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Time Ordering Theorem and Calculational Recipes for Thermal Field Dynamics

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We show that in thermo-field-dynamics (TFD), with a particular choice of the parameters, internal times cannot be larger than the largest external time for a Feynmann diagram with any number of external points. We present a simple recipe which finally makes Feynmann diagram calculations in TFD feasible. In fact, frequency integration becomes trivial compared to Matsubara approach when the time property is utilized.

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I. INTRODUCTION

Among equilibrium finite temperature quantum field theories, the imaginary time approach is the most popular [1]. The time variable is rotated and shrunk into a finite segment on the imaginary axis and the frequency is thus discretized also on the imaginary axis. When a two point retarded function is needed, an analytic continuation is performed at the end of the calculation. This approach is straightforward and easy to learn. However there are certain unpleasant features related to it: (1) One must be very careful in performing the frequency continuation from the discrete imaginary axis to the real axis since this requires that the function be analytic especially when numerical computations are involved. (2) It is not certain when one should perform this continuation. It turns out that in certain cases such as broken symmetry the continuation at the very end of the calculation leads to results that violate the spectral representation properties of Green's functions [2]. (3) When a diagram is complicated the frequency summation itself becomes very tedious.

A relatively less known theory is the Thermo-Field-Dynamics (TFD). This theory is based on a theorem that an ensemble average can be replaced by an expectation value over a properly chosen pure state vacuum [3], when the degree of freedom is doubled, i.e., to every operator A is associated an operator $\tilde{A}$ so that $[A,\tilde{A}]_\sigma = 0$ where $\sigma = \pm 1$ for boson and fermion like operators respectively. If H denotes the dynamical energy operator in the usual quantum field theory, the total Hamiltonian in TFD is given by $\hat{H} = H - \tilde{H}$. If a, a^+, $\tilde{a}$, and $\tilde{a}^+$ denote the free operators in TFD that satisfy the usual operator algebra $[a,a^+]_\sigma = [\tilde{a},\tilde{a}^+]_\sigma = 1$, $[a,a]_\sigma = [\tilde{a},\tilde{a}]_\sigma = [a,\tilde{a}]_\sigma = [a^+,\tilde{a}^+]_\sigma = 0$, and $|0>$ denotes the thermal ground state (or vacuum) in terms of these operators, then $a|0> \neq 0$ and $\tilde{a}|0> \neq 0$. However we can find another set of free operators ξ and $\tilde{\xi}$ through a linear Bogoliubov transformation:

$$\begin{pmatrix} \xi \\ \tilde{\xi}^+ \end{pmatrix} = B \begin{pmatrix} a \\ \tilde{a}^+ \end{pmatrix}, (\xi^+, \sigma\tilde{\xi}) = (a^+, \sigma\tilde{a})\tau_3 B^{-1},$$

so that $\xi|0>= \tilde{\xi}|0>= 0$ where the matrix B will be given later. This approach has been proven very useful and powerful in dealing with non-equilibrium situations. This is so because unlike the Matsubara approach, time is treated on equal footing as the space variables. TFD shares some common features with the Schwinger-Keldysh approach [4]. The Green's functions in both approaches are 2×2 matrices. People feel reluctant to use TFD because of its matrix structure. However we show in this paper that when the time property is utilized, it is the very matrix structure that actually makes the Feynmann diagram calculations very easy in TFD. Frequency integrations can be easily taken care of.

