\mu \bar{a}_l(t')^\nu]\tau_3|0\rangle. \tag{61}$$

Eqs. (29) and (30) give

$$\Delta_k(t,t') = B_k^{-1}(t)\Theta(\omega_k : t-t') B_k(t')\tau_3. \tag{62}$$

Note that the time-dependent s_k-parameter explicitly contribute to this unperturbed propagator of the physical particles.

We need an explanation for the presence of τ_3 in the above definition of the physical particle propagators, because the usual definition does not include τ_3. According to the notations for the thermal doublet in (13), we have $\bar{a}_k(t)^2 = -\sigma[a_k(t)^2]^\dagger$. Since the interaction Hamiltonian is expressed in terms of $a_k, \tilde{a}_k$ and their dagger conjugate, the Feynman internal line is written by these operators. Furthermore, the tilde conjugate of $\bar{a}^1 a^1 = a^\dagger a$ is not $\bar{a}^2 a^2$ but $\tilde{a}^\dagger \tilde{a} = a^2 \bar{a}^2$. To make use of the thermal doublet notations in the Feynman diagrams, we need the factor σ to take care of this change in this standing order. Thus the above factor $-\sigma$ is taken care of by τ_3 in the propagators of the physical particles. Furthermore, this definition of the propagator including τ_3 simplifies the computational rules which will be introduced in section 5.

A remarkable fact is that a form similar to (62) holds true also for the corrected propagator of the observed particles:

$$G_k(t,t')\delta(\vec{k}-\vec{l}) \equiv \langle 0|T[a_{Hk}(t)\bar{a}_{Hl}(t')]\tau_3|0\rangle. \tag{63}$$

It has been shown that this has the form [18,19]

$$G_k(t,t') = B^{-1}[N_k(t,t')] \begin{pmatrix} g(t,t':\vec{k}) & 0 \\ 0 & g^*(t',t:\vec{k}) \end{pmatrix} \Theta(0 : t-t') B[N_k^*(t',t)]\tau_3, \tag{64}$$

with a c-number function $g(t,t':\vec{k})$. The corrected number parameter $N_k(t,t')$ with $t=t'$ has the property

$$N_k(t=t') = \langle 0|a_{Hk}^\dagger(t)\, a_{Hk}(t)|0\rangle \tag{65}$$

which is the observed number. The temporal behavior of this quantity will be studied in section 5.

IV. SELF-ENERGY AND ON-SHELL DEFINITION

In this section we express the proper self-energy in terms of the quasi-particle propagator Θ. This will help us to find a reasonable definition of the on-shell self-energy.

A. General Properties of the Self-Energies

We consider the self-energy diagrams with two vertices connected by some lines. Since the interaction Hamiltonian has the counter term $\hat{Q}$ according to (50) with (46), the proper self-energy has the form

$$\Sigma_{prop}(t,t':\vec{k}) = \Sigma_k(t,t') + P_k(t)\delta(t-t'). \tag{66}$$

A calculation showed [18,19,10] that, when $\Sigma_k(t,t')$ is given by a self-energy diagram with two vertices, it has the form

$$\Sigma_k(t,t') = -i\int[dq]\, B^{-1}[N(t)] \begin{pmatrix} s(t') & 0 \\ 0 & s(t) \end{pmatrix} \Theta(W:t-t')B[N(t')]. \tag{67}$$

Here $[dq]$ denotes the momentum integration with respect to the momenta associated with the internal lines which join the two vertices. The quantities $N(t)$ and $s(t)$ depend on these momenta. Suppose that n lines propagate from t to t' with momenta $(\vec{q}_i\ : i = 1,\cdots,n)$ and m lines move in the opposite direction with momenta $(\vec{q}_j\ : j = 1,\cdots,m)$. Here the direction points from ϕ to ϕ. We then have

$$[dq] = \prod_{i=1}^{n}\left(\frac{d^3q_i}{(2\pi)^3}\right)\prod_{j=1}^{m}\left(\frac{d^3q_j}{(2\pi)^3}\right)(2\pi)^3 \delta(\vec{k}-\sum_{i=1}^{n}\vec{q}_i+\sum_{j=1}^{m}\vec{q}_j), \tag{68}$$

$$s(t) = C\prod_{i=1}^{n} n_{q_i}(t)\prod_{j=1}^{m}\sigma_j[1+\sigma_j n_{q_j}(t)]\frac{1}{N(t)}, \tag{69}$$

$$N(t) = \frac{1}{F^{-1}(t)-\sigma_0} \tag{70}$$

$$F(t) = \prod_{i=1}^{n} f_{q_i}(t)\prod_{j=i}^{m} f_{q_j}^{-1}(t), \tag{71}$$

$$W = \sum_{i=1}^{n}\omega_{q_i} - \sum_{j=1}^{m}\omega_{q_j}. \tag{72}$$

Here $n_q(t)$ is the unperturbed number parameter of the line with momentum $\vec{q}$. The σ_i and σ_j are the σ-parameters of the i-th and j-th lines respectively. The sign factor σ_0 is the σ-sign parameter of the external lines. The coefficient C is a positive numerical factor determined by the coupling constant. We present in section 5 an extremely simple derivation of this result. Eq.(67) indicates that the proper self-energy does propagate through the quasi-particle propagator.

