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Towards a Theory of Macroscopic Gravity

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By averaging out Cartan's structure equations for a fo dimensional Riemannian space over space regions, the struct equations for the averaged space have been derived with procedure being valid on an arbitrary Riemannian space. averaged space is characterized by a metric, Riemannian non-Riemannian curvature 2-forms, and correlation 2-, 3- 4-forms, an affine deformation 1-form being due to non-metricity of one of two connection 1-forms. Using procedure for the space-time averaging of the Einstein equatic brings the averaged ones with the terms of geometric correcti by the correlation tensors. The equations of motion for averag energy-momentum, obtained by averaging out the contracted Bianc identities, do also include such terms. Considering t gravitational induction tensor to be the Riemannian curvatu tensor (the non-Riemannian one is then the field tensor), theorem is proved which relates the algebraic structure of tl averaged microscopic metric with that of the induction tensor. I is shown that the averaged Einstein equations can be put in th form of the Einstein equations with the conserved macroscopi energy-momentum tensor of a definite structure including th correlation functions. By using the high-frequency approximatio of Isaacson with second-order correction to the microscopi metric, the self-consistency and compatibility of the equation and relations obtained have been shown. Macrovacuum turns out t be Ricci non-flat, the macrovacuum source being defined in terms of the correlation functions. In the high-frequency limit the equations are shown to become Isaacson's ones with the source as Isaacson's stress tensor for gravitational waves.

TRODUCTION

In general relativity there exists a problem of self-stent macroscopic description of gravity [1-3]. Just as in rodynamics, the process of constructing the field equations acroscopic gravity consists in averaging out the Einstein ions and includes two steps: a space-time averaging of the operator and a microscopic matter distribution. However, in al relativity, unlike electrodynamics, doing even the first one meets a number of difficulties of principle. Until now problem of the space-time averaging of the Einstein equations dered as microscopic in their physical meaning has not been d to some considerable content. All the attempts have been mainly in the framework of linearized theory and no results in fact been achieved for a full theory of macroscopic ity (see Refs. 2, p.215-288, 3, 4 and references therein[2]). Due to a correct setting of the problem [4] one should y, first of all, the problem of how to average out a ido-)Riemannian space itself, i.e. Cartan's structure tions. The questions are which average geometrical object - ic, connection or curvature - can characterize an averaged e and how to define a metric and a connection on the space. her important topic is the splitting of the averages of ucts of the objects, being found in averaging out Cartan's cture equations. It is the problem of introducing the

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In Ref. 4 a full list of the references is given and the ting approaches to treat the macroscopic gravitational fields related topics are discussed.

correlation functions. Upon deriving the structure equations for the averaged manifold, the Einstein equations, following the Riemannian character of the space-time geometry and known to be additional conditions to Cartan's structure equations, can successfully be averaged out. Then, the main problem one encounters is which average object plays the role of the average gravitational field and what object plays that of induction, if exists. The nonlinear character of the Riemannian geometry and, as a consequence, the Einstein equations may cause the change of geometrical and physical meaning of macroscopic gravity compared with microscopic Einsteinian one. Moreover, averaging out the equations of motion for matter (the energy-momentum conservation law) may lead to the appearance of additional terms of geometric correction (correlation functions) in the averaged ones.

This paper presents an approach for solving the problem (part of the results has been given in Refs. 3-6) and provides a system of the structure and field equations of a full geometric theory of macroscopic gravity. The paper is organized as follows. The formalism necessary for treating the space-time averages in a covariant manner is given in Sections 2-7. Sections 8 and 9 present a bilocal formulation of Cartan's structure equations of a Riemannian manifold. By averaging out the Cartan's equations the structure equations for the averaged space have been derived in Sections 10-12. It is in these sections that the correlation forms are introduced and a geometric theory of splitting out the averages of products of the metric, connection and curvature has been developed. The averaged Einstein equations, the equations of motion for the averaged matter, and the structure of the averaged space-time are discussed in Sections 13 and 14. The averaged

Einstein equations can be shown to be put in the form of the Einstein equations with the conserved macroscopic energy-momentum tensor of a definite structure including the correlation functions. Two last sections are devoted to the high-frequency approximation analysis of the equations and relations of the theory up to the fourth order in the amplitude of metric perturbation for the case of second-order corrected microscopic metric. The self-consistency and compatibility of them have been shown. In Section 17 an important result has been established: the macrovacuum equations being Ricci non-flat become Isaacson's ones in the high-frequency limit with the macrovacuum source coinciding with Isaacson's stress tensor for gravitational waves. It is a realization of the correspondence principle in the framework of the theory of macroscopic gravity.

2. THE AVERAGING SCHEME

The averaging scheme proposed in [7,8] and applied in [3,5] will be used throughout the paper, being generalized for an n-dimensional differentiable manifold $\mathcal{M}$. Let us consider a region $\Sigma\subset\mathcal{M}$ with a supporting point $x\in\Sigma$ chosen arbitrarily. Any bilocal operator $\mathcal{A} = \mathcal{A}^{\alpha'}_{\beta}\, e_{\alpha'}\, dx^{\beta}$, $\mathcal{A}^{\alpha'}_{\beta} = \mathcal{A}^{\alpha'}_{\beta}(x,x')$, satisfying the properties

$$\lim_{x'\to x} \mathcal{A} = d\mathcal{P} \quad , \tag{1a}$$

and, in $\mathcal{M}$ there locally exists a coordinate system in which

$$\mathcal{A} = e_{\alpha'}\, dx^{\alpha} \tag{1b}$$

(summation over α' and α), can be considered as an averaging operator (for an affine connection space the condition (1b) can be required globally in the limit of vanishing curvature; see

Refs. 3,8 for a Riemannian case). Such an operator, a bilo extension of Cartan's unit tensor $d\mathcal{P} = e_{\alpha}\otimes dx^{\alpha}$, establishes mapping of the tensor's values at $x'\in\Sigma$ to the supporting point For a vector field v on $\mathcal{M}$ it acts as $\mathcal{A}$: $v'\to \mathcal{V} = \mathcal{A}(v')$ for ev $x'\in\Sigma$, $\mathcal{V}$ being a scalar at x' and a vector at x. The average va of a tensor field $p = p^{\alpha\cdots}_{\beta\cdots} e_{\alpha}\otimes\ldots\otimes dx^{\beta}\otimes\ldots$ over Σ at supporting point $x\in\Sigma$ is defined as

$$\bar{p} = \frac{1}{V_{\Sigma}} \int_{\Sigma} p'(\mathcal{A}^{-1},\ldots,\mathcal{A},\ldots)\varepsilon' \equiv \langle p'(\mathcal{A}^{-1},\ldots,\mathcal{A},\ldots)\rangle \tag{2}$$

where $V_{\Sigma} = \int_{\Sigma}\varepsilon'$ is the volume of Σ and ε is the volume n-for both the ways of denoting the average (2), a bar and angul brackets, will be used throughout the paper. The condition (1 ensures the existence of an inverse operator, $\mathcal{A}(\mathcal{A}^{-1}) = d\mathcal{P}$ a $\mathcal{A}(\mathcal{A}^{-1})' = d'\mathcal{P}'$, and a correct limit of (2) as $V_{\Sigma}\to 0$, the lim $x'\to x$ for all $x'\in\Sigma$. The condition (1b) provides the "standar definition of the average (2)

$$\bar{p}^{\alpha\cdots}_{\beta\cdots}(x) = \frac{1}{V_{\Sigma}} \int_{\Sigma} p^{\alpha\cdots}_{\beta\cdots}(x')d^{n}x' \tag{3}$$

in the coordinate system with considering subsequently $\bar{p}^{\alpha\cdots}_{\beta\cdots}$ a the tensor components at x in any other coordinate system.

The average of a sum of tensors is equal to the sum of th averages and, consequently, the averaging commutes with contrac tion, as well as preserving the algebraic symmetry properties o a tensor. A tensor field q is shown [8] to be taken out of th averaging sign, $\overline{pq} = \bar{p}q$, if and only if this field is $\mathcal{A}$-constant, $q'(\mathcal{A}^{-1},\ldots,\mathcal{A},\ldots) = q$. Note that the concept of $\mathcal{A}$-constant fiel generalizes that of a covariantly constant field for which $\mathcal{A}^{\alpha'}_{\beta}$ is the parallel-displacement bivector $g^{\alpha'}_{\beta}$ [9].

To average out differential equations on a differentiable fold in accordance with the averaging scheme proposed, it is sary to be able, firstly, to write a bilocal extension of equations under study and, secondly, to express the average terior derivative of a form through the exterior derivative e average of the form.