II. PROPERTIES OF THE GREEN'S FUNCTIONS

TFD has the symmetry of a three parameter (α, s, n) group [3]. The best choice for α is $\alpha = 1$ with which causality is preserved and the calculations become the simplest. It has been successfully applied to non-equilibrium situations [5,6]. Hence we shall use $\alpha = 1$ throughout this paper unless stated otherwise. The Green's function for a free oscillator is

$$G(t,t') = B^{-1}(t) \begin{pmatrix} \Delta_0(t-t') & \\ & \Delta_0^*(t'-t) \end{pmatrix} B(t')\tau_3 \quad (1)$$

where $\Delta_0(t,t') = -i\theta(t-t')e^{-i\omega(t-t')}$,

$$\tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, B(t) = \begin{pmatrix} 1+\sigma n(t) & -n(t) \\ -\sigma & 1 \end{pmatrix},$$

$$B^{-1}(t) = \begin{pmatrix} 1 & n(t) \\ \sigma & 1+\sigma n(t) \end{pmatrix}.$$

The τ_3 matrix is a trivial factor with no physical significance but a vital importance for computational convenience. This factor comes about because in the original

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formulation of TFD, the doublet notation is defined as $\tilde{a}^{(2)} = -\sigma\tilde{a}$, however the Hamiltonian $\tilde{H}$ is written in terms of

$$\tilde{a}_i^+\tilde{a}_j = \delta_{i,j} + \sigma\tilde{a}_j\tilde{a}_i^+ = \delta_{i,j} - \tilde{a}_j^{(2)}\tilde{a}_i^{(2)}.$$

The choice of our Green's function is equivalent to a redefinition of $\tilde{a}^{(2)}$ as $\sigma\tilde{a}$.

Notice that in general the number parameter cannot be expressed as a function of energy ω. Instead we should simply write it as $n(t) = 1/[f(t) - \sigma]$ where f in equilibrium becomes e^ω when $\beta = 1$. Therefore the number density is determined by f through $n(t) = n[f(t)]$. The free Green's function can be easily cast into the following form by calculating the matrix elements explicitly:

$$\begin{aligned} G_k(\omega;t,t') &= \Delta_0(t,t')n[f_k(t')]T_\sigma T_0 T(f_k(t')) \\ &+ \Delta_0^*(t',t)n[f^{-1}f_k(t)]T(f_k^{-1}(t))T_0T_\sigma \end{aligned} \tag{2}$$

with the following definitions

$$T_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}, T_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, T(f) = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

The first term is the retarded part of the Green's function and the second term is the advanced part. The Green's function has the property that if a forward propagator is denoted as $G(\omega, f; t, t')$ a backward propagator is

$$G(\omega, f; t', t)^T = -G(-\omega, f^{-1}; t, t') \tag{3}$$

as shown in Fig. 1. This is very useful in practical calculations since we can fix the direction of a propagator arbitrarily as we see convenient. At the end of the calculation we put back the proper signs for the Green's function and its energy.

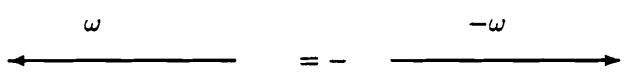

FIG. 1. A backward propagator is replaced by a forward propagator

Owing to the particular structure of TFD, many matrix manipulations are rather simple. To facilitate our calculations we define the following tensors

$$\Gamma^{(1)} \equiv \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

which is just a two component vector and a 2×2 matrix

$$\Gamma^{(2)} \equiv \begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix} = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} T_0 \begin{pmatrix} b_1 & \\ & b_2 \end{pmatrix}$$

is denoted by

$$\Gamma^{(2)} \equiv \left(\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \right).$$

This form is very convenient in calculations. For instance the product of two matrices is

$$\begin{aligned} &\left(\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \right) \left(\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \right) \\ &= (b_1c_1 + b_2c_2) \left(\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \right), \end{aligned}$$

which simply amounts to a contraction of two columns. This is generalized to higher dimensions

$$\Gamma^{(n)} \equiv \left(\begin{Bmatrix} a_1^{(1)} \\ a_2^{(1)} \end{Bmatrix} \begin{Bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{Bmatrix} \cdots \begin{Bmatrix} a_1^{(n)} \\ a_2^{(n)} \end{Bmatrix} \right) \equiv \prod_j^n a_{i_j}^j$$

which is related to an n point Green's function. A free Green's function can be thus denoted as

$$\begin{aligned} G_k(\omega;t,t') &= \Delta_0(t,t')n[f_k(t')] \left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \begin{Bmatrix} f_k(t') \\ 1 \end{Bmatrix} \right) \\ &+ \Delta_0^*(t',t)n[f_k^{-1}(t')] \left(\begin{Bmatrix} f_k^{-1}(t') \\ 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \right). \end{aligned} \tag{4}$$