B. Definition of On-Shell Self-Energy and Renormalization

The above structure of the proper self-energy leads to a reasonable definition of the on-shell energy. Since the quasi-particles are stable, we can apply the usual definition of the on-shell energy to their self-energies. Thus write the Fourier representation of the quasi-particle propagator:

$$\Theta(\nu:t-t') = \int\frac{dk_0}{2\pi}e^{-ik_0(t-t')}\Theta(\nu:k_0). \tag{73}$$

We then have

$$\Theta(\nu:k_0) = \frac{i}{k_0-\nu+i\epsilon\tau_3}. \tag{74}$$

Now the well-known definition of the on-shell part can be immediately applied to this by putting $k_0 = \omega_k$. Thus the on-shell part $\Theta_0(\nu:t-t')$ is obtained by replacing k_0 by ω_k in (73):

$$\Theta_0(\nu:t-t') = \delta(t-t')\Theta(\nu:\omega_k) \tag{75}$$

To split this into the real and imaginary parts, it is convenient to write

$$\Theta_0(\nu:t-t') = i\delta(t-t')\left[\mathcal{P}\frac{1}{\omega_k-\nu} - i\pi\delta(\omega_k-\nu)\tau_3\right]. \tag{76}$$

With this we can define the on-shell self-energy of physical particles even when they are not stable. Put the self-energy in the form

$$\Sigma_k(t,t') = \int d\nu\, B^{-1}[N(t,t')]\begin{pmatrix} s(t,t') & 0 \\ 0 & s^*(t',t)\end{pmatrix}\Theta(\nu:t-t')B[N^*(t',t)], \tag{77}$$

which is always possible. Here the tilde conjugation rule is considered. The quantity $N(t,t')$ and $s(t,t')$ depend on ν. Then, the on-shell self-energy of the physical particles is obtained by replacing Θ by Θ_0, so we have the on-shell part

$$\Sigma_k^0(t,t') = \delta(t-t')\int d\nu\, B^{-1}[N(t)]\begin{pmatrix} s(t) & 0 \\ 0 & s^*(t)\end{pmatrix}\Theta_0(\nu:\omega_k)B[N^*(t)]. \tag{78}$$

Here $N(t)$ and $s(t)$ are respectively $N(t,t')$ and $s(t,t')$ at equal time $t = t'$. *This coincidence of time is due to the delta function in* $\Theta_0(\nu:t-t')$.

We now apply this definition of the on-shell self-energy to the above example in (67). Its on-shell part is

$$\Sigma_k^0(t,t') = \delta(t-t')\Sigma_k^0(t) \tag{79}$$

with

$$\Sigma_k^0(t) = -i\int[dq]\, s(t)\, B^{-1}[N(t)]\Theta_0(W:\omega_k)B[N(t)]. \tag{80}$$

Then the relation (66) shows that the total on-shell self-energy is

$$\Sigma^0_{prop}(t,t':\vec{k}) = \Sigma^0_{prop}(t:\vec{k})\,\delta(t-t') \tag{81}$$

with

$$\Sigma^0_{prop}(t:\vec{k}) = \Sigma^0_k(t) + P_k(t). \tag{82}$$

With Eq. (76) we can split this into the real and imaginary on-shell parts:

$$\mathcal{R}\left[\Sigma^0_{prop}(t:\vec{k})\right] = \int [dq]\, s(t)\, \mathcal{P}\frac{1}{\omega_k - \nu} \tag{83}$$

$$\mathcal{I}\left[\Sigma^0_{prop}(t:\vec{k})\right] = P_k(t) - i\int [dq]\, s(t)\, \delta(\omega_k - W) A[N(t)] \tag{84}$$

with

$$A \equiv B^{-1}\tau_3 B. \tag{85}$$

We can rewrite the imaginary part of this on-shell total self-energy as

$$\mathcal{I}\left[\Sigma^0_{prop}(t:\vec{k})\right] = \sigma\dot{n}_k(t)T_0 - \int [dq]\, s(t)\, \delta(\omega_k - W) A[N(t)], \tag{86}$$

where (56) is considered.

The on-shell self-energy contributes to the renormalized quasi-particle Hamiltonian $\hat{H}^r_0$ as

$$\hat{H}^r_0 = \hat{H}_0 + \bar{a}_k(t)\, \Sigma^0_{prop}(t:\vec{k})\, a_k(t). \tag{87}$$

The renormalized quasi-particle Hamiltonian should be diagonal in quasi-particle operators (the ξ-operators). The real part of the on-shell self-energy replaces ω_k with

$$\omega_k(t) = \omega_k + \mathcal{R}\left[\Sigma^0_{prop}(t:\vec{k})\right], \tag{88}$$

preserving the diagonal nature of the free Hamiltonian. Since the Hamiltonian should change sign under the tilde conjugation, its diagonal imaginary term can only take the form $i(\bar{\xi}_k\tau_3\xi_k)$ which is equal to $i(\bar{a}_k(t)A[n_k]a_k)$. Thus the above imaginary part of the on-shell self-energy should be proportional to such a term:

$$\sigma\dot{n}_k(t)T_0 - \int [dq]\, s(t)\, \delta(\omega_k - W) A[N(t)] = -\kappa_k(t) A[n_k(t)] \tag{89}$$

with an unknown constant $\kappa_k(t)$. This two by two matrix equation gives the following solution:

$$\dot{n}_k(t) = -2\kappa_k(t) n_k(t) + 2\pi \int [dq]\, \delta(\omega_k - W) N(t) s(t) \tag{90}$$

with

$$\kappa_k(t) = \pi \int [dq]\, \delta(\omega_k - W) s(t). \tag{91}$$

This is the Boltzman equation obtained in [18,19,10], where it was shown that the entropy law follows from this Boltzman equation. In other words, the above Boltzman equation leads to

$$\dot{S} \geq 0, \tag{92}$$

where the entropy S is defined in terms of the number parameters as

$$S(t) = \int d^3k \{[1+\sigma n_k(t)]\ln[1+\sigma n_k(t)] - n_k(t)\ln n_k(t)\}. \tag{93}$$

This illustrates a usefulness of the on-shell self-energy condition in the stable quasi-particle picture. The constant $\kappa_k(t)$ is the dissipative constant. Though the quasi-particles are stable, the observed particles are dissipative.

Eq.(90) can be put in the form

$$\dot{n}_k(t) = 2\pi \int [dq]\, \delta(\omega_k - W)\, [N(t) - n_k(t)]. \tag{94}$$

Thus the condition for $\dot{n}_k = 0$ is $N(t) = n_k(t)$ for $W = \omega_k$. According to (70) this condition is equivalent to the relation $F(t) = f_k(t)$ for $W = \omega_k$. Then (71) shows that $f_k(t)$ should have the form

$$f_k(t) = e^{-\beta\omega_k} \tag{95}$$

with a parameter β which is independent of $\vec{k}$. This implies that a stationary situation without any external interaction is an equilibrium state. The concept of temperature ($T = 1/\beta$) emerges from the condition that a system be stationary.

