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AVERAGING OUT THE EINSTEIN EQUATIONS AND MACROSCOPIC SPACE-TIME GEOMETRY*

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Abstract

By averaging out Cartan's structure equations for a four-dimensional Riemannian space over space regions, the structure equations for the averaged space have been derived with the procedure being valid on an arbitrary Riemannian space. The averaged space is characterized by a metric, non-Riemannian and Riemannian curvature 2-forms, and correlation 2-, 3- and 4-forms. The procedure allows the space-time averaging of the Einstein equations. It is shown that the averaged Einstein equations can be put in the form of the Einstein equations with the conserved macroscopic energy-momentum tensor of a definite structure including, in particular, the correlation functions.

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In general relativity there exists the problem of self-consistent macroscopic description of gravity.[1] The way of constructing the equations for macroscopic gravity consists in averaging out Cartan's structure equations and the Einstein equations over space-time regions.

Averaging out Cartan's structure equations of a Riemannian space $\mathcal{M}$ for matrix-valued metric 0-form g, connection 1-form ω and curvature 2-form r by using the averaging scheme[2] and bilocal exterior calculus[3] yields the structure equations for an averaged space $\bar{\mathcal{M}}$. The connection 1-form on $\bar{\mathcal{M}}$ is taken as $\bar{\Omega} \equiv \langle\Omega\rangle$ where Ω is a bilocal extension[3] of ω with the coincidence limit $\lim \Omega = \omega$. The first equation for a coordinate 1-form basis θ takes the form $\bar{\Omega}\wedge\theta = 0$ that means the absence of torsion. Averaging out the second one gives the following structure relation:

$$M = R - \langle\Omega\wedge\Omega\rangle + \bar{\Omega}\wedge\bar{\Omega} \qquad (1)$$

where M is the curvature 2-form for the connection 1-form $\bar{\Omega}$, $M = d\bar{\Omega} + \bar{\Omega}\wedge\bar{\Omega}$, and R is the average of the microscopic curvature 2-form r, $R = \langle\mathcal{R}\rangle$ ($\mathcal{R}$ is a bilocal extension of r, $\lim \mathcal{R} = r$). With the 2-form R assumed to be a curvature 2-form for another connection 1-form Π, $R = d\Pi + \Pi\wedge\Pi$, the formula (1) can be considered to establish a relation between the curvature 2-forms M and R for the two connections that are distinguished by an affine deformation 1-form A. There are two alternative choices to identify the curvatures: (I) R is a non-Riemannian curvature and M is a Riemannian one, $A = \bar{\Omega} - \Pi$, and (II) conversely, $A = \Pi - \bar{\Omega}$. For the metric G of the averaged space the structure equations $D_{\Pi}G = N$ and $D_{\bar{\Omega}}G = 0$ hold (I), and conversely for (II), 1-form A being defined in terms of the non-metricity 1-form N (D_{Π} and $D_{\bar{\Omega}}$ are the exterior covariant derivatives with respect to Π and $\bar{\Omega}$).

The last equations for G (I) have the integrability conditions

$$GR + R^{T}G = -D_{\Pi}N \quad , \quad GM + M^{T}G = 0 \quad , \qquad (2a,b)$$

(T is a transposition sign) and there are similar ones for (II).

The equations for A (variants (I) and (II), respectively)

$$D_{\bar{\Omega}}A - A\wedge A = Q \quad , \quad D_{\Pi}A - A\wedge A = -Q \quad , \qquad (3)$$

where 2-form $Q = -\langle\Omega\wedge\Omega\rangle + \bar{\Omega}\wedge\bar{\Omega}$ (see (1)), are shown to be always integrable on $\bar{\mathcal{M}}$ and hence the affine deformation 1-form does always exist!

The algebraic identities for R, $R\wedge\theta = 0$ and $\mathrm{tr}R = 0$, together with $Q\wedge\theta = 0$ and $\mathrm{tr}Q = 0$ provide, in addition to (2b), the algebraic identities $M\wedge\theta = 0$ and $\mathrm{tr}M = 0$. Hence the Ricci tensors $M_{\alpha\beta} = M^{\gamma}{}_{\alpha\beta\gamma}$ and $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\beta\gamma}$ are symmetric. Note that the non-Riemannian geometry for the two cases (I) and (II) is always equiaffine since due to (3) Weyl's 1-form $\mathrm{tr}A$ is always gradient.