However this form of the Green's function is not unique to the choice of $\alpha = 1$. Green's functions with other choices of α can also take similar structure with different elements. For instance if $\alpha = 1/2$ we have

$$\begin{aligned} G &= \Delta_0 \left(\begin{Bmatrix} u(t) \\ \sigma v(t) \end{Bmatrix} \begin{Bmatrix} u(t') \\ v(t') \end{Bmatrix} \right) \\ &+ \Delta_0^* \left(\begin{Bmatrix} v(t) \\ u(t) \end{Bmatrix} \begin{Bmatrix} v(t') \\ \sigma u(t') \end{Bmatrix} \right) \end{aligned}$$

where $u = \sqrt{1 + \sigma n}$ and $v = \sqrt{n}$. We should not pursue the matter any further.

In equilibrium situations, for notational convenience we denote frequency by q instead of q_0 . The Green's functions are expressed in terms of frequency q and energy ω since the momentum $\vec{k}$ should appear through the momentum suffix of energy $\omega = \omega_k$. With the above definition and the choice $\alpha = 1$, and with energy measured in k_BT, Green's functions for free oscillator operators in equilibrium TFD are

$$\begin{aligned} G &= \frac{n(\omega)}{q - \omega + i\epsilon} \left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \begin{Bmatrix} e^\omega \\ 1 \end{Bmatrix} \right) \\ &+ \frac{n(-\omega)}{q - \omega - i\epsilon} \left(\begin{Bmatrix} e^{-\omega} \\ 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \right). \end{aligned} \tag{5}$$

Since the number density n matters only for the imaginary part which is proportional to a $\delta(q - \omega)$ function, we can thus replace the energy in $n(\omega)$ by $n(q)$ and hence the above Green's function can be written as

$$\begin{aligned} G(q,\omega) &= \frac{n(q)}{q - \omega + i\epsilon} \left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \begin{Bmatrix} e^q \\ 1 \end{Bmatrix} \right) \\ &+ \frac{n(-q)}{q - \omega - i\epsilon} \left(\begin{Bmatrix} e^{-q} \\ 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \right). \end{aligned} \tag{6}$$

We caution that this replacement of energy ω by frequency q cannot be done freely. We must use the energy or the frequency consistently for any propagator so that the real part is not affected. It is more convenient to prove certain relations using frequency. However it is more convenient to do practical calculations when energy is used since the frequency integration involving $n(q)$ is formidable. Again if a forward propagator is represented by $G(q,\omega)$ a backward propagator is (see Eq. (3) and Fig. 1)

$$G^T(-q,\omega) = -G(q,-\omega).$$

When $\alpha = 1$, the Green's functions are identical to the ones in Schwinger-Keldysh formalism in equilibrium situations. However the difference becomes evident in non-equilibrium situations. We shall leave the discussion to a future article [7].

III. FEYNMANN DIAGRAM CALCULATIONS

It has been shown that the vacuum state $<0|$ is stable in TFD, $<0|S =<0|$. All closed diagrams are zero. Feynmann diagram technique can be proven valid without using Gell-Mann-Low theorem.

In addition to the ordinary Feynman diagram rules, the rules for calculating diagrams in TFD are: (1) each vertex carries a matrix τ_3 since the total Hamiltonian is $\hat{H} = H - \tilde{H}$. This is similar to the minus sign related to the time path integration below the real axis in the Schwinger-Keldysh formalism. It is important to note that there is no mixing of tilde and non-tilde operators at all vertices. (2) At each vertex, multiply the contributions from all the internal lines for each thermal index (1,2) separately. (3) Sum over the thermal doublet index for all internal vertices. When applied to loop diagrams, this rule can be shown to be identical to the loop rule defined by Eqs. (9.78) in Ref. [3] which cannot be used for more complicated diagrams.