However a more rigorous proof is from the entropy law. For an interaction of the form $\int dxdx' V(x-x')\psi^\dagger(x)\psi^\dagger(x')\psi(x')\psi(x)$, it has can be shown that

$$\begin{aligned}\dot{n}_k = \pi \int & dk_1 dk_2 dq \delta(\omega_k + \omega_q - \omega_{k_1} - \omega_{k_2})\delta(k+q-k_1-k_2) \\ & \{[V(k-k_1) + \sigma V(k-k_2)]^2 + 4sV(k-k_1)^2\} \\ & n_{k_1} n_{k_2} n_k n_q (f_q^{-1} f_k^{-1} - f_{k_1}^{-1} f_{k_2}^{-1}),\end{aligned} \tag{96}$$

where s is the particle spin. The entropy change is

$$\dot{S} = -\int dk \dot{n}_k \ln(f_k). \tag{97}$$

Through the exchange of variables it can be written as

$$\begin{aligned}\dot{S} = \frac{1}{4}\pi \int & dk_1 dk_2 dq \delta(\omega_k + \omega_q - \omega_{k_1} - \omega_{k_2}) \\ & \delta(k+q-k_1-k_2) \\ & \{[V(k-k_1) + \sigma V(k-k_2)]^2 + 4SV(k-k_1)^2\} \\ & n_{k_1} n_{k_2} n_k n_q (f_q^{-1} f_k^{-1} - f_{k_1}^{-1} f_{k_2}^{-1}) \ln(f_{k_1} f_{k_2}/f_q f_k).\end{aligned} \tag{98}$$

It is zero if and only if

$$(f_q f_k - f_{k_1} f_{k_2})\delta(\omega_k + \omega_q - \omega_{k_1} - \omega_{k_2}) = 0.$$

This requires that $(\ln f)$ as a function of ω be linear.

C. Equilibrium Situations

In equilibrium situations, the above consideration becomes extremely simple. Then the unperturbed number parameter has the equilibrium distribution

$$n(\nu) = \frac{1}{e^{\beta\nu} - 1}. \tag{99}$$

The Fourier amplitude of the two-point functions $G_k(t - t')$ has the spectral representation

$$G_k(k_0) = \int_{-\infty}^{\infty} d\nu\, B^{-1}[n(\nu)] \frac{\rho_k(\nu)}{k_0 - \nu + i\epsilon\tau_3} B[n(\nu)]\tau_3, \tag{100}$$

The function ρ is the spectral function. The integration over k_0 gives

$$G_k(t - t') = -i \int d\nu\, B^{-1}[n(\nu)]\Theta(\nu : t - t')\rho_k(\nu) B[n(\nu)]\tau_3. \tag{101}$$

In this way the physical particle propagator is expressed in terms of the propagator of the stable quasi-particles. However, the structure of the spectral function is such that the physical particles are dissipative, as is well known in thermal physics.

The Dyson equation shows that the proper self-energy has the same form with a different spectral function:

$$\Sigma_k(t - t') = -i \int d\nu\, B^{-1}[n(\nu)]\Theta(\nu : t - t')\sigma_k(\nu) B[n(\nu)]. \tag{102}$$

The spectral function ρ is related to the spectral function σ through

$$\rho_k(\nu) = \frac{\sigma_k(\nu)}{(\nu - \omega_k)^2 + \pi^2\sigma_k^2(\nu)}. \tag{103}$$

Thus the proper self-energy is also expressed in terms of the propagator of the stable quasi-particle. The on-shell self-energy is obtained from the on-shell part of the quasi-particle propagator as

$$\begin{aligned} & i\Sigma_k^0(t - t') \\ &= \int d\nu\, B^{-1}[n(\nu)]\Theta_0(\nu : t - t')\sigma_k(\nu) B[n(\nu)] \\ &= \delta(t - t') \int d\nu\, B^{-1}[n(\nu)]\Theta(\nu : \omega_k)\sigma_k(\nu) B[n(\nu)]. \end{aligned} \tag{104}$$

The real part gives the energy shift, while the imaginary part is the dissipative constant which never vanishes when the fields have nonlinear interactions.

We write the Green's function in momentum space in a slightly different form

$$G_k(k_0) = B^{-1}(k_0) \begin{pmatrix} \bar{g}_k & \\ & \bar{g}_k^* \end{pmatrix} B(k_0) \tag{105}$$

where

$$\bar{g}_k = \int d\nu \frac{\rho_k(\nu)}{k_0 - \nu + i\epsilon}$$

which completely determines the Green's function. g_k does not have any poles in the complex plane, to see this we assume that the spectral function has Lorentzian form

$$\rho_k = \frac{1}{\pi} \frac{\Gamma_k}{(\nu - E_k)^2 + \Gamma_k^2}$$

where E_k is the energy pole and Γ_k is the width of the spectral function. Extend the frequency k_0 to the complex domain $k_0 = k_r + ik_i$ then

$$\bar{g}_k = \frac{1}{k_r + ik_i - E_k + iSGN(k_i)\Gamma_k} \tag{106}$$

which apparently has no complex poles. We shall now consider an electron phonon system assuming that the electron-density is zero. With the use of full Green's function the electron self-energy is

$$\Sigma_k = B^{-1}(k_0) \begin{pmatrix} \Pi_k & \\ & \Pi_k^* \end{pmatrix} B(k_0)$$

with

$$\begin{aligned} \Pi_k = & \int d\nu dq \frac{g_{k-q}^2}{2\omega_{k-q}} \rho_q(\nu) \\ & \left[\frac{1 + n_{k-q}}{k_0 - \nu - \omega_{k-q} + i\epsilon} + \frac{n_{k-q}}{k_0 - \nu + \omega_{k-q} + i\epsilon} \right] \end{aligned} \tag{107}$$