In averaging out the Bianchi identities $\mathcal{D}r = 0$, the equation $\mathcal{D}g = 0$ and its integrability condition $gr + r^{T}g = 0$, there appears the problem of splitting out $\langle\mathcal{R}\wedge\Omega\rangle$, $\langle\mathcal{G}\Omega\rangle$ and $\langle\mathcal{G}\mathcal{R}\rangle$ ($\mathcal{G}$ is a bilocal extension of g, $\lim \mathcal{G} = g$). The first splitting rule is achievable by introducing a 2-matrix-valued correlation 2-form

$$\mathsf{Z} = \langle\Omega\wedge\Omega\rangle - \bar{\Omega}\wedge\bar{\Omega} \quad , \qquad (4)$$

so that $Q = -\mathbb{C}\mathsf{Z}$. The structure equation for Z reads

$$D_{\bar{\Omega}}\mathsf{Z} = -\mathbb{C}\mathsf{Y} + 2\mathbb{P}(\langle\mathcal{R}\wedge\Omega\rangle - R\wedge\bar{\Omega}) \qquad (5)$$

which is simultaneously the splitting rule for $\langle\mathcal{R}\wedge\Omega\rangle$. Here $\mathbb{C}$ is a matrix contraction operator acting on a k-matrix-valued k-form $\mathsf{M}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\rho}{}_{\sigma}{}^{\dots}{}_{\dots}$ as $\mathbb{C}\mathsf{M} = \mathsf{M}^{\alpha}{}_{\delta}{}^{\delta}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\dots}{}_{\dots} - \mathsf{M}^{\alpha}{}_{\beta}{}^{\mu}{}_{\delta}{}^{\delta}{}_{\nu}{}^{\dots}{}_{\dots} + \dots$ and $\mathbb{P}$ is a matrix permutation one $\mathbb{P}\mathsf{M} = \frac{1}{k!}(\mathsf{M}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\rho}{}_{\sigma}{}^{\dots}{}_{\dots} - \mathsf{M}^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta}{}^{\rho}{}_{\sigma}{}^{\dots}{}_{\dots} + \mathsf{M}^{\rho}{}_{\sigma}{}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\dots}{}_{\dots}\dots)$.
The 3-matrix-valued correlation 3-form Y defined as

$$\mathsf{Y} = \langle\Omega\wedge\Omega\wedge\Omega\rangle - 3\mathbb{P}\mathsf{Z}\wedge\bar{\Omega} - \bar{\Omega}\wedge\bar{\Omega}\wedge\bar{\Omega} \quad . \qquad (6)$$

The contracted structure equation (5) turns out to average out the differential Bianchi identities $\mathrm{D}r = 0$, to yield the Bianchi identities for the curvature 2-form $\mathcal{M}$, $\mathrm{D}_{\bar{\Omega}}\mathcal{M} = 0$!

The correlation 3-form Y satisfies the structure equation

$$\mathrm{D}_{\bar{\Omega}}Y = -\mathbb{C}X + 6\mathbb{P}Z\wedge Z$$
$$- 3\mathbb{P}(\langle\Omega\wedge\mathcal{R}\wedge\Omega\rangle - \bar{\Omega}\wedge\langle\mathcal{R}\wedge\Omega\rangle - \langle\Omega\wedge\mathcal{R}\rangle\wedge\bar{\Omega} + R\wedge Z + \bar{\Omega}\wedge R\wedge\bar{\Omega}) \qquad (7)$$

which is simultaneously the splitting rule for $\langle\Omega\wedge\mathcal{R}\wedge\Omega\rangle$. Here the 4-matrix-valued correlation 4-form X defined as

$$X = \langle\Omega\wedge\Omega\wedge\Omega\wedge\Omega\rangle - 3\mathbb{P}Z\wedge Z - 4\mathbb{P}Y\wedge\bar{\Omega} - 6\mathbb{P}Z\wedge\bar{\Omega}\wedge\bar{\Omega} - \bar{\Omega}\wedge\bar{\Omega}\wedge\bar{\Omega}\wedge\bar{\Omega} \quad . \qquad (8)$$

The equations for Z and Y are shown to be always integrable on $\bar{\mathcal{M}}$. In a four-dimensional space there are no other correlation forms and structure equations. Given X and the structure of correlators in the right-hand sides of (5) and (7), these should be taken as differential equations to find Z and Y. For example, one possible choice is $\mathrm{D}_{\bar{\Omega}}Z = 0$ with integrability condition $\mathbb{P}(R\wedge Z - Z\wedge R) = 0$, which permits using in fact only Z and restricting the geometry of curvature R to a class defined by $\mathrm{D}_{\bar{\Omega}}R = 0$ and the condition.