In the following we shall state the rules in more detail taking advantage of the tensors defined in the previous section by using the retarded and advanced parts of the Green's function separately. This is very useful because when a propagator is connected to two vertices, for each vertex, only the corresponding column vector in Eq. (4) is required, according to the rule that if a propagator leaves a vertex use the first column (the first index), if a propagator comes into a vertex use the second column (the second index) independent of the other column vector. This remarkable property simplifies calculations in TFD tremendously. Due to the above reasoning we only need the operations for one column tensors. There are two operations. For an external vertex, we take the corresponding column vector from all the propagators attached to this vertex and multiply all the elements in each row, according to rule (2)

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} \times \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} \cdots \times \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \begin{Bmatrix} \prod_i a_i \\ \prod_i b_i \end{Bmatrix} \tag{7}$$

which is an overall column vector factor to be inserted into the proper position designated for that vertex for a diagram. For internal vertices only one more additional operation is required according to rules (2) and (3) which is to sum up the two elements obtained above

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} \cdot \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} \cdots \begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \prod_i a_i - \prod_i b_i \tag{8}$$

which is an overall numerical factor for the diagram.

Since TFD is a real time formulation of quantum field theory, it is most often more convenient to calculate Feynmann diagrams in time space rather than in frequency space. Before we state the recipes for a general multi-point diagram we shall prove a time ordering theorem in TFD which states that *no internal time can be larger than all external times.* This reflects causality even in calculations of time ordered quantities which cannot be shared in other quantum field theories.

Fig. 2 shows an internal vertex with time t connected to n incoming lines from $t_1, \cdots, t_n$ and m outgoing lines to $t'_1, \cdots, t'_m$. We use the convention that the arrow corresponds to $\bar{\psi}$. The $n+m$ point Green's function is

$$G_{n,m}(\mu_1, \cdots, \mu_n; \mu'_1, \cdots, \mu'_m)$$

$$= \int dt \sum_{l=1}^{2} (-1)^{l-1} \prod_i^n G^{\mu_i, l}(t_i, t) \prod_j^m G^{l, \mu'_j}(t, t'_j)$$

where $l = 1, 2$ is the thermal index which represents contributions from H_i and $\tilde{H}_i$ respectively. If the internal time $t > t_i, t'_j$, $(i = 1, \cdots, n; j = 1, \cdots, m)$, we take the advanced part of $G(t_i, t)$ with the second column factor T_{σ_i}, and the retarded part of $G(t'_j, t)$ with the first column factor $T_{\sigma,j}$. Using the diagram rules stated above, we have a factor $1 - \prod_i^n \sigma_i \prod_j^m \sigma_j = 0$ due to the summation of the thermal indices, by requiring that the interaction Hamiltonian be Boson type observable physical quantity. The subscripts i and j are for incoming and outgoing variables respectively. This shows that the internal time cannot be larger than all external times. By induction we can prove that no internal times can be larger than all external times. We shall call this the *time ordering theorem.*

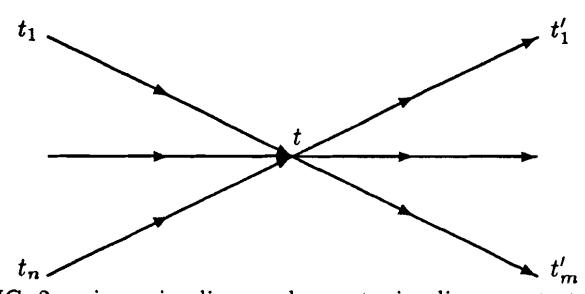

FIG. 2. n incoming lines and m outgoing lines meet at t.

This property is unique to the choice of $\alpha = 1$. When $\alpha = 0$ it is when the internal time is before all external times that the contribution is zero. Nothing can be said about other choices of α. This property will also simplifies calculations significantly when we count the number of possible time arrangements for internal vertices.