where ω_k is the phonon energy, g_q is the electron-phonon interaction matrix element and n_q is the phonon number density. To make the problem manageable we again use the Lorentzian approximation for the spectral weight function we have

$$\begin{aligned} \Pi_k = & \int dq \frac{g_{k-q}^2}{2\omega_{k-q}} \\ & \left[\frac{1 + n_{k-q}}{k_0 - E_q - \omega_{k-q} + i\Gamma_q} + \frac{n_{k-q}}{k_0 - E_q + \omega_{k-q} + i\Gamma_q} \right]. \end{aligned} \tag{108}$$

where E_k and Γ_k are to be determined later. Hence

$$\bar{g}^{-1} = k_0 - \epsilon_k - \Pi_k(k_0) = k_0 - E_k + i\Gamma_k$$

where ϵ_k is the bare electron energy. Since we assumed the full Green's function to have the Lorentzian form, replacing k_0 by E_k in the self-energy with the above on-shell definition, we must have

$$E_k - \epsilon_k + i\Gamma_k - \Pi_k(E_k) = 0$$

which gives the self-consistent equation for E_k and Γ_k

$$E_k + i\Gamma_k = \epsilon_k - \int dq \frac{g_{k-q}^2}{2\omega_{k-q}}$$
$$\left[\frac{1+n_{k-q}}{E_k - E_q - \omega_{k-q} + i\Gamma_q} + \frac{n_{k-q}}{E_k - E_q + \omega_{k-q} + i\Gamma_q}\right]. \quad (109)$$

This defines explicitly the on-shell approximation for the problem. This constitutes the crudest self-consistent approximation for the energy poles of an electron moving in a phonon field. This has been called a polaron. Note that a common mistake is to replace k_0 by $E_k - i\Gamma_k$ in the self-energy part which is

$$E_k + i\Gamma_k = \epsilon_k - \int dq \frac{g_{k-q}^2}{2\omega_{k-q}} \left[\frac{1+n_{k-q}}{E_k - E_q - \omega_{k-q} + i(\Gamma_q - \Gamma_k)}\right.$$
$$\left. + \frac{n_{k-q}}{E_k - E_q + \omega_{k-q} + i(\Gamma_q - \Gamma_k)}\right].$$

We would have no way of ensuring the imaginary part Γ_k to be positive definite.

In a hot quark gluon plasma system [21], it has been shown that if the present on-shell definition is used, the thermal infrared divergence disappears though the authors believed that complex poles should be used. However if the complex poles are used, the infra-red divergence reappears.

V. SIMPLE COMPUTATIONAL RECIPE IN TFD

In this section we formulate a doublet vector algebra and show that the formalism provides us with an extremely simple computational recipe in both equilibrium and nonequilibrium TFD.

A. Doublet Vector Algebra

Let us begin with the general time-dependent situation. In this case the unperturbed propagator of the physical particle was given in (62):

$$\Delta_k(t,t') = B^{-1}[n(t)]\Theta(\omega_k : t-t')B[n(t')]\tau_3, \quad (110)$$

When (60) is considered, this reads as

$$\Delta_k(t,t') = e^{-i\omega_k(t-t')}\left[n_k(t')\,\theta(t-t')T_\sigma T_1 T[f_k(t')]\right.$$
$$\left. + \sigma[1+\sigma n_k(t)]\theta(t'-t)T[f_k^{-1}(t)]T_1 T_\sigma\right]. \quad (111)$$

Here f is related to n as

$$n_k = \frac{1}{f_k^{-1} - \sigma}, \quad (112)$$

which gives

$$\sigma[1+\sigma n_k] = \sigma f_k^{-1} n_k = -\frac{1}{f_k - \sigma} \quad (113)$$

The notations used are

$$T_\sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}, \; T_1 \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; T[f] \equiv \begin{pmatrix} f^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad (114)$$

with any function f.

Eq.(111) has the following Fourier expansion:

$$\Delta_k(t,t') = i\int dk_0 e^{-ik_0(t-t')}\left[\frac{n_k(t')}{k_0 - \omega_k + i\epsilon}T_\sigma T_1 T[f_k(t')]\right.$$
$$\left. - \frac{\sigma[1+\sigma n_k(t)]}{k_0 - \omega_k - i\epsilon}T[f_k^{-1}(t)]T_1 T_\sigma\right]. \quad (115)$$

Note that the Fourier amplitude still depends, not on two time, but on one time; the first (second) term is the Fourier transform with respect to t (t') and the Fourier amplitude does depend on t' (t).

In an equilibrium situation n_k and f_k are independent of time and we have (with the unit $\beta = 1$)

$$f_k(t) = e^{-\omega_k} \quad (116)$$

Thus, when we introduce the notation

$$n(\nu) \equiv \frac{1}{e^\nu - \sigma}, \quad (117)$$

the unperturbed propagator of the physical particle in (62) reads as

$$\Delta_k(t-t') = B^{-1}[n(\omega_k)]\Theta(\omega_k : t-t')B[n(\omega_k)]\tau_3, \quad (118)$$

In this case Eq.(111) takes the following form:

$$\Delta_k(t-t') = e^{-i\omega_k(t-t')}\left[n(\omega_k)\theta(t-t')T_\sigma T_1 T(\omega_k)\right.$$
$$\left. - n(-\omega_k)\theta(t'-t)T(-\omega_k)T_1 T_\sigma\right]. \quad (119)$$

Here use was made of the relation

$$n(-\nu) = -[n(\nu) + \sigma]. \quad (120)$$

We used also the following notation:

$$T(\nu) \equiv \begin{pmatrix} e^\nu & 0 \\ 0 & 1 \end{pmatrix}. \quad (121)$$

To facilitate our calculations it is convenient to define the following tensors:

$$\Gamma^{(1)} \equiv \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \quad (122)$$

which is just a two component vector. This will be called a doublet column vector with components (a_1, a_2). A 2×2 matrix

$$\Gamma^{(2)} \equiv \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} T_0 \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad (123)$$

is denoted by

$$\Gamma^{(2)} \equiv \left(\left\{ \begin{array}{c} a_1 \\ a_2 \end{array} \right\} \left\{ \begin{array}{c} b_1 \\ b_2 \end{array} \right\} \right). \tag{124}$$

This is generalized to higher dimensions

$$\Gamma^{(n)} \equiv \left(\left\{ \begin{array}{c} a_1^{(1)} \\ a_2^{(1)} \end{array} \right\} \left\{ \begin{array}{c} a_1^{(2)} \\ a_2^{(2)} \end{array} \right\} \cdots \left\{ \begin{array}{c} a_1^{(n)} \\ a_2^{(n)} \end{array} \right\} \right) \equiv \prod_j^n a_{i_j}^j \tag{125}$$

which is related to an n point Green's function.