The solution of the first problem requires a bilocal rior calculus [10] to be developed. For a differentiable fold the exterior derivative is known to be the only local erential operation. Considering on $\mathcal{M}$ a bilocal (k,l')-form . a k-form at x and an l-form at x', $1 \leqslant k,l \leqslant n$)

$$\Phi = \frac{1}{k!\,l!}\,\Phi_{\alpha\ldots\beta'\ldots}\,dx^{\alpha}\wedge\ldots dx^{\beta'}\wedge\ldots \quad ,$$

s introduce the antisymmetrized-at-x exterior derivative $\mathcal{W}$) acting according to the rule

$$d'\Phi(\wedge\mathcal{W}) = \frac{1}{k!\,l!}\,\Phi_{\alpha\ldots\beta'\ldots,\gamma'}\,\mathcal{W}^{\gamma'}_{\delta}\,dx^{\delta}\wedge dx^{\alpha}\wedge\ldots dx^{\beta'}\wedge\ldots \quad .$$

$\mathcal{W} = \mathcal{W}^{\alpha'}_{\beta}\,e_{\alpha'}\,dx^{\beta}$, $\mathcal{W}^{\alpha'}_{\beta} = \mathcal{W}^{\alpha'}_{\beta}(x,x')$, is another bilocal operator iding the displacement of d at x' to point x

$$\mathcal{W}(dx^{\alpha'}) \equiv \mathcal{W}^{\alpha'} = \mathcal{W}^{\alpha'}_{\beta}\,dx^{\beta} \quad . \tag{4}$$

same properties (1a,b) as for $\mathcal{A}$ should be required for $\mathcal{W}$.

On defining now a bilocal exterior derivative, $\dot{d}$, as

$$\dot{d} = d + d'(\wedge\mathcal{W}) \quad , \tag{5}$$

following important theorem takes place.

rem 1. The bilocal exterior derivative $\dot{d}$ is nilpotent, $\dot{d}\dot{d}$ = if and only if

$$\dot{d}\mathcal{W}^{\alpha'} = 0 \quad . \tag{6}$$

Proof is straightforward, using the property $(dd' + d'd)\Phi(\wedge\mathcal{W}) = 0$. It should be noted that in an analogy with (5) one can introduce the antisymmetrized-at-x' exterior derivative $d(\wedge\mathcal{W}^{-1})$ and another bilocal exterior derivative $\underset{+}{q} = d' + d(\wedge\mathcal{W}^{-1})$ which is also nilpotent, $\underset{+}{q}\underset{+}{q} = 0$, if and only if (6) holds. Both the bilocal devivatives are related by $\dot{d}(\wedge\mathcal{W}^{-1}) = \underset{+}{q}$ or $\underset{+}{q}(\wedge\mathcal{W}) = \dot{d}$.

There is a bilocal extension of the Maurer-Cartan equations

$$\dot{d}\mathcal{E}^{\alpha'} = -\tfrac{1}{2}\mathcal{B}^{\alpha'}_{\cdot\beta'\gamma'}\,\mathcal{E}^{\beta'}\wedge\mathcal{E}^{\gamma'} = -\tfrac{1}{2}\mathcal{B}^{\alpha'}_{\cdot\beta\gamma}\,dx^{\beta}\wedge dx^{\gamma}$$

for the (1,0')-form $\mathcal{E}^{\alpha'} = \mathcal{E}^{\alpha'}_{\beta}\,dx^{\beta}$, a bilocal generalization of the coordinate 1-form basis. Here $\mathcal{B}^{\alpha'}_{\beta\gamma} = 2(\mathcal{E}^{\alpha'}_{[\beta,\gamma]} + \mathcal{E}^{\alpha'}_{[\beta,\delta'}\mathcal{E}^{\delta'}_{\gamma]})$ are the bianholonomicity coefficients which can be shown [10] to serve as a measure of anholonomicity of the bilocal vector basis $e_{\alpha} + \mathcal{E}^{\beta'}_{\alpha}e_{\beta'}$, a bilocal generalization of the ordinary (local) one.

An operator $\mathcal{E}$ will be said to be biholonomic if $\dot{d}\mathcal{E}^{\alpha'} = 0$. In such a case $\mathcal{B}^{\alpha'}_{\beta\gamma} = 0$. Any operator $\mathcal{E}$ satisfying (1a,b) is biholonomic since in the coordinate system (1b) $\dot{d}\mathcal{E}^{\alpha'} = 0$ and $\mathcal{B}^{\alpha'}_{\beta\gamma} = 0$, these relations being tensorial. Futhermore, $\dot{d}\mathcal{E}^{\alpha'} = 0$ taken as the differential equations to find $\mathcal{E}^{\alpha'}_{\beta}$ turn out to be always integrable on $\mathcal{M}$ with the biholonomic operator $\mathcal{W}$ (6).

4. AN AVERAGING-REGION COORDINATION

To solve the second task and obtain the commutation formulae for derivation and averaging, it is necessary to define a scheme of comparison (coordination) of neighbouring averaging regions (a correspondence between the regions' points).

Let $\xi = d/d\lambda$ be a vector field on $\mathcal{M}$. Taking the ξ field at

the supporting point x of a region Σ, let us define the shift field for every point x' of the region due to [3,7]

$$S^{\alpha'} = \mathcal{W}^{\alpha'}(\xi) \tag{7}$$

where the same bilocal operator $\mathcal{W}$ as in (4) is suitable to be applied. Then, Lie-transporting Σ along the integral lines of the $S^{\alpha'}$ field on a parametric length $\Delta\lambda$ defines a family of regions $\Sigma(\Delta\lambda)$. Using such a family enables one to calculate a family of the average values (2) of a tensor and, upon applying the usual procedure from analysis, to define the directional and partial derivatives of the average tensor.

Thus, it is the bilocal operator $\mathcal{W}$ that allows one to calculate the shift field $S^{\alpha'}$ (7) for any averaging region in accordance with a given field ξ at the region's supporting point, and, by choosing such n linear independent fields, one can build a covering of a differentiable manifold by the averaging regions for all points $x \in \mathcal{M}$ as the supporting ones.

5. THE COMMUTATION FORMULA FOR EXTERIOR DERIVATIVE AND AVERAGING

As a result of using the averaging-region coordination the exterior derivative and averaging of a k-form α permute due to

$$\begin{aligned} d\bar{\alpha} = {} & \langle d'\alpha'(\mathcal{W},\mathcal{A},\ldots,\mathcal{A})\rangle + \langle \mathrm{div}_\varepsilon \mathcal{W} \wedge \alpha'(\mathcal{A},\ldots,\mathcal{A})\rangle - \\ & \langle \mathrm{div}_\varepsilon \mathcal{W}\rangle \wedge \bar{\alpha} + \tfrac{1}{2}\langle \alpha'(\mathcal{A}_\beta,\ldots,\mathcal{A}) \wedge B^\beta{}_\gamma\rangle \wedge dx^\gamma \end{aligned} \tag{8}$$

where $\mathrm{div}_\varepsilon \mathcal{W}$ is the ε-divergence [11] of the 1-form-valued-at-x 1-vector $\mathcal{W}^{\alpha'}$ at x', the $(0,0')$-form $\mathcal{A}_\beta = \mathcal{A}^{\alpha'}_\beta e_{\alpha'}$ and the bianholonomicity $(1,0')$-form $B^\beta{}_\gamma = \mathcal{A}^{-1\beta}{}_\varepsilon B^{\varepsilon'}{}_{\gamma\delta} dx^\delta$.

Upon taking the biholonomic averaging operator, $d\mathcal{A}^{\alpha'} = 0$, and keeping the regions' volumes constant during their coordination

$$\mathrm{div}_\varepsilon \mathcal{W} = 0 \tag{9}$$

the exterior derivative and averaging are seen to commute.

The commutation formula (8) proves to yield a natural property of the exterior derivative of an average form.

Theorem 2. For any k-form α on $\mathcal{M}$ the exterior derivative of the averaged k-form $\bar{\alpha}$ is closed, $dd\bar{\alpha} = 0$, if and only if the coordination operator $\mathcal{W}$ is biholonomic, $d\mathcal{W}^{\alpha'} = 0$.

Proof is straightforward, using (8). The geometric meaning of Theorem 2 is that, after shifting a region Σ along a contour built from the Lie-dragged vector fields, the region $\tilde{\Sigma}$ obtained coincides with the original one, $\tilde{\Sigma} = \Sigma$. If $d\mathcal{W}^{\alpha'} \neq 0$ it is not the case (an analogous theorem has been given for the case of Riemannian space in [8]; see also Ref. 3).

Thus, the biholonomocity of $\mathcal{W}$ provides a standard exterior calculus for the averaged forms, they behaving as the local functions of supporting point.