The time ordering theorem is just a consequence of the thermal state condition proven in section (7.2.5) of Ref. [3]. It asserts that when $\alpha = 1$, $< 0|(A^+ - \tilde{A}) = 0$ for a boson like operator A. An internal vertex carries nothing but the interaction Hamiltonian $H_i - \tilde{H}_i$. If its time is larger than all other times it's placed to the left-most and thus the matrix element becomes zero.

We are now ready to state the general procedures in calculating a multi-point Green's function with some internal vertices in a Feynmann diagram technique. We shall point out that in a diagram, as far as the frequency integration is concerned, a point is defined by time not space. Hence an instantaneous space like interaction should be treated as a point in the following and the interaction potential should be treated as an overall numerical factor.

(1) Fix the time ordering of the external points. After this is done arrange all possible time ordering for the internal vertices. Arrange the time ordering in a descending order. Redraw the Feynmann diagram in a manner that vertices with larger time are placed to the left of the vertices with smaller time. This procedure is only to help visualize things. Since we are using $\alpha = 1$, no internal times should be larger than the largest external time. Thus an arrangement having some internal vertices with no lines to the left is excluded, the left-most vertex has to be an external one.

(2) Since in non-equilibrium situations the energy distribution may have no meaning, we shall use f in place of e^{ω} and denote n as $n(f)$. Assign arrows to all the lines according to the time ordering so that arrows point to smaller time. If the arrow of the time direction does not agree with the arrow of the Green's function itself f is replaced by f^{-1} and $n(f)$ replaced by $-n(f^{-1})$.

(3) Put an overall factor

$$\prod_l n(f_l(t_{l,<})) \prod_{l'} [-n(f_{l'}^{-1}(t_{l',<}))].$$

The first factor is for propagators with time ordering agreeing with the arrow of the propagator and the second for the ones with opposite directions. $t_<$ stands for the smaller of the two times at the two ends of a propagator. This is because every Green's function carries a factor n according to Eq. (2).

(4) For each external vertex t_j, accoring to Eq. (7), assign a column factor

$$\begin{pmatrix} \prod_l f_l(t_j) \\ \prod_{l'} \sigma_{l'} \end{pmatrix}$$

to be placed to the appropriate position designated for the vertex and for each internal vertex assign a numerical factor

$$\prod_l f_l(t_i) - \prod_{l'} \sigma_{l'} = \prod_l f_l(t_i) - \prod_l \sigma_l$$

according to Eq. (8), where l represents lines with arrows pointing into the vertex and m represents lines leaving the vertex according to the time arrows. This is because we only need to use the retarded part of a Green's function, since the advanced part can be replaced by the retarded part

$$\left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \begin{Bmatrix} f(t') \\ 1 \end{Bmatrix} \right).$$

When the time arrow points to a vertex the second column is used and when the time arrow points away from a vertex the first column is used. Particularly for two point diagrams if we assume that $t > t'$ and if $l > 0$, we have a matrix factor

$$\left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \begin{Bmatrix} \prod_l f_l(t') \\ \prod_{l'} \sigma_{l'} \end{Bmatrix} \right) = \sigma' \left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix} \begin{Bmatrix} \sigma' \prod_l f_l(t') \\ 1 \end{Bmatrix} \right)$$

$$= \sigma' n^{-1}(F(t')) B^{-1}(F(t)) \prod \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B(F(t'))\tau_3,$$

where $\sigma' = \prod_m \sigma_m = \sigma \prod_l \sigma_l$ and $F(t') = \sigma' \prod_l f_l(t')$,

$$\sigma' n^{-1}(F) = \prod_l f_l - \prod_l \sigma_l.$$

Thus for the retarded part of a two point function to be sandwiched between the B-matrices, the factor $\prod_l f_l - \prod_l \sigma_l$ denoted as $n^{-1}(\sum_l \omega_l)$ in equilibrium is to be attached to all vertices except the vertex t, the one with the largest time.