To average out $\mathrm{D}g = 0$, for the fields c changing slowly on $\mathcal{M}$ (such as covariantly constant, Killing ones and all that) it is assumed to be the following splitting rule:

$$\langle\Omega\wedge C\rangle = \bar{\Omega}\wedge\bar{c} \quad . \qquad (9)$$

($\lim C = c$). The rule provides immediately for $\bar{g} = \langle\mathcal{G}\rangle$ and $\langle\mathcal{G}^{-1}\rangle$

$$\mathrm{D}_{\bar{\Omega}}\bar{g} = 0 \quad , \quad \mathrm{D}_{\bar{\Omega}}\langle\mathcal{G}^{-1}\rangle = 0 \qquad (10a,b)$$

(similary, a Killing tensor on $\mathcal{M}$ becomes that on $\bar{\mathcal{M}}$), which is clear from the geometric point of view - any averaging must conserve the symmetries of original space. Choosing without loss of generality $\bar{g} = G$, one has $\langle\mathcal{G}^{-1}\rangle \neq G^{-1}$ in general.

With an additional assumption $\langle\Omega\wedge\Omega\wedge C\rangle = \langle\Omega\wedge\Omega\rangle\wedge\bar{c}$ in agreement with (9), exterior derivative of (9) gives the following rule for splitting out $\langle\mathcal{R}\wedge C\rangle$, written here in indices for the case $\mathrm{D}c = 0$

$$\langle \mathcal{R}^{\alpha}{}_{\beta}\wedge \mathcal{C}^{\mu}{}_{\nu}\rangle - R^{\alpha}{}_{\beta}\wedge \bar{C}^{\mu}{}_{\nu} = - Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\delta}\wedge \bar{C}^{\delta}{}_{\nu} + \bar{C}^{\mu}{}_{\delta}\wedge Z^{\alpha}{}_{\beta}{}^{\delta}{}_{\nu} \quad . \tag{11}$$

This rule applied for g averages out $g\Gamma + \Gamma^{T} g = 0$, giving

$$GM + M^{T}G = 0 \quad , \quad M\langle \mathcal{G}^{-1}\rangle + \langle \mathcal{G}^{-1}\rangle M^{T} = 0 \quad , \tag{12a,b}$$

where the identification $\bar{g} = G$ is used, these equations being the integrability conditions of (10a,b)! Compare them also with (2b).

The splitting rule (11) turns out to allow one to average out the Einstein equations in the mixed form, which yields

$$G^{\alpha\epsilon}M_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}G^{\mu\nu}M_{\mu\nu} = - \ae T^{\alpha}_{\beta} \quad . \tag{13}$$

where the macroscopic energy-momentum tensor T^{α}_{β} is of the form

$$\ae T^{\alpha}_{\beta} = \ae\langle \mathcal{T}^{\alpha(\mathrm{micro})}_{\beta}\rangle - (Z^{\alpha}{}_{\mu\nu\beta} + \frac{1}{2}\delta^{\alpha}_{\beta}Q_{\mu\nu})\bar{g}^{\mu\nu} + U^{\alpha\epsilon}M_{\epsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}U^{\mu\nu}M_{\mu\nu} . \tag{14}$$

Here $Z^{\epsilon}{}_{\mu\nu\gamma} = 2Z^{\epsilon}{}_{\mu\alpha}{}^{\alpha}{}_{\nu\gamma}$ is a Ricci-tensor like object for 2-form Z, $Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} = Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma}\theta^{\gamma}\wedge\theta^{\sigma}$, $Q_{\mu\nu} = Q^{\epsilon}{}_{\mu\nu\epsilon}$, $\langle \mathcal{T}^{\alpha(\mathrm{micro})}_{\beta}\rangle$ is the averaged energy-momentum tensor and $U^{\alpha\beta} = \bar{g}^{\alpha\beta} - G^{\alpha\beta}$ is the tensor defined by simultaneous analysis[3] of (12b) and $MG^{-1} + G^{-1}M^{T} = 0$ (I) or $MG^{-1} + G^{-1}M^{T} = - G^{-1}D_{\bar{\Omega}}NG^{-1}$ (II) by means of algebraic classification of $\bar{g}^{\alpha\beta}$. Averaging out the contracted Bianchi identities provides[3] the equation of motion for the averaged energy-momentum

$$\ae\langle \mathcal{T}^{\epsilon(\mathrm{micro})}_{\beta}\rangle_{|\epsilon} = (Z^{\epsilon}{}_{\mu\nu\beta|\epsilon} + \frac{1}{2}Q_{\mu\nu|\beta})\bar{g}^{\mu\nu} \quad . \tag{15}$$

(bar is the covariant $\bar{\Omega}$-derivative). Taking the choice (I) with M and R being the induction and field tensors, one arrives[3] at

$$T^{\epsilon}_{\beta|\epsilon} = 0 \tag{16}$$

and the Einstein equations (13) as macroscopic ones with the macroscopic (continuously distributed) source of the form (14).

References

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