6. COMMUTATION OF COVARIANT EXTERIOR DERIVATIVE AND AVERAGING

Let us consider an affine connection space. For a tensor-valued k-form $p = p^{\alpha\ldots}_{\beta\ldots}\, e_\alpha \otimes \ldots \otimes dx^\beta \otimes \ldots$ with a k-form $p^{\alpha\ldots}_{\beta\ldots}$ there is the following commutation formula:

$$\begin{aligned} D\bar{p}^{\alpha\ldots}_{\beta\ldots} = {} & \langle d'\mathcal{P}^{\alpha\ldots}_{\beta\ldots}(\mathcal{W},\mathcal{A},\ldots,\mathcal{A})\rangle + \langle \mathrm{div}_\varepsilon \mathcal{W} \wedge \mathcal{P}^{\alpha\ldots}_{\beta\ldots}(\mathcal{A},\ldots,\mathcal{A})\rangle - \\ & \langle \mathrm{div}_\varepsilon \mathcal{W}\rangle \wedge \bar{p}^{\alpha\ldots}_{\beta\ldots} + (-1)^k \langle \mathcal{P}^{\alpha\ldots}_{\delta\ldots}(\mathcal{A},\ldots,\mathcal{A}) \wedge \mathcal{S}^\delta{}_\beta\rangle + \ldots - \\ & \langle \mathcal{S}^\alpha{}_\delta \wedge \mathcal{P}^{\delta\ldots}_{\beta\ldots}(\mathcal{A},\ldots,\mathcal{A})\rangle - \ldots + \tfrac{1}{2}\langle \mathcal{P}^{\alpha\ldots}_{\beta\ldots}(\mathcal{A}_\delta,\ldots,\mathcal{A}) \wedge B^\delta{}_\gamma\rangle \wedge dx^\gamma , \end{aligned} \tag{10}$$

where $\mathcal{P}^{\alpha\ldots}_{\beta\ldots} = p'(\mathcal{A}^{-1\alpha},\ldots,\mathcal{A}_\beta,\ldots)$ and the covariant exterior derivative is defined as $Dp^{\alpha\ldots}_{\beta\ldots} = dp^{\alpha\ldots}_{\beta\ldots} + (-1)^k p^{\alpha\ldots}_{\varepsilon\ldots} \wedge \omega^\varepsilon{}_\beta +$

$\beta\ldots$ with $\omega^{\alpha}{}_{\beta}$ being the connection 1-form. The so-called tural (1,0′)-form

$$S^{\alpha}{}_{\beta} = \mathcal{A}^{-1\alpha}{}_{\delta'}[D\mathcal{A}^{\delta'}_{\beta} + D'\mathcal{A}^{\delta'}_{\beta}(\mathcal{W})] \tag{11}$$

not vanish in a curved space and is expressible through the ture tensor of it. Presence of them in the formula (10) is tal importance and reflects the fact of non-commutativity of veraging and covariant exterior derivation of a tensor even e requires (9). (A commutation formula for the case of the iant derivative of a tensor in a Riemannian space has been ned in [7]; see also Ref. 3.)

Under condition (9) the formulae (8) and (10) reads

$$d\bar{p}^{\alpha\ldots}_{\beta\ldots} = \langle \mathring{d}[\mathcal{P}^{\alpha\ldots}_{\beta\ldots}(\mathcal{A},\ldots,\mathcal{A})]\rangle \quad . \tag{12}$$

the formulae have a correct local limit as $V_{\Sigma} \to 0$ and the cidence limit of (11) is $\lim\limits_{x'\to x} S^{\alpha}{}_{\beta} = 0$ (similar coincidence ts can be determined by using a theorem of Christensen [12]).

Upon assuming the coordinate systems (1b) for $\mathcal{A}$ and $\mathcal{W}$ to be same and the volume n-form ε to be standard, one arrives at

$$d\bar{p}^{\alpha\ldots}_{\beta\ldots} = \frac{1}{V_{\Sigma}}\int_{\Sigma} dp^{\alpha\ldots}_{\beta\ldots}(x')d^{n}x' \tag{13}$$

permutation of the operations of derivation and averaging. flat space (3) and (13) are "standard".

THE CHOICE OF THE OPERATORS

The biholonomicity condition (6) for the operator $\mathcal{W}$

$$\mathring{d}\mathcal{W}^{\alpha'} \equiv [d\mathcal{W}^{\alpha'}_{\beta} + d'\mathcal{W}^{\alpha'}_{\beta}(\mathcal{W})]\wedge dx^{\beta} = 0$$

ıld be considered as the equations in the unknowns $\mathcal{W}^{\alpha'}_{\beta}$ with boundary conditions (1a), these equations being integrable

due to $\mathring{d}\mathring{d} = 0$ and serving to find the x-dependence of $\mathcal{W}$. The "inverse" equations

$$q\mathcal{W}^{-1\alpha} \equiv [d'\mathcal{W}^{-1\alpha}{}_{\beta'} + d\mathcal{W}^{-1\alpha}{}_{\beta'}(\mathcal{W}^{-1})]\wedge dx^{\beta'} = 0$$

are integrable too and serve to find the x'-dependence of $\mathcal{W}$. A factorized solution for the equations has been obtained in [8] (see also Ref. 3)

$$\mathcal{W}^{\alpha'}_{\beta} = e^{\alpha'}_{i}e^{i}_{\beta} \quad \text{or} \quad \mathcal{W}^{\alpha'}_{\beta} = (\partial x^{\alpha'}/\partial\phi^{i})(\partial\phi^{i}/\partial x^{\beta}) \quad , \tag{14a,b}$$

(summation over $i = 1,\ldots,n$) where e^{α}_{i} is any viel-bien field with constant anholonomicity coefficients $C^{i}{}_{jk}$. The solution (14b) depends on n arbitrary scalar functions ϕ^{i}, freedom of choice of which allows one to satisfy the condition (9) as well. It should be noted that the well-known bivectors $g^{\alpha'}_{\beta}$ and $\sigma^{\alpha'}_{\beta}$ [9] have turned out to be neither biholomomic (6) nor volume-preserving (9) ones.

The question is how to choose the two bilocal operators $\mathcal{A}$ and $\mathcal{W}$. It is reasonable to take

$$\mathcal{A} = \mathcal{W} \quad ,$$

for, firstly, the only operator for averaging and averaging-region coordination is determined by the equations $\mathring{d}\mathcal{W}^{\alpha'} = 0$ (the averages can be treated as ordinary local functions) and $\mathrm{div}_{\varepsilon}\mathcal{W} = 0$ (the volume-dependent terms disappear) and, secondly, the factor $(\mathcal{W} - \mathcal{A})$ disappears (the first term in the right-hand sides of the commutation formulae is exactly the average of exterior derivative). Then, the commutation formula (12) reads

$$d\bar{p}^{\alpha\ldots}_{\beta\ldots} = \langle \mathring{d}\mathcal{P}^{\alpha\ldots}_{\beta\ldots}(\mathcal{W},\ldots,\mathcal{W})\rangle \quad . \tag{15}$$

8. A BILOCAL EXTENSION OF CONNECTION 1-FORM

Hereafter we will consider a 4-dimensional pseudo-Riemannian space $(\mathcal{M}, g_{\alpha\beta})$ with a metric $g_{\alpha\beta}$ of signature $(-,+,+,+)$.

Writing the structural (1,0')-form $S^{\alpha}{}_{\beta}$ (11) as $S^{\alpha}{}_{\beta} = \Omega^{\alpha}{}_{\beta} - \omega^{\alpha}{}_{\beta}$, one can observe that the bilocal object

$$\Omega^{\alpha}{}_{\beta} = W^{-1\alpha}{}_{\delta'}[dW^{\delta'}_{\beta} + D'W^{\delta'}_{\beta}(W)] \qquad (16)$$

behaves as a connection 1-form under coordinate transformations at x. The coincidence limit of (16) is

$$\lim_{x'\to x} \Omega^{\alpha}{}_{\beta} = \omega^{\alpha}{}_{\beta} \quad .$$

Therefore, it can be considered as a bilocal extension of the ordinary (local) connection 1-form. When averaged out the object $\langle\Omega^{\alpha}{}_{\beta}\rangle \equiv \bar{\Omega}^{\alpha}{}_{\beta} = \bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma}dx^{\gamma}$ is supposed to be an affine connection 1-form of an averaged space $\bar{\mathcal{M}}$, the affine connection coefficients $\bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma}$ being symmetric due to (6). Note that for the factorized form of $W^{\alpha'}_{\beta}$ (14b) there always exists the coordinate system (1b), $x^{\alpha} = \phi^{t}$, where $W^{\alpha'}_{\beta} = \delta^{\alpha}_{\beta}$ so that in the system $\bar{\Omega}^{\alpha}{}_{\beta}$ is manifestly the averaged $\omega^{\alpha}{}_{\beta}$ defined due to (3).