The unperturbed propagator now can be written as

$$\Delta_k(t-t') = e^{-i\omega_k(t-t')} \left[n(\omega_k)\, \theta(t-t') \left(\left\{ \begin{array}{c} 1 \\ \sigma \end{array} \right\} \left\{ \begin{array}{c} e^{\omega_k} \\ 1 \end{array} \right\} \right) - n(-\omega_k)\, \theta(t'-t) \left(\left\{ \begin{array}{c} e^{-\omega_k} \\ 1 \end{array} \right\} \left\{ \begin{array}{c} 1 \\ \sigma \end{array} \right\} \right) \right]. \tag{126}$$

Suppose that a vertex at t is connected to n vertices at $(t_i : \ i = 1 \cdots n)$ through incoming lines and to m vertices at $(t'_j : \ j = 1 \cdots m)$ through outgoing lines (see fig.1). The vertex at t is the end points of the incoming lines and the beginning points of the outgoing lines. We denote the momenta of the i-th and j-th lines by $\vec{k}_i$ and $\vec{k}_j$ respectively. The energy ω_{k_i} will be simply denoted by ω_i.

Consider the i-th line which is incoming to this vertex. Then the t-vertex is the end point of this line. When the starting point of this line is before t, the second term in (126) contributes and the column vector with the two components $(1, \sigma_i)$ is attached to the vertex at t, leaving us with another column vector with the components $(e^{-\omega_i}, 1)$. When the starting point of this line is after t, the incoming line contribution comes from the first term in (126). Then the column vector with the components $(e^{\omega_i}, 1)$ is attached to the t-vertex, leaving us with the column vector with the components $(1, \sigma_i)$.

Consider now the outgoing i-th line joined to the t-vertex. Then the t-vertex becomes the beginning point of this line. When the end point of this line is before t, the first term in (126) contributes and the column vector with the two components $(1, \sigma_i)$ is attached to the t-vertex, leaving us with the column vector with the components $(e^{\omega_i}, 1)$. When the end point of this line is after t, the outgoing line contribution comes from the second term in (126). Then the column vector with the components $(e^{-\omega_k}, 1)$ is attached to the t-vertex, leaving us with the column vector with the components $(1, \sigma_i)$.

Thus $l(= n+m)$-lines create l column vectors attached to the t-vertex, leaving us with other l-column vectors. Those attached to the t-vertex form the inner product:

$$\left\{ \begin{array}{c} a_1 \\ b_1 \end{array} \right\} \cdot \left\{ \begin{array}{c} a_2 \\ b_2 \end{array} \right\} \cdot \left\{ \begin{array}{c} a_3 \\ b_3 \end{array} \right\} \cdots \left\{ \begin{array}{c} a_l \\ b_l \end{array} \right\} \equiv \prod_{i=1}^{l} a_i - \prod_{i=1}^{l} b_i \tag{127}$$

which is a c-number factor. Here the minus sign in front of the second term on the right hand side comes from the minus sign in front of $\tilde{H}_{int}$ in the interaction Hamiltonian $\hat{H}_I$ The other columns stand in the form of $\Gamma^{(l)}$. Many column vectors in this $\Gamma^{(l)}$ form inner products with the column vectors attached to other vertices, while some remain in the form of $\Gamma^{(n)}$-tensor when we study n-point functions.

A general rule is that all the columns attached to an internal vertex form the inner product, creating a c-number factor while the columns attached to an external vertex form the out-product defined by

$$\left\{ \begin{array}{c} a_1 \\ b_1 \end{array} \right\} \times \left\{ \begin{array}{c} a_2 \\ b_2 \end{array} \right\} \times \cdots \times \left\{ \begin{array}{c} a_n \\ b_n \end{array} \right\} = \left\{ \begin{array}{c} a_1 \cdots a_n \\ b_1 \cdots b_n \end{array} \right\} \tag{128}$$

Let us come back to the t-vertex in fig.1 and derive an important theorem. When t is after all of the $n+m$ vertices , i.e. $t > t_i, t'_j$ for all (i,j), all the incoming lines contribute with the second term in (126) while the outgoing lines with the first terms. Thus each line creates the column vector with the components $(1, \sigma)$ according to the above consideration and these column vectors form the inner product. This inner product is the factor $(1 - \prod_i \sigma_i \prod_j \sigma_j)$, which vanishes because the interaction Hamiltonian is a bosonic operator. This theorem reflects the relation $\langle 0 | H_I(t) = 0$ at any time t.

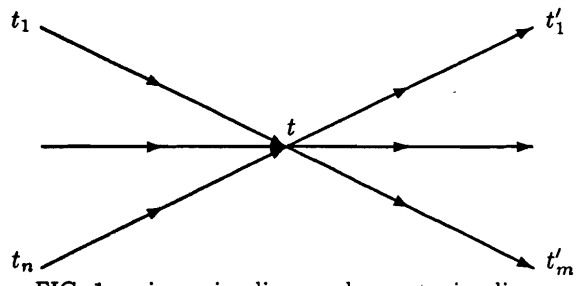

FIG. 1. n incoming lines and m outgoing lines meet at t.