(5) Put back the factors $\Delta_0(t, t')$ from the Green's functions and integrate over all the internal vertex times.

(6) Method of projection for equilibriums. Project the Green's functions onto the time ordering determined in (1) utilizing the properties of the step-function $\theta(t - t')$. If a time segment (t_{i+1}, t_i) with $t_i > t_{i+1}$ and no other internal time in between, belongs to the time domain of a particular propagator $G_l(t_l, t_l')$ with energy ω_l, i.e., $t_l > t_i$ and $t_l' < t_{i+1}$, assign a factor

$$e^{-i\omega_l(t_i - t_{i+1})}.$$

Repeat for all the Green's functions satisfying the above condition, multiply them together we can build a factor

$$-i\theta(t_i - t_{i+1}) e^{-i \sum_l \omega_l (t_i - t_{i+1})}.$$

For all the external points we multiply by a factor $e^{iq_j t_j}$. For convenience we also multiply the internal lines by $e^{iq_i t_i}$ with the understanding that $q_i = 0$. Hence we have

$$\int dt_0 \prod_{l=1}^{n} dt_l e^{iq_l t_l} [-i\theta(t_l - t_{l+1}) e^{-iW_l(t_l - t_{l+1})}$$

$$= (q_1 - W_1 + i\epsilon)^{-1} (q_1 + q_2 - W_l + i\epsilon)^{-1}$$

$$\cdots (\sum_j^n q_j - W_n + i\epsilon)^{-1} \delta(\sum q_j).$$

Note that $W_j = \sum_l \omega_{j,l}$ with $\omega_{j,l}$ the energy of a propagator whose time domain contains the time segment l.

This method of time projection also works for zero temperature quantum field theory. However there can be a great many more terms involved since there is no time ordering theorem at work.

(7) Repeat the process for all other possible time ordering arrangements and add up all the contributions.

Example 1. A loop diagram with n lines joining two external points. Assume that all lines point from t to t'. Otherwise we use Eq. (3) which means that f is replaced by f^{-1} and $n(f)$ is replaced by $-n(1/f)$. There are no internal vertices. According to (3), we have $F(t) = \prod_j f_j(t)$ an overall factor

$$s = n^{-1}(F(t'))\prod_j n(f_j) = \prod_j [1+\sigma n(f_j)] - \prod_j n(f_j).$$

The rest comes from the factor

$$-i\theta(t-t')e^{-i\sum_j \omega_j(t-t')}$$

whose Fourier transform is

$$(q_0 - \sum_j \omega_j + i\epsilon)^{-1}.$$

Hence in equilibrium $F = \prod_j f_j = \prod_j e^{\omega_j} = e^{\sum_j \omega_j}$ we have

$$B^{-1}(\sum_j \omega_j)\frac{s}{q_0 - \sum_j \omega_j + i\epsilon\tau_3}B(\sum_j \omega_j)\tau_3$$
$$= B^{-1}(q_0)\frac{s}{q_0 - \sum_j \omega_j + i\epsilon\tau_3}B(q_0)\tau_3$$

which is the spectral form.

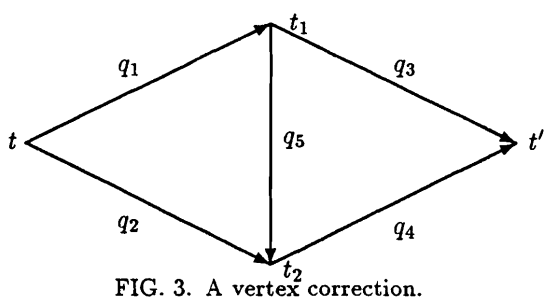

FIG. 3. A vertex correction.