9. A BILOCAL EXTENSION OF CARTAN'S STRUCTURE EQUATIONS

Let us consider the vector-valued-at-x' (0,0')-form $W_{\gamma} \equiv W^{\alpha'}_{\gamma} e_{\alpha'}$, a bilocal generalization of the vector basis e_{α}. Then, the definition of the bilocal connection 1-form (16) is

$$dW_{\gamma} = W_{\beta}\Omega^{\beta}{}_{\gamma} \qquad (17a)$$

and the value of the last expression on (0,1')-form $W^{-1\alpha} = W^{-1\alpha}{}_{\beta'}dx^{\beta'}$ (the bilocal 1-form basis dual to W_{β}) is equal to

$$dW_{\gamma}(W^{-1\alpha}) = \Omega^{\alpha}{}_{\gamma} \quad . \qquad (17b)$$

The first and second bilocal structure equations read

$$\Omega^{\alpha}{}_{\beta}\wedge dx^{\beta} = 0 \quad , \qquad (18)$$

$$d\Omega^{\alpha}{}_{\beta} + \Omega^{\alpha}{}_{\varepsilon}\wedge\Omega^{\varepsilon}{}_{\beta} = \mathcal{R}^{\alpha}{}_{\beta} \quad , \qquad (19)$$

where the (2,0')-form $\mathcal{R}^{\alpha}{}_{\beta} = W^{-1\alpha}{}_{\gamma'}W^{\delta'}_{\beta}r^{\gamma'}{}_{\delta'}(W,W)$ is a bilocal extension of the curvature 2-form $r^{\alpha}{}_{\beta} = \frac{1}{2}r^{\alpha}{}_{\beta\gamma\delta}dx^{\gamma}\wedge dx^{\delta}$, $\lim_{x'\to x}\mathcal{R}^{\alpha}{}_{\beta} = r^{\alpha}{}_{\beta}$. The bilocal extension of the covariant constancy condition of the metric 0-form $g_{\alpha\beta}$, $Dg_{\alpha\beta} = 0$, is found to be

$$d\mathcal{G}_{\alpha\beta} - \mathcal{G}_{\alpha\varepsilon}\Omega^{\varepsilon}{}_{\beta} - \mathcal{G}_{\varepsilon\beta}\Omega^{\varepsilon}{}_{\alpha} = 0 \qquad (20)$$

where (0,0')-form $\mathcal{G}_{\alpha\beta} = W^{\gamma'}_{\alpha}W^{\delta'}_{\beta}g_{\gamma'\delta'}$, $\lim_{x'\to x}\mathcal{G}_{\alpha\beta} = g_{\alpha\beta}$.

The integrability conditions of (18)-(20) are of the form

$$\mathcal{R}^{\alpha}{}_{\beta}\wedge dx^{\beta} = 0 \quad , \qquad (21)$$

$$d\mathcal{R}^{\alpha}{}_{\beta} - \mathcal{R}^{\alpha}{}_{\varepsilon}\wedge\Omega^{\varepsilon}{}_{\beta} + \Omega^{\alpha}{}_{\varepsilon}\wedge\mathcal{R}^{\varepsilon}{}_{\beta} = 0 \quad , \qquad (22)$$

$$\mathcal{G}_{\alpha\varepsilon}\mathcal{R}^{\varepsilon}{}_{\beta} + \mathcal{G}_{\varepsilon\beta}\mathcal{R}^{\varepsilon}{}_{\alpha} = 0 \quad . \qquad (23)$$

10. METRIC, CONNECTION AND CURVATURE OF THE AVERAGED SPACE

Averaging out the structure equations (18)-(23) yields the structure equations for the averaged space $\bar{\mathcal{M}}$. The connection 1-form of the space is defined from (17a,b) as

$$\bar{\Omega}^{\alpha}{}_{\beta} = \langle dW_{\beta}(W^{-1\alpha})\rangle$$

The first equation is

$$\bar{\Omega}^{\alpha}{}_{\beta}\wedge dx^{\beta} = 0 \quad , \qquad (24)$$

that means the absence of torsion. The second one takes the form

$$\bar{R}^{\alpha}{}_{\beta} = R^{\alpha}{}_{\beta} - \langle\Omega^{\alpha}{}_{\varepsilon}\wedge\Omega^{\varepsilon}{}_{\beta}\rangle + \bar{\Omega}^{\alpha}{}_{\varepsilon}\wedge\bar{\Omega}^{\varepsilon}{}_{\beta} \qquad (25)$$

$M^\alpha{}_\beta$ is the curvature 2-form for the connection 1-form $\bar\Omega^\alpha{}_\beta$,

$$\Gamma^\alpha{}_\beta = d\bar\Omega^\alpha{}_\beta + \bar\Omega^\alpha{}_\epsilon\wedge\bar\Omega^\epsilon{}_\beta \quad , \tag{26a}$$

$^\alpha{}_\beta$ is the average of the microscopic curvature 2-form $\Gamma^\alpha{}_\beta$, $\langle\mathcal{R}^\alpha{}_\beta\rangle$. The averaged 2-form $R^\alpha{}_\beta$ is assumed to be a curvature m for another connection 1-form, $\Pi^\alpha{}_\beta$,

$$R^\alpha{}_\beta = d\Pi^\alpha{}_\beta + \Pi^\alpha{}_\epsilon\wedge\Pi^\epsilon{}_\beta \quad . \tag{26b}$$

according to [13] the formula (25) can be considered to lish a relation between the curvature 2-forms for the two ctions that are distinguished by an affine deformation m $A^\alpha{}_\beta = A^\alpha{}_{\beta\gamma}dx^\gamma$. There are two alternative possibilities to ify the curvature 2-forms:

$R^\alpha{}_\beta$ is a non-Riemannian curvature 2-form and $M^\alpha{}_\beta$ is a Riemannian one, $A^\alpha{}_\beta = \bar\Omega^\alpha{}_\beta - \Pi^\alpha{}_\beta$;

The forms are identified conversely, with $A^\alpha{}_\beta = \Pi^\alpha{}_\beta - \bar\Omega^\alpha{}_\beta$.

The affine deformation 1-form is defined in terms of the metricity 1-form $N_{\alpha\beta} = N_{\alpha\beta\gamma}dx^\gamma$ as

$$A^\alpha{}_{\beta\gamma} = \frac{1}{2}\, G^{\alpha\delta}(N_{\delta\beta\gamma} + N_{\delta\gamma\beta} - N_{\beta\gamma\delta}) \quad ,$$

e $G_{\alpha\beta}$ and $G^{\alpha\beta}$ are metric and inverse metric tensors of $\bar{\mathcal{M}}$ and

$$D_\Pi G_{\alpha\beta} = N_{\alpha\beta} \quad , \quad D_{\bar\Omega}G_{\alpha\beta} = 0 \quad , \tag{27a,b}$$

he case (a) and conversely in (b). (D_Π and $D_{\bar\Omega}$ denote the riant exterior derivatives with respect to connections $\Pi^\alpha{}_\beta$ $\bar\Omega^\alpha{}_\beta$, respectively). The integrability conditions of (27a,b) found to be

$$G_{\alpha\epsilon}R^\epsilon{}_\beta + G_{\epsilon\beta}R^\epsilon{}_\alpha = -\,D_\Pi N_{\alpha\beta} \quad , \quad G_{\alpha\epsilon}M^\epsilon{}_\beta + G_{\epsilon\beta}M^\epsilon{}_\alpha = 0 \tag{28a,b}$$

h analogous equations in the case (b).

According to (25) there are the following equations for $A^\alpha{}_\beta$:

$$\text{(a)}\ D_{\bar\Omega}A^\alpha{}_\beta - A^\alpha{}_\epsilon\wedge A^\epsilon{}_\beta = Q^\alpha{}_\beta \quad , \quad \text{(b)}\ D_\Pi A^\alpha{}_\beta - A^\alpha{}_\epsilon\wedge A^\epsilon{}_\beta = -\,Q^\alpha{}_\beta \tag{29}$$

where 2-form $Q^\alpha{}_\beta = -\langle\Omega^\alpha{}_\epsilon\wedge\Omega^\epsilon{}_\beta\rangle + \bar\Omega^\alpha{}_\epsilon\wedge\bar\Omega^\epsilon{}_\beta$. These equations can be shown to be always integrable on the averaged manifold $\bar{\mathcal{M}}$ and hence the affine deformation 1-form does always exist.