B. Equilibrium Calculations

To illustrate the above recipe of computation of the Feynman diagrams in equilibrium TFD, we consider a self-energy diagram with two vertices joined by two lines. The time of the two vertices are t and t'. The one line points from t to t', while the another line moves in the opposite direction. Here the directions is pointing from ϕ to $\bar{\phi}$. This self-energy diagram has been calculated (see, for example, page 137 in [20]). However, our recipe gives the same result in a much easier way.

According to (126) the unperturbed propagator is

$$\Delta_k(t,t') = e^{-i\omega_k(t-t')}\, d_k(t-t') \tag{129}$$

with

$$d_k(t-t') = \theta(t-t') n(\omega_k) \left(\left\{ \begin{array}{c} 1 \\ \sigma \end{array} \right\} \left\{ \begin{array}{c} e^{\omega_k} \\ 1 \end{array} \right\} \right)$$

$$- n(-\omega_k)\theta(t'-t)\left(\left\{\begin{matrix} e^{-\omega_k} \\ 1 \end{matrix}\right\}\left\{\begin{matrix} 1 \\ \sigma \end{matrix}\right\}\right). \quad (130)$$

Let us first consider the case $t > t'$. Then the propagator pointing from t to t' contributes with the first term in (130), while the other line contribution is given by the transposition of the second term with exchange $t \leftrightarrow t'$ which is

$$-n(-\omega_q)\theta(t-t')\left(\left\{\begin{matrix} 1 \\ \sigma \end{matrix}\right\}\left\{\begin{matrix} e^{-\omega_q} \\ 1 \end{matrix}\right\}\right). \quad (131)$$

Here the momentum of the latter line is denoted by $\vec{q}$. Then the momentum of the first line is $(\vec{k}-\vec{q})$, where $\vec{k}$ is the external momentum. Since we take the cross product of the columns at each external vertices, the contribution of the two internal lines become

$$-\theta(t-t')n(\omega_{k-q})n(-\omega_q)\left(\left\{\begin{matrix} 1 \\ \sigma_0 \end{matrix}\right\}\left\{\begin{matrix} e^{\omega_{k-q}-\omega_q} \\ 1 \end{matrix}\right\}\right), \quad (132)$$

which can be rewritten as

$$\theta(t-t')[n(\omega_q)-n(\omega_{k-q})]\, n(\omega_{k-q}-\omega_q) \left(\left\{\begin{matrix} 1 \\ \sigma_0 \end{matrix}\right\}\left\{\begin{matrix} e^{\omega_{k-q}-\omega_q} \\ 1 \end{matrix}\right\}\right), \quad (133)$$

There is no need to calculate the case $t < t'$, because this case can be obtained from the case $t > t'$ by means of the tilde conjugation rules. Comparing (133) with (130) we see that the self-energy diagram under consideration gives

$$\Sigma_k(t-t') = \int d\nu\, \sigma_0(\nu)\, B^{-1}[n(\nu)]\, \Theta(\nu : t-t')B[n(\nu)] \quad (134)$$

with

$$\sigma_0 \equiv \delta(\nu-\omega_q+\omega_{k-q})\,[n(\omega_q)-n(\omega_{k-q})]. \quad (135)$$

In this way the result takes the form of the spectral representation. The simplicity of the computation is remarkable. We performed nothing more than the outproducts of columns at each external vertices. Furthermore, we did not use any temporal Fourier amplitutdes. The calculation was made in the time-representation (t-representation). This example illustrates an application of the doublet vector algebra for calculation of self-energies in the t-representation. A similar calculation can be applied to a self-energy diagram with two external vertices joined by any number of lines.

C. Non-Equilibrium Calculations

The above calculation of the self-energy can easily be extended to a time-dependent nonequilibrium case. To show this we calculate the two vertex self-energy discussed in section IV. Thus we consider a self-energy diagram with two vertices joined by $(n+m)$ lines. The time of the two vertices are t and t'. The n lines go from t to t', while the other m lines move in the opposite direction. Here the directions is pointing from ϕ to $\bar{\phi}$. We apply the recipe to this diagram and derive the result (67). The calculation turns out to be as easy as the one in the equilibrium case.

The unperturbed propagator in time-dependent cases was given by (111). This can be written in terms of the column tensor notations as

$$\Delta_k(t,t') = e^{-i\omega_k(t-t')}\, d_k(t,t') \quad (136)$$

with

$$d_k(t,t') = \theta(t-t')n_k(t')\left(\left\{\begin{matrix} 1 \\ \sigma \end{matrix}\right\}\left\{\begin{matrix} f_k^{-1}(t') \\ 1 \end{matrix}\right\}\right) + \sigma[1+\sigma n_k(t)]\theta(t'-t)\left(\left\{\begin{matrix} f_k(t) \\ 1 \end{matrix}\right\}\left\{\begin{matrix} 1 \\ \sigma \end{matrix}\right\}\right). \quad (137)$$

Let us first consider the case $t > t'$. Then the i-th propagator contributes with the first term in (137), while the j-th line contribution is given by the transposition of the second term with exchange $t \leftrightarrow t'$ which is

$$\sigma[1+\sigma n_k(t')]\theta(t-t')\left(\left\{\begin{matrix} 1 \\ \sigma \end{matrix}\right\}\left\{\begin{matrix} f_k(t') \\ 1 \end{matrix}\right\}\right). \quad (138)$$

Since we take the cross product of the columns at each external vertices, the contribution of all the internal lines become

$$\theta(t-t')\prod_{i=1}^{n} n_{q_i}(t')\prod_{j=1}^{m}\sigma_j[1+\sigma_j n_{q_j}(t')] \left(\left\{\begin{matrix} 1 \\ \sigma_0 \end{matrix}\right\}\left\{\begin{matrix} F^{-1}(t') \\ 1 \end{matrix}\right\}\right), \quad (139)$$

where σ_0 specifies the external propagator and

$$F(t) = \prod_{i=1}^{n} f_{q_i}(t)\prod_{j=1}^{m} f_{q_j}^{-1}(t). \quad (140)$$

When we define

$$N(t) = \frac{1}{F^{-1}(t)-\sigma_0}, \quad (141)$$

$$s(t) = \frac{1}{N(t)}\prod_{i=1}^{n} n_{q_i}(t')\prod_{j=1}^{m}\sigma_j[1+\sigma_j n_{q_j}(t')], \quad (142)$$

the above contribution for $t > t'$ reads as

$$\theta(t-t')s(t')N(t')\left(\left\{\begin{matrix} 1 \\ \sigma_0 \end{matrix}\right\}\left\{\begin{matrix} F^{-1}(t') \\ 1 \end{matrix}\right\}\right). \quad (143)$$

The case $t < t'$ can be calculated in a similar manner. We see from (143) that the self-energy under consideration is

given by (67). The computation in this section is much simpler than the one given in [18,19,10].