Example 2. Fig. 3, one vertex correction in equilibrium calculations. We work out the retarded part with $t > t'$. If all the lines connected to vertex t' are to the right according to the time direction, there is a contribution of the form

$$\left(\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix}\begin{Bmatrix} 1 \\ \sigma \end{Bmatrix}\right) = \sigma B^{-1}(n)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} B(n)\tau_3$$

for any n and thus it gives no contribution to the diagonal part. This is to be discarded since if the spectral representation is valid it can be cancelled by other terms using the following

$$\sum_l B^{-1}(n(\nu_l))\begin{pmatrix} D_l(q_0) & O_l(q_0) \\ & D_l'(q_0) \end{pmatrix} B(n(\nu_l))$$
$$= B^{-1}(n(q_0))\begin{pmatrix} D_t & O_0 \\ & D_t' \end{pmatrix} B(n(q_0))$$

where l represents a time ordering and by assumption

$$O_0 = \sum_l \{O_l + [n(q_0) - n(\nu_l)](D_l - D_l')\} = 0,$$
$$D_t' = \sum_l D_l' = \sum_l D_l^* = D_t^*.$$

The spectral representation of a Feynmann diagram can be proven in two ways. First, it has been proven that any two point function satisfies spectral representation. If perturbation is valid, it should be satisfied order by order. The second proof relies on two stages. (1) the independence of theory in equilibrium on the choice of α which can be proven using the frequency formulation of the Green's functions Eq. (6). (2) the relationship among the two point functions. We have already proven that the matrix sandwiched between the B-matrices has only upper-right off-diagonal term when $\alpha = 1$ which we denote by O. When $\alpha = 1/2$, the off-diagonal terms have the following form

$$\begin{pmatrix} & C \\ \sigma C^* & \end{pmatrix}$$

which must be related to the $\alpha = 1$ term in the following manner

$$\begin{pmatrix} & C \\ \sigma C^* & \end{pmatrix} = e^{q_0\tau_3/4}\begin{pmatrix} & O \\ 0 & \end{pmatrix} e^{-q_0\tau_3/4}.$$

Hence $C^* = 0$, which leads to $O = 0$. Since the theory with $\alpha = 1/2$ is symmetric, we must have $D_t' = D_t^*$. In a separate paper we shall prove that only D is relevant in the calculation of the response functions [7].

In our present case there are only 4 possible ways to order the internal vertices.

(1) $t_1 > t_2 > t'$. All time directions are the same as the directions of the propagators we have a factor

$$n(\omega_1)n(\omega_2)n(\omega_3)n(\omega_4)n(\omega_5),$$

vertex t_1 with line 1 going in a factor $n^{-1}(\omega_1)$, vertex t_2 with lines 2 and 5 going in a factor $n^{-1}(\omega_2+\omega_5)$, vertex t' with lines 3 and 4 going in a factor $n^{-1}(\omega_3+\omega_4)$. Lines 1 and 2 project onto (t, t_1) a factor $(q_0 - \omega_1 - \omega_2 + i\epsilon)^{-1}$, lines 3, 5 and 2 project onto (t_1, t_2) a factor $(q_0 - \omega_3 - \omega_5 - \omega_2 + i\epsilon)^{-1}$, lines 3 and 4 onto (t_2, t') a factor $(q_0 - \omega_3 - \omega_4 + i\epsilon)^{-1}$. Collect all the factors we have

$$\frac{n^{-1}(\omega_3+\omega_4)n(\omega_3)n(\omega_4)}{(q_0-\omega_1-\omega_2+i\epsilon)(q_0-\omega_3-\omega_4+i\epsilon)}$$
$$\frac{n(\omega_5)n(\omega_2)n^{-1}(\omega_2+\omega_5)}{q_0-\omega_2-\omega_3-\omega_5+i\epsilon}$$

(2) Similarly when $t_2 > t_1 > t'$, the time direction of line 5 is opposite to the time direction of the propagator, so the sign of its energy is to be reversed, this gives an overall term

$$\frac{n^{-1}(\omega_3+\omega_4)n(\omega_3)n(\omega_4)}{(q_0-\omega_1-\omega_2+i\epsilon)(q_0-\omega_3-\omega_4+i\epsilon)}$$
$$\frac{-n(-\omega_5)n(\omega_1)n^{-1}(\omega_1-\omega_5)}{q_0+\omega_5-\omega_1-\omega_4+i\epsilon}.$$