The algebraic identities for $R^\alpha{}_\beta$ which follow from the averaging out of the algebraic identitites for 2-form $\Gamma^\alpha{}_\beta$ are

$$R^\alpha{}_\beta\wedge dx^\beta = 0 \quad , \quad R^\alpha{}_\alpha = 0 \tag{30a,b}$$

and, together with $Q^\alpha{}_\beta\wedge dx^\beta = 0$ and $Q^\alpha{}_\alpha = 0$, they provide, in addition to (28b) or their analog in the case (b), the following algebraic identities for $M^\alpha{}_\beta$:

$$M^\alpha{}_\beta\wedge dx^\beta = 0 \quad , \quad M^\alpha{}_\alpha = 0 \tag{31a,b}$$

Therefore the Ricci tensors $M_{\alpha\beta} = M^\gamma{}_{\alpha\beta\gamma}$ and $R_{\alpha\beta} = R^\gamma{}_{\alpha\beta\gamma}$ are symmetric. It should be noted that the non-Riemannian geometry in the two cases (a) and (b) is always equi-affine since due to (29) Weyl's 1-form $A = A^\alpha{}_\alpha$ is always gradient.

Now there is a puzzle of how to average out the other structure equations: the Bianchi identities (22),

$$dR^\alpha{}_\beta - \langle\mathcal{R}^\alpha{}_\epsilon\wedge\Omega^\epsilon{}_\beta\rangle + \langle\Omega^\alpha{}_\epsilon\wedge\mathcal{R}^\epsilon{}_\beta\rangle = 0 \quad , \tag{32}$$

the conditions (20) of the covariant constancy of metric,

$$d\bar g_{\alpha\beta} - \langle\mathcal{G}_{\alpha\epsilon}\Omega^\epsilon{}_\beta\rangle - \langle\mathcal{G}_{\epsilon\beta}\Omega^\epsilon{}_\alpha\rangle = 0 \quad , \tag{33}$$

$\bar g_{\alpha\beta} \equiv \langle\mathcal{G}_{\alpha\beta}\rangle$, and their integrability conditions (23),

$$\langle\mathcal{G}_{\alpha\epsilon}\mathcal{R}^\epsilon{}_\beta\rangle + \langle\mathcal{G}_{\epsilon\beta}\mathcal{R}^\epsilon{}_\alpha\rangle = 0 \quad . \tag{34}$$

The relations seem mysterious and evidently it it necessary to somehow split out the averages of products of the geometrical objects - metric, connection and curvature.

11. CORRELATION FORMS AND THE DIFFERENTIAL BIANCHI IDENTITIES

Thus, the averaged manifold $\bar{\mathcal{M}}$ has two curvature 2-forms, $\mathbb{M}^\alpha{}_\beta$ and $R^\alpha{}_\beta$, and they satisfy the differential Bianchi identities

$$d\mathbb{M}^\alpha{}_\beta - \mathbb{M}^\alpha{}_\varepsilon \wedge \bar{\Omega}^\varepsilon{}_\beta + \bar{\Omega}^\alpha{}_\varepsilon \wedge \mathbb{M}^\varepsilon{}_\beta = 0 \quad , \tag{35}$$

$$dR^\alpha{}_\beta - R^\alpha{}_\varepsilon \wedge \Pi^\varepsilon{}_\beta + \Pi^\alpha{}_\varepsilon \wedge R^\varepsilon{}_\beta = 0 \quad . \tag{36}$$

Let us introduce a correlation 2-form[3]

$$Z^\alpha{}_\beta{}^\mu{}_\nu = \langle \Omega^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \rangle - \bar{\Omega}^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu \quad , \tag{37}$$

so that $Q^\alpha{}_\beta = - Z^\alpha{}_\varepsilon{}^\varepsilon{}_\beta$ [see (25) and (29)]. Taking the exterior derivative of the 2-form gives the structure equation for $Z^\alpha{}_\beta{}^\mu{}_\nu$

$$D_{\bar{\Omega}} Z^\alpha{}_\beta{}^\mu{}_\nu = - 2\mathbb{P} Y^\alpha{}_\delta{}^\delta{}_\beta{}^\mu{}_\nu + 2\mathbb{P}(\langle \mathcal{R}^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \rangle - R^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu) \tag{38}$$

which is simultaneously the splitting rule sought. Here $\mathbb{P}$ is a matrix permutation operator acting on a k-matrix-valued k-form as $\mathbb{P}M^\alpha{}_\beta{}^\mu{}_\nu{}^\rho{}_{\sigma\ldots} = \frac{1}{k!}(M^\alpha{}_\beta{}^\mu{}_\nu{}^\rho{}_{\sigma\ldots} - M^\mu{}_\nu{}^\alpha{}_\beta{}^\rho{}_{\sigma\ldots} + M^\rho{}_\sigma{}^\alpha{}_\beta{}^\mu{}_{\nu\ldots} - \ldots)$. Here $Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa$ is the correlation 3-form defined as

$$Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa = \langle \Omega^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \wedge \Omega^\theta{}_\varkappa \rangle - 3\mathbb{P}(Z^\alpha{}_\beta{}^\mu{}_\nu \wedge \bar{\Omega}^\theta{}_\varkappa) - \bar{\Omega}^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu \wedge \bar{\Omega}^\theta{}_\varkappa \ . \tag{39}$$

A marvellous fact is that the equation (38) contracted over β and μ, with (25) taken into account, turns out to average out (32), to yield the Bianchi identities (35) for $\mathbb{M}^\alpha{}_\beta$.

The correlation 3-form satisfies the structure equation

$$D_{\bar{\Omega}} Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa = - 3\mathbb{P} X^\alpha{}_\delta{}^\delta{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa + 6\mathbb{P}(Z^\alpha{}_\beta{}^\mu{}_\delta \wedge Z^\delta{}_\nu{}^\theta{}_\varkappa) + 3\mathbb{P}(\langle \mathcal{R}^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \wedge \Omega^\theta{}_\varkappa \rangle -$$
$$\langle \mathcal{R}^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \rangle \wedge \bar{\Omega}^\theta{}_\varkappa + \langle \mathcal{R}^\alpha{}_\beta \wedge \Omega^\theta{}_\varkappa \rangle \wedge \bar{\Omega}^\mu{}_\nu - R^\alpha{}_\beta \wedge Z^\mu{}_\nu{}^\theta{}_\varkappa + R^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu \wedge \bar{\Omega}^\theta{}_\varkappa) \tag{40}$$

---

[3] Note that the correlation 2-form, as well as the correlation forms (39) and (41), is a tensor object built up from the affine connection 1-form, which differs from an affine curvature 2-form.

which can be considered as a splitting rule for $\langle \mathcal{R}^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \wedge \Omega$
Here the correlation 4-form $X^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa{}^\tau{}_\varphi$ is defined as

$$X^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa{}^\tau{}_\varphi = \langle \Omega^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \wedge \Omega^\theta{}_\varkappa \wedge \Omega^\tau{}_\varphi \rangle - 3\mathbb{P}(Z^\alpha{}_\beta{}^\mu{}_\nu \wedge Z^\theta{}_\varkappa{}^\tau{}_\varphi) -$$
$$4\mathbb{P}(Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa \wedge \bar{\Omega}^\tau{}_\varphi) - 6\mathbb{P}(Z^\alpha{}_\beta{}^\mu{}_\nu \wedge \bar{\Omega}^\theta{}_\varkappa \wedge \bar{\Omega}^\tau{}_\varphi) - \bar{\Omega}^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu \wedge \bar{\Omega}^\theta{}_\varkappa \wedge \bar{\Omega}^\tau{}_\varphi$$

The equations for $Z^\alpha{}_\beta{}^\mu{}_\nu$ and $Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa$ have been shown to be alw integrable on $\bar{\mathcal{M}}$. Due to four-dimensionality of the space th are no other correlation forms and structure equations. Given correlation 4-form $X^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa{}^\tau{}_\varphi$ and the structure of correlators the right-hand sides of (38) and (40), these equations should taken to find $Z^\alpha{}_\beta{}^\mu{}_\nu$ and $Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa$. For example, one possible choi is to fix

$$D_{\bar{\Omega}} Z^\alpha{}_\beta{}^\mu{}_\nu = 0 \tag{42}$$

with the integrability conditions

$$\mathbb{P}(R^\alpha{}_\delta \wedge Z^\delta{}_\beta{}^\mu{}_\nu - Z^\alpha{}_\beta{}^\mu{}_\delta \wedge R^\delta{}_\nu) = 0 \quad , \tag{4}$$

which enables one to use in fact only the correlation 2-fo $Z^\alpha{}_\beta{}^\mu{}_\nu$, take

$$Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa = 0 \quad , \quad X^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\varkappa{}^\tau{}_\varphi = 0 \tag{42b,}$$

and restrict the geometry of the curvature 2-form $R^\alpha{}_\beta$ to a cla defined by the conditions

$$D_{\bar{\Omega}} R^\alpha{}_\beta \equiv dR^\alpha{}_\beta - R^\alpha{}_\varepsilon \wedge \bar{\Omega}^\varepsilon{}_\beta + \bar{\Omega}^\alpha{}_\varepsilon \wedge R^\varepsilon{}_\beta = 0 \tag{4}$$

and (43) [the integrability conditions of (44) are in (43)].