When we consider diagrams with many internal vertices, calculation is considerably complicated. However, use of the recipe simplifies treatment of the internal vertices. Without the doublet vector algebra, calculations with internal vertices would be intractable. We are planning a further development of the recipe for still more simplification of computations.

The last example of application of the doublet vector algebra recipe is a study of the time-change of the particle number. In section IV we derived the Boltzman equation for the time change of the number parameter $n_k(t)$. The equation was given by (94). However, the observed number is not this number parameter but the particle number defined by the Heisenberg operator:

$$N_k^r(t)\delta(\vec{k}-\vec{l}) = \langle 0|a_{Hk}^\dagger(t)a_{Hl}(t)|0\rangle. \tag{144}$$

In equilibrium cases it is known that this observed number is a frequency average of $n(\nu)$ over the spectral function $\rho(\nu)$:

$$N_k^r = \int d\nu\ \rho(\nu)n(\nu), \tag{145}$$

where the number parameter $n(\nu)$ has the Boltzman distribution with the energy ν; the observed number fluctuates around the Boltzman distribution. This shows that the number parameter has a simple property, but the observed one behaves in a complicated way. It is therefore interesting to study the equation for $N_k^r(t)$ in time-dependent cases. In the following, we study this equation. To do this we consider the corrected propagator $G_k(t,t')$, because its (1,1)-component in the limit $t \to t'$ with the condition $t < t'$ is $\sigma N_k^r(t)$, while the same quantity at the limit $t \to t'$ with $t > t'$ is $[1+\sigma N_k^r(t)]$. Therefore, the quantity $\dot{N}_k^r(t)$ is given by the (1,1)-component of $\sigma[(\partial/\partial t)+(\partial/\partial t')]G_k(t,t')$ at $t=t'$ without any condition for ordering among t and t'.

We calculate this in the second order approximation of the coupling constant. Thus, we approximately write the Dyson relation as

$$G_k(t,t') = \int dt_1 dt_2\ \Delta_k(t,t_1)\Sigma_{prop}(t_1,t_2:\vec{k})\Delta_k(t_2,t') + \Delta_k(t,t'). \tag{146}$$

The unperturbed propagator satisfies the equations:

$$[\frac{d}{dt}+i\omega_k(t)-iP_k(t)]\Delta_k(t,t') = \delta(t-t'), \tag{147}$$

$$\Delta_k(t,t')[\frac{\overleftarrow{d}}{dt'}-i\omega_k(t')+iP_k(t')] = -\delta(t-t'). \tag{148}$$

Thus

$$\lim_{t\to t'}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\Delta_k(t,t') = 0. \tag{149}$$

When the t- and t'-derivatives act on the second term in (146), they creat P_k-terms which are proportional to $\dot{n}_k$ Since $\dot{n}_k$ is of the second or higher order in coupling constant, so are the P_k-terms. Thus product of these P_k-terms and Σ_{prop} are of the fourth or higher order of the coupling constant. We therefore ignore these terms. because their contribution is of the same order as the terms ignored in the above approximate Dyson equation. We then have

$$-i\lim_{t\to t'}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)G_k(t,t') = \int dt'\,[\Sigma_{prop}(t,t')\Delta_k(t',t)-\Delta_k(t,t')\Sigma_{prop}(t',t)]. \tag{150}$$

Recall now that no internal vertex is in future of t, implying that t_1 and t_2 in the above equation must before t. We therefore consider only those terms which are proportional to $\theta(t-t_1)$ or $\theta(t-t_2)$ in the integrand. Note that t_1 and t_2 appear separately in two terms on the right hand side of the above relation. Therefore, to simplify notations, we will denote t_2 in the first term by t_1. Since the P_k-term in Σ_{prop} in (66) is proportional to $\delta(t-t_1)$, integration domain for this term shrinks to vanish; this P_k-term does not have any contribution. Therefore, we can replace Σ_{prop} by Σ_k. According to (67) Σ_k has the form

$$\begin{aligned} i\Sigma_k(t,t') = &\int [dq]\ e^{-iW(t-t')} \\ &\left[\theta(t-t')s(t')N(t')\ \left(\left\{\begin{array}{c}1\\ \sigma\end{array}\right\}\left\{\begin{array}{c}F^{-1}(t')\\ 1\end{array}\right\}\right)\right. \\ &\left. +\ s(t)\sigma[1+\sigma N(t)]\theta(t'-t)\left(\left\{\begin{array}{c}F(t)\\ 1\end{array}\right\}\left\{\begin{array}{c}1\\ \sigma\end{array}\right\}\right)\right] \end{aligned} \tag{151}$$