(3) When $t_1 > t' > t_2$ we have a term

$$\frac{-n(\omega_2)n(-\omega_4)n(\omega_5)}{(q_0-\omega_1-\omega_2+i\epsilon)(q_0-\omega_2-\omega_3-\omega_5+i\epsilon)}$$
$$\frac{n^{-1}(\omega_2+\omega_5-\omega_4)}{(\omega_4-\omega_2-\omega_5)}.$$

(4) When $t_2 > t' > t_1$ we have a term

$$\frac{n(\omega_1)n(-\omega_3)n(-\omega_5)}{(q_0-\omega_1-\omega_2+i\epsilon)(q_0-\omega_1-\omega_4+\omega_5+i\epsilon)}$$
$$\frac{n^{-1}(\omega_1-\omega_5-\omega_3)}{(\omega_3+\omega_5-\omega_1)}.$$

Add up all the terms obtained above we have

$$B^{-1}(q_0)\begin{pmatrix} D(q_0) & \\ & D^*(q_0)\end{pmatrix}B(q_0)\tau_3$$

where

$$D=\frac{1}{(q_0-\omega_1-\omega_2+i\epsilon)}\left\{\frac{n^{-1}(\omega_3+\omega_4)n(\omega_3)n(\omega_4)}{(q_0-\omega_3-\omega_4+i\epsilon)}\right.$$
$$\left[\frac{n(\omega_5)n(\omega_2)/n(\omega_2+\omega_5)}{q_0-\omega_2-\omega_3-\omega_5+i\epsilon}-\frac{n(-\omega_5)n(\omega_1)/n(\omega_1-\omega_5)}{q_0+\omega_5-\omega_1-\omega_4+i\epsilon}\right]$$
$$+\frac{-n(\omega_2)n(-\omega_4)n(\omega_5)}{(q_0-\omega_2-\omega_3-\omega_5+i\epsilon)}\frac{n^{-1}(\omega_2+\omega_5-\omega_4)}{(\omega_4-\omega_2-\omega_5)}$$
$$\left.+\frac{n(\omega_1)n(-\omega_3)n(-\omega_5)}{(q_0-\omega_1-\omega_4+\omega_5+i\epsilon)}\frac{n^{-1}(\omega_1-\omega_5-\omega_3)}{(\omega_3+\omega_5-\omega_1)}\right\}. \qquad (9)$$

The rest is just momentum integration.

It is important to point out that we have only four terms to work with, while in Matsubara we need 10 terms to begin with. Another point to note is that we have terms of the following form

$$n^{-1}(\sum_i\omega_i)/\sum_i\omega_i=(e^{\sum_i\omega_i}-1)/\sum_i\omega_i.$$

This term does not give an imaginary part and does not diverge, the limit that $\sum_i\omega_i=0$ is β. This point is non-trivial when one calculates the correction of the density number N_H to the bare Bogolibov number n. Take the simplest interaction $H_i=\delta\omega a^+a$ for instance, the first order correction is

$$N_H(t)\approx n(\omega)-n^2(\omega)e^{\omega}i\delta\omega\lim_{\epsilon\to 0}\int_{-\infty}^{0}dt'(e^{\beta\epsilon}-1)e^{-i\epsilon t'}$$
$$=n(\omega)+n^2(\omega)e^{\omega}\beta\delta\omega,$$

provided that $\beta\delta\omega$ is sufficiently small. ϵ is the energy transfer. The same can happen in other situations.

In summary, we have presented a simple recipe to calculate Feynmann diagrams in TFD. Practical calculations become possible for more complicated diagrams. We have shown that calculations in TFD can be simpler than the imaginary time approach and causality is preserved even for quantum systems.

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