AVERAGED MICROSCOPIC METRIC TENSOR AND SPLITTING OUT THE
:RAGE OF METRIC TIMES CURVATURE

· average the metric structure equations (33) and (34), it
essary to be able to split out the averages of metric times
tion and of metric times curvature. Consider on a
nian manifold a class of slowly changing tensor fields,
ing the covariantly constant tensors (i.e. g-constant ones;
ction 2), Killing ones, and similar objects, which have
large-scale mode of length L and a small-scale mode λ,
$\ll L$, where d is the typical diameter of averaging regions.
e fields denoted as a k-form $c^{\mu\cdots}_{\nu\ldots}$ it has been assumed
] to be the following splitting rule:

$$\langle\Omega^{\alpha}{}_{\beta}\wedge c^{\mu\cdots}_{\nu\ldots}\rangle = \bar{\Omega}^{\alpha}{}_{\beta}\wedge\bar{c}^{\mu\cdots}_{\nu\ldots} \quad , \tag{45}$$

is a bilocal extension of $c^{\mu\cdots}_{\nu\ldots}$, $\lim\limits_{x'\to x} C^{\mu\cdots}_{\nu\ldots} = c^{\mu\cdots}_{\nu\ldots}$ (due to
ile a W-constant field is taken out of the averaging sign).
rule provides immediately for (33) and its analogue for $g^{\alpha\beta}$

$$D_{\bar{\Omega}}\bar{g}_{\alpha\beta} = 0 \quad , \quad D_{\bar{\Omega}}\bar{g}^{\alpha\beta} = 0 \tag{46a,b}$$

$\bar{g}^{\alpha\beta} \equiv \langle g^{\alpha\beta}\rangle$ (similary, a Killing tensor on $\mathcal{M}$ becomes that
), which is clear from the geometric point of view - any
ging must conserve the symmetries of an original space. The
rison of (46a,b) with (27b) shows that one can always put

$$\bar{g}_{\alpha\beta} = G_{\alpha\beta} \tag{47}$$

the case (a) from Section 10 with $\bar{g}^{\alpha\beta} \neq G^{\alpha\beta}$ and $\bar{g}_{\beta\gamma}\bar{g}^{\alpha\beta} \neq \delta^{\alpha}_{\gamma}$
neral.

The exterior derivative of (45) and an additional assumption

$$\langle\Omega^{\alpha}{}_{\beta}\wedge\Omega^{\mu}{}_{\nu}\wedge c^{\sigma\cdots}_{\rho\ldots}\rangle = \langle\Omega^{\alpha}{}_{\beta}\wedge\Omega^{\mu}{}_{\nu}\rangle\wedge\bar{c}^{\sigma\cdots}_{\rho\ldots} \tag{48}$$

in agreement with (45) (see Section 15 below) yield the following remarkable splitting rule:

$$\begin{aligned} -\langle\Omega^{\alpha}{}_{\beta}\wedge Đ_{\Omega}c^{\mu\cdots}_{\nu\ldots}\rangle + \bar{\Omega}^{\alpha}{}_{\beta}\wedge D_{\bar{\Omega}}\bar{c}^{\mu\cdots}_{\nu\ldots} + \langle\mathcal{R}^{\alpha}{}_{\beta}\wedge c^{\mu\cdots}_{\nu\ldots}\rangle - R^{\alpha}_{\beta}\wedge\bar{c}^{\mu\cdots}_{\nu\ldots} = \\ - Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\delta}\wedge\bar{c}^{\delta\cdots}_{\nu\ldots} - \ldots + \bar{c}^{\mu\cdots}_{\delta\ldots}\wedge Z^{\alpha}{}_{\beta}{}^{\delta}{}_{\nu} + \ldots \end{aligned} \tag{49}$$

where $Đ_{\Omega}$ is the bilocal covariant exterior derivative built from $đ$ and $\Omega^{\alpha}{}_{\beta}$. When applied for the case of metric tensor this rule averages out (34) and its analogue for $g^{\alpha\beta}$, giving

$$\bar{g}_{\alpha\epsilon}M^{\epsilon}{}_{\beta} + \bar{g}_{\epsilon\beta}M^{\epsilon}{}_{\alpha} = 0 \quad , \quad M^{\alpha}{}_{\epsilon}\bar{g}^{\epsilon\beta} + M^{\beta}{}_{\epsilon}\bar{g}^{\alpha\epsilon} = 0 \tag{50a,b}$$

that are exactly the integrability conditions of the equations (46a,b). Compare them also with (28b) and its analogue for $G^{\alpha\beta}$.

Thus, the scheme proposed allows the averaging out of a Riemannian space, which results in the appearance of the averaged one with a metric and two equi-affine connections, the affine deformation tensor being a result of non-metricity of one of the connections. The geometry of the space is described by the non-Riemannian and Riemannian curvatures and correlation tensors. It is remarkable that the number of the tensors is finite due to finite dimensionality of the space. Moreover, the procedure is seen to be valid on an arbitrary Riemannian manifold and does not require any restrictions on its curvature.

13. THE MACROSCOPIC EINSTEIN EQUATIONS

An intent look at the Einstein equations written in the form

$$g^{\alpha\epsilon}r_{\epsilon\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}g^{\mu\nu}r_{\mu\nu} = -\varkappa t^{\alpha(\mathrm{micro})}_{\beta} \tag{51}$$

reveals that the rule (49) is the single one needed to average out them. Then, the space-time averaging of (51) (see Ref. 3 for

details) yields the following equations [5,6]:

$$G^{\alpha\epsilon}\mathcal{M}_{\epsilon\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}G^{\mu\nu}\mathcal{M}_{\mu\nu} = -\varkappa T^{\alpha(macro)}_{\beta} \tag{52}$$

where the macroscopic stress tensor $T^{\alpha(macro)}_{\beta}$ is of the form

$$\varkappa T^{\alpha(macro)}_{\beta} = \varkappa\langle T^{\alpha(micro)}_{\beta}\rangle - (Z^{\alpha}{}_{\mu\nu\beta} + \tfrac{1}{2}\delta^{\alpha}_{\beta}Q_{\mu\nu})\bar{g}^{\mu\nu} + U^{\alpha\epsilon}\mathcal{M}_{\epsilon\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}U^{\mu\nu}\mathcal{M}_{\mu\nu} \tag{53}$$

Here $Z^{\alpha}{}_{\mu\nu\beta} = 2Z^{\alpha}{}_{\mu\rho}{}^{\rho}{}_{\nu\beta}$ is a Ricci-tensor like object for the correlation 2-form $Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} = Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\delta}dx^{\gamma}\wedge dx^{\delta}$, $Q_{\mu\nu} = Q^{\epsilon}{}_{\mu\nu\epsilon}$, $\langle T^{\alpha(micro)}_{\beta}\rangle$ is the averaged microscopic energy-momentum tensor, $\lim_{x'\to x} T^{\alpha(micro)}_{\beta} = t^{\alpha(micro)}_{\beta}$, and $U^{\alpha\beta} = \bar{g}^{\alpha\beta} - G^{\alpha\beta}$ (see next section to inquire about the structure of the tensor).

An averaging of the energy-momentum conservation law means that of the contracted Bianchi identities $g^{\epsilon\gamma}r_{\epsilon\beta;\gamma} - \tfrac{1}{2}g^{\mu\nu}r_{\mu\nu;\beta} = 0$. In order to carry out that, it is necessary to be able to split out the averages $\langle\mathcal{R}^{\alpha}{}_{\beta}\wedge\Omega^{\mu}{}_{\nu}\wedge C^{\sigma\ldots}_{\rho\ldots}\rangle$. Such a rule is obtainable by the exterior derivation of (48) [see relations (24) of Ref. 3] and when applied gives the averaged contracted Bianchi identities in the form

$$\bar{g}^{\alpha\epsilon}\mathcal{M}_{\alpha\beta|\epsilon} - \tfrac{1}{2}\bar{g}^{\mu\nu}\mathcal{M}_{\mu\nu|\beta} = 0 \tag{54}$$

and, as a consequence, the equations of motion for the averaged energy momentum

$$\varkappa\langle T^{\epsilon(micro)}_{\beta}\rangle_{|\epsilon} = (Z^{\epsilon}{}_{\mu\nu\beta|\epsilon} + \tfrac{1}{2}Q_{\mu\nu|\beta})\bar{g}^{\mu\nu} \tag{55}$$

(the bar denotes the covariant derivative with respect to the connection coefficients $\bar{\Gamma}^{\alpha}_{\beta\gamma}$)

Taking the choice (a) from Section 10 with $R^{\alpha}{}_{\beta\gamma\delta}$ and $\mathcal{M}^{\alpha}{}_{\beta\gamma\delta}$ as the average field and induction tensors, respectively, one has been shown by using $D_{\bar{\Omega}}G^{\alpha\beta} = 0$, (46b), (54) and (55) to arrive at the conservation law

$$T^{\epsilon(macro)}_{\beta|\epsilon} = 0$$

and the Einstein equations (52) as macroscopic ones with macroscopic (continuously distributed) source of the form (53)

The appearance of the additional terms in (55) compared w the microscopic conservation law $t^{\epsilon(micro)}_{\beta;\epsilon} = 0$ is connected w the changing of the geometrical structure of the macrosco space-time as a whole, although the field equations (52) macroscopic gravity remain the Einstein equations and have k their Riemannian character. The space-time acquires some properties described, in particular, by the tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}$ which has emerged as an independent geometric object. From t physical point of view this object is likely to represent tensor of elastic moduli for the macroscopic space-time. Due eqs. (55) it influences the behaviour of continuously distribut matter on the averaged manifold. Being included into t macroscopic stress tensor (53) these correlation terms can considered as providing a geometric correction of the averag energy momentum.