The structure of Δ_k was given in (137). At the t_1-vertex in the first term in (151) Σ_{prop} creates the column with the components $(F^{-1}(t_1),1)$ and Δ_k gives the column with the components $(f_k(t_1),1)$. The inner product of these two columns attached to t_1-vertex gives rise to a factor $[F^{-1}(t_1)f_k(t_1)-1]$. At t_1 in the second term Σ_{prop} creates the column with $(F(t_1),1)$ and Δ_k gives a column with $(f_k^{-1}(t_1),1)$. Their inner product is the factor $[F(t_1)f_k^{-1}(t_1)-1]$. We are then left with two columns attached to t. Their components are both $(1,\sigma)$. In this way the vector algebra recipe simplifies the computation and we obtain

$$\begin{aligned} &\lim_{t\to t'}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)G_k(t,t') = \left(\left\{\begin{array}{c}1\\ \sigma\end{array}\right\}\left\{\begin{array}{c}1\\ \sigma\end{array}\right\}\right) \\ &\int [dq]\int_{-\infty}^{t} dt_1\ s(t_1) \\ &\times\left(N(t_1)\sigma[1+\sigma n_k(t_1)][F^{-1}(t_1)f_k(t_1)-1]e^{i(\omega_k-W)(t-t_1)}\right. \\ &\left.-\ n_k(t_1)\sigma[1+\sigma N(t_1)][f_k^{-1}(t_1)F(t_1)-1]e^{-i(\omega_k-W)(t-t_1)}\right). \end{aligned} \tag{152}$$

Picking the (1,1)-element and using the relations

$$N(t)[1+\sigma n_k(t)][F^{-1}(t)f_k(t)-1] = n_k(t) - N(t) \quad (153)$$
$$n_k(t)[1+\sigma N(t)][F(t)f_k^{-1}(t)-1] = N(t) - n_k(t), \quad (154)$$

we obtain

$$\lim_{t\to t'}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)G_k(t,t')^{11}$$
$$= \sigma\int[dq]\int_{-\infty}^{t} dt_1\ s(t_1)[n_k(t_1)-N(t_1)]$$
$$\times[e^{i(\omega_k-W)(t-t_1)}+e^{-i(\omega_k-W)(t-t_1)}]. \quad (155)$$

We thus have

$$\dot{N}_k^r(t) = \int[dq]\int_{-\infty}^{t} dt_1\ s(t_1)[n_k(t_1)-N(t_1)]$$
$$\times\ [e^{i(\omega_k-W)(t-t_1)}+e^{-i(\omega_k-W)(t-t_1)}]. \quad (156)$$

Note the resemblance of this equation to the Boltzman equation (94) for $n_k(t)$. When we approximately replace $[n_k(t_2)-N(t_2)]$ by the same quantity at time t, the above equation (156) becomes

$$\dot{N}_k^r(t) \approx 2\pi\int[dq]\delta(\omega_k-W)s(t)[N(t)-n_k(t)], \quad (157)$$

the right hand side of which is the same as the one in (94). This implies that in this crude approximation the observed number agrees with the number parameter and the Boltzman equation obtained from the self-consistent renormalization (based on our on-shell definition) is reproduced by the Heisenberg equation. The derivation of Boltzmann equations in CPF is usually done this way. However this equation does not relate the change of number density n_k to the collision integral in terms of themselves as is obvious from the notations in the above equation.

VI. RESPONSE FUNCTIONS

TFD has the advantage of calculating response functions without resorting to the Kubo formula. Suppose that there added to the system a weak perturbing potential $H_i = \int dxU(x)B(x)$ where $B(x)$ is an operator for a measurable physical quantity and $U(x)$ is an externally added potential. The corresponding potential in TFD is $\hat{H}_i = \int dxU(x)[B(x)-\hat{B}(x)]$. T Use the Feynmann diagram approach, the change of another measurable physical quantity A is

$$<A_t(x)> - A_0(x)$$
$$= -i\int dydt'U(y) < T[A(x)(B(y)-\tilde{B}(y))] >$$
$$= -i\int dy\int_{t_0}^{t} dt'U(y) < A(x)(B(y)-\tilde{B}(y)) > . \quad (158)$$

Only the retarded part is needed. Similar to the self-energy the retarded part of the correlation function can be written as

$$-i < T[A^\mu B^\nu] >$$
$$= \int d\nu B^{-1}(N(t,t'))\begin{pmatrix} s_1(\nu;t,t') & O \\ & s_2(\nu;t,t')\end{pmatrix}B(N(t,t')).$$

Eq. (158) becomes

$$<A_t(x)> - A_0(x) = \int dydt'\theta(t-t')d\nu U(y)s_1(\nu;t,t')$$
$$= \int dydt'\theta(t-t')U(y)s_1(t,t'). \quad (159)$$

This explains why it is important to keep only the diagonal terms. A comparison shows that this is the same in form as the retarded function in CPF, though the explicit structures of these functions are different due to the reasons stated near the end of section 2.

VII. SUMMARY

In this paper we formulated TFD in terms of stable quasi-particles, which are different from the dissipative physical particles. In this formalism, all operators are realized in a Fock space associated with the stable quasi-particles. Even the operators of the dissipative physical particles are expressed in terms of operators of the stable quasi-particles. The propagators of the physical particles were expressed in terms of the stable quasi-particle propagators. Although the quasi-particles are stable, the weight probability of the stable quasi-particles in the physical particles diminishes in time, making the physical particles dissipative. In the stable quasi-particle picture we inherit the definition of the on-shell self-energy from the usual field theory without thermal degree of freedom, finding a reasonable formulation of the energy renormalization. Through the course of this consideration a difference between the closed path formalism and TFD was discussed. The difference is due to the freedom in the choice of parameter s_k and explicit time dependence of the number parameter n_k. The difference becomes visible in time-dependent problems. Due to this difference it is not easy to compare our results with those in the closed path formalism. The definition of the on-shell self-energy leads us to a self-consistent renormalization, from which the Boltzman equation followed. This equation determines the temporal behavior of the number parameter $n_k(t)$ and leads to the entropy law.

With the doublet vector algebra we obtained a simple recipe for treatment of the internal vertices and the external vertices in the Feynman diagrams. We showed some examples of self-energy calculation which are tremendously simplified by this recipe. With the help of this recipe we also derived the equation which determines the temporal behavior of the observed number. The latter

equation resembles the Boltzman equation for the number parameter derived from the self-consistent renormalization, providing a support for our on-shell definition. Further elaboration of the recipe is under study.

In this paper we used mostly the fields of the type 1. The entire consideration can be extended to the fields of the type 2 by introducing diagonal propagators of both the quasi- particles and their anti-particles. Such a consideration may follow the steps presented in [17].

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