Upon defining the macrovacuum state of macroscop gravitation as the state corresponding to the microvacuum o after an averaging out, the macrovacuum equations following fr the microvacuum equations $r_{\alpha\beta} = 0$ and from eqs. (52) read

$$\mathcal{M}_{\alpha\beta} = Q_{\alpha\beta} \quad , \quad \bar{g}^{\alpha\epsilon}Q_{\beta\epsilon} = Z^{\alpha}{}_{\mu\nu\beta}\bar{g}^{\mu\nu} \quad . \tag{57a,}$$

It follows from (57b) that $\bar{g}^{\alpha\beta}Q_{\alpha\beta} = 0$.

In view of the importance of the correlation 2-form $Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}$ is worth writing here its algebraic properties

$$Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} = -Z^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta} \quad ,$$

$$Z^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}\wedge dx^{\nu} = 0 \quad ,$$

$$^{\alpha\ \varepsilon}_{\ \varepsilon\ \nu} = - Q^{\alpha}_{\ \nu} \quad ,$$

$$^{\varepsilon\ \mu}_{\ \varepsilon\ \nu} = 0 \quad ,$$

st relation being a consequence of equi-affinity. The $Z^{\mu}_{\ \nu\rho\sigma}$ has the following properties:

$$^{\alpha}_{\ [\beta\sigma]\nu} = - \tfrac{1}{2} Q^{\alpha}_{\ \nu\beta\sigma} \quad ,$$

$$^{\alpha}_{\ \beta[\nu\sigma]} = 0 \quad ,$$

$$^{\varepsilon}_{\ \varepsilon\nu\sigma} = 0 \quad ,$$

$$^{\varepsilon}_{\ \beta\varepsilon\sigma} = - Q_{\beta\sigma} \ .$$

IE STRUCTURE OF MACROSCOPIC SPACE-TIME

The choice (a) taken allows one to prove an important em on the algebraic structure of the averaged (macroscopic) -time and of the tensor $U^{\alpha\beta} = \bar{g}^{\alpha\beta} - G^{\alpha\beta}$ appearing in the scopic stress tensor (53). Indeed, the simultaneous analysis 0b) and the relations $M^{\alpha}_{\ \varepsilon} G^{\varepsilon\beta} + M^{\beta}_{\ \varepsilon} G^{\alpha\varepsilon} = 0$, the integrability tions of $D_{\bar{\Omega}} G^{\alpha\beta} = 0$ [note that (28b) and (50a) are identical dopting (47)], by means of the invariant algebraic ification [14] of the symmetric tensor $\bar{g}^{\alpha\beta}$ through the scopic metric tensor $G^{\alpha\beta}$ of signature (−,+,+,+) yields the wing theorem.

em 3. Due to relations (50b) there is the following ondence between the algebraic structure of averaged tensor nd that of a Riemannian space with curvature tensor $M^{\alpha}_{\ \beta\rho\sigma}$:

All four algebraic types A_1, A_2, A_3 and B without generations lead to the existence of four linearly independent covariantly constant vectors so that $M^{\alpha}_{\ \beta\rho\sigma} = 0$.

(B) For all the cases with generations, excepting $A_1[(111,1)]$ and $A_3[(11,2)]$, the non-trivial components of $M^{\alpha}_{\ \beta\rho\sigma}$ are expressible through the Ricci tensor $M_{\alpha\beta}$ and the space-times are algebraically special and reducible (there are different types,[4] depending on the algebraic types of $\bar{g}^{\alpha\beta}$).

(C) For the case $A_3[(11,2)]$ when

$$\bar{g}^{\alpha\beta} = \text{const.} G^{\alpha\beta} - 2\xi^{\alpha}\xi^{\beta} \tag{58a}$$

space-time has a covariantly constant null vector ξ^{α}, i.e. $G^{\alpha\beta}$ is a pp-wave metric [14]; for the case $A_1[(111,1)]$ when

$$\bar{g}^{\alpha\beta} = \text{const.} G^{\alpha\beta} \tag{58b}$$

there are no conditions on the curvature tensor and space-time structure.

The theorem gives a classification of possible macroscopic space-times and manifests the dependence between the algebraic properties of microscopic and macroscopic space-times - the more symmetric the original manifold, the more symmetric the final manifold. In the case (A) the microscopic space-time evidently has no order and therefore the induction $M^{\alpha}_{\ \beta\gamma\delta}$ vanishes. For all the cases (B) and (C) it has symmetries, which results in a non-trivial structure of macroscopic space-times, and the form of the tensor $U^{\alpha\beta}$ from (53) is manifestly established, in the most symmetric case (58b) this tensor being equal to zero (const.= 1).

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4 See Appendix of Ref. 3 with changing $\bar{g}_{\alpha\beta}$ for $\bar{g}^{\alpha\beta}$, which provides the full list of all the possibilities.

15. THE HIGH-FREQUENCY APPROXIMATION ANALYSIS

Let us study the self-consistency and compatibility of the equations and relations obtained, using the high-frequency approximation of R. Isaacson [15-17]. According to the approach there exists a family of the geometries for which the microscopic metric takes the form

$$g_{\alpha\beta} = G_{\alpha\beta} + h_{\alpha\beta} \tag{59}$$

where $G_{\alpha\beta}$ is the macroscopic metric (called the background one in Ref. 15) slowly changing over a characteristic macroscopic length L and $h_{\alpha\beta}$ is the high-frequency oscillating part (a ripple) with a microscopic lenght λ, $\lambda \ll d \ll L$, where d is the typical diameter of averaging regions. The Brill-Hartle (BH) space-time averaging of Refs. 16,17 was carried out over the background's regions, i.e. the integration measure was $[-\det(G_{\alpha\beta})]^{1/2}$, and $\mathcal{A}^{\alpha'}_{\beta} = g^{\alpha'}_{\beta}$, whereas the averaging over the microscopic space-time regions with the measure $[-\det(g_{\alpha\beta})]^{1/2}$ and $\mathcal{A}^{\alpha'}_{\beta} = W^{\alpha'}_{\beta}$ is applied in this paper. For convenience, a coordinate system (59) can be chosen, without loss of generality, as the system (1b) for both the cases, which enables one to use (3) and (13) in what follows.

The averages of $h_{\alpha\beta}$ and its derivatives evidently vanish

$$\langle h_{\alpha\beta}\rangle_{BH} = 0 \quad , \quad \langle \partial_\gamma h_{\alpha\beta}\rangle_{BH} = 0 \quad , \ldots , \tag{60a}$$

$$\langle h_{\alpha\beta}\rangle = 0 \quad , \quad \langle \partial_\gamma h_{\alpha\beta}\rangle = 0 \quad , \ldots , \tag{60b}$$

while averaging out the microscopic metric (59) gives

$$\langle g_{\alpha\beta}\rangle_{BH} = G_{\alpha\beta} - \tfrac{1}{2}\langle h_{\alpha\beta}h^{\mu}_{\mu}\rangle_{BH} - \tfrac{1}{8}G_{\alpha\beta}\langle h_{[\nu}{}^{\nu}h^{\mu}_{\mu]}\rangle_{BH} + \ldots \quad , \tag{61a}$$

$$\langle g_{\alpha\beta}\rangle \equiv \bar{g}_{\alpha\beta} = G_{\alpha\beta} \quad , \tag{61b}$$

Here the extra terms in (61a) compared with (61b) emerge due to

$$\det(g_{\alpha\beta}) = \det(G_{\alpha\beta})[1 + h^{\alpha}_{\alpha} + \tfrac{1}{2}h_{[\alpha}{}^{\alpha}h^{\beta}_{\beta]} + \tfrac{1}{3!}h_{[\alpha}{}^{\alpha}h^{\beta}_{\beta}h^{\gamma}_{\gamma]} + \det(h_{\alpha\beta})] \ .$$

The averages of the inverse microscopic metric

$$g^{\alpha\beta} = G^{\alpha\beta} - G^{\alpha\rho}G^{\beta\sigma}h_{\rho\sigma} + G^{\alpha\rho}G^{\epsilon\sigma}G^{\beta\varkappa}h_{\rho\sigma}h_{\varkappa\epsilon} - \ldots$$

have the following forms

$$\langle g^{\alpha\beta}\rangle_{BH} = G^{\alpha\beta} + \tfrac{1}{2}\langle h^{\alpha\beta}h^{\mu}_{\mu}\rangle_{BH} - \tfrac{1}{8}G^{\alpha\beta}\langle h_{[\nu}{}^{\nu}h^{\mu}_{\mu]}\rangle_{BH} \quad , \tag{62a}$$

$$\langle g^{\alpha\beta}\rangle = G^{\alpha\beta} + \langle h^{\alpha\epsilon}h^{\beta}_{\epsilon}\rangle - \ldots \tag{62b}$$

where $\langle g^{\alpha\beta}\rangle \equiv \bar{g}^{\alpha\beta}$. Compare (61b) and (62a,b) with the results of Sections 12 and 14 to elucidate the origin of tensor $U^{\alpha\beta}$.

Now let us consider the second-order corrected metric

$$g_{\alpha\beta} = G_{\alpha\beta} + h_{\alpha\beta} + j_{\alpha\beta} \tag{63}$$

Taking the amplitudes of $g_{\alpha\beta}$, $h_{\alpha\beta}$ and $j_{\alpha\beta}$ as $O(1)$, a and a^2, respectively, $a \ll 1$, let us expand all microscopic objects in powers of a up to a^4, keeping for each object only dominant terms

$$\underset{O(1)}{g} = \underset{a}{G} + \underset{a^2}{h} + j \quad , \tag{64a}$$

$$\underset{O(1)}{g^{-1}} = \underset{a}{G^{-1}} + \underset{a^2}{G^{-2}h} + \underset{a^3}{(g^{-1})^{(2)}(h^2,j)} + \underset{a^4}{(g^{-1})^{(3)}(h^3,hj)} + (g^{-1})^{(4)}(h^4,h^2j,j^2) \quad , \tag{64b,c,d}$$

$$\underset{L^{-1}}{\gamma} = \bar{\mathcal{F}} + \underset{a\lambda^{-1}}{\gamma^{(1)}(h)} + \underset{a^2\lambda^{-1}}{\gamma^{(2)}(h^2,j)} + \underset{a^3\lambda^{-1}}{\gamma^{(3)}(h^3,hj)} + \underset{a^4\lambda^{-1}}{\gamma^{(4)}(h^4,h^2j,j^2)} \ ,$$

$$\underset{L^{-2}}{r} = \mathcal{M} + \underset{a\lambda^{-2}}{r^{(1)}(h)} + \underset{a^2\lambda^{-2}}{r^{(2)}(h^2,j)} + \underset{a^3\lambda^{-2}}{r^{(3)}(h^3,hj)} + \underset{a^4\lambda^{-2}}{r^{(4)}(h^4,h^2j,j^2)} \ .$$

Here all the expansions are obtained by using symbolically the structure of the inverse metric $g^{-1}g = 1$, the Christoffel symbols $\gamma = g^{-1}\partial g$ and the Riemann tensor $r = g^{-1}\partial^2 g + g^{-2}(\partial g)^2$; $\bar{\mathcal{F}}$ and $\mathcal{M}$ are the connection coefficients and curvature of background (see

tions 10 and 12)

$$\bar{\mathcal{F}} = G^{-1}\partial G \quad , \quad \mathcal{M} = G^{-1}\partial^2 G + G^{-2}(\partial G)^2 \quad . \tag{65a,b}$$

aking into account the structure of (63) with h of frequency nd j of ω and 2ω [16], the averaging of (64a-d) brings

$$\rangle = \underset{O(1)}{G} \, , \tag{66a}$$

$$^{-1}\rangle = \underset{O(1)}{G^{-1}} + \underset{a^2}{\langle (g^{-1})^{(2)}(h^2)\rangle} + \underset{a^4}{\langle (g^{-1})^{(4)}(h^4, h^2 j, j^2)\rangle} \, , \tag{66b}$$

$$\rangle = \underset{L^{-1}}{\bar{\mathcal{F}}} + \underset{a^2 L^{-1}}{\langle \gamma^{(2)}(h^2)\rangle} + \underset{a^3\lambda^{-1}}{\langle \gamma^{(3)}(hj)\rangle} + \underset{a^4 L^{-1}}{\langle \gamma^{(4)}(h^4, h^2 j, j^2)\rangle} \, , \tag{66c}$$

$$\rangle \equiv R = \underset{L^{-2}}{\mathcal{M}} + \underset{a^2\lambda^{-2}}{\langle r^{(2)}(h^2)\rangle} + \underset{a^3\lambda^{-1}L^{-1}}{\langle r^{(3)}(hj)\rangle} + \underset{a^4\lambda^{-2}}{\langle r^{(4)}(h^4, h^2 j, j^2)\rangle}, \tag{66d}$$

e the properties $\langle h^2\rangle \neq 0$, $\langle h\partial h\rangle = 0$, $\langle(\partial h)^2\rangle \neq 0$, $\langle j\rangle = 0$, $\rangle = 0$, $\langle jh\rangle = 0$, $\langle j\partial h\rangle \neq 0$, $\langle h\partial j\rangle \neq 0$ and so on have been d. Note that although the leading terms for $\gamma^{(2)}$ and $\gamma^{(4)}$ in c) and $r^{(3)}$ in (64d) are $a^2\lambda^{-1}$, $a^4\lambda^{-1}$ and $a^3\lambda^{-2}$, respectively, se for $\langle\gamma^{(2)}\rangle$, $\langle\gamma^{(4)}\rangle$ and $\langle r^{(3)}\rangle$ in (66c,d) turn out to be $^{-1}$, $a^4 L^{-1}$ and $a^3\lambda^{-1}L^{-1}$. The formula (66d) clears up the origin the relations (25). The use of (64a-d), (65a,b) and (66a-d) the structure equations (38) and (40) leads to the following er-of-magnitude estimations:

$$\underset{\lambda^{-2}L^{-1}+a^3\lambda^{-1}L^{-2}+a^4\lambda^{-2}L^{-1}}{\partial Z} = \underset{a^4\lambda^{-2}L^{-1}}{-2Y} + \underset{a^3\lambda^{-3}+a^4\lambda^{-2}L^{-1}}{2\langle\gamma r\rangle - 2\langle\gamma\rangle\langle r\rangle} \, , \tag{67a,b}$$

$$\underset{\lambda^{-2}L^{-2}}{\partial Y} = \underset{a^4\lambda^{-4}}{-3X + 6Z\wedge Z} - 3\langle\gamma\gamma r\rangle + \underset{a^4\lambda^{-4}}{3\langle\gamma\gamma\rangle\langle r\rangle - 6\langle\gamma\rangle\langle\gamma r\rangle} + 6\langle\gamma\rangle\langle\gamma\rangle\langle r\rangle .$$

e that Y has not any terms in a^3 power. The expansions for the itting rules (45), (48) and (49) are

$$\langle\gamma\rangle\langle g\rangle \simeq a^2 L^{-1} + a^3\lambda^{-1} + a^4 L^{-1} \quad , \tag{68a}$$

$$\langle\gamma g\rangle - \langle\gamma\gamma\rangle\langle g\rangle \simeq a^2 L^{-2} + a^3\lambda^{-1}L^{-1} + a^4\lambda^{-2} \quad , \tag{68b}$$

$$\underset{a^2\lambda^{-2}+a^3\lambda^{-1}L^{-1}+a^4\lambda^{-2}}{\langle rg\rangle - \langle r\rangle\langle g\rangle} = \underset{a^2\lambda^{-2}+a^3\lambda^{-1}L^{-1}+a^4\lambda^{-2}}{2Z\langle g\rangle} \quad , \tag{68c}$$

where sign $\simeq$ denotes the order of magnitude.

The formulae show that the leading terms in every power in a under study are a^2/λ^2, a^3/λ^3 and a^4/λ^4, respectively. Due to (68a,b) the rules (45) and (48) are seen to take place always in powers a^2, a^3 and a^4 with the error less than that of the averaging, λL^{-1}. The expressions (67a,b) and (68c) manifest the self-consistency of the structure equations for Z and Y and the splitting rule for $\langle rg\rangle$. Besides, one can see validity of the choice (42a-c) in considering effectively the theory in a^2 or a^3 powers in the amplitude of the metric corrections of the form (59) and (63). This high-frequency analysis carried out has revealed the compatibility of the equations and relations within the theory of macroscopic gravity and the results of the high-frequency analysis can be shown to remain valid for the higher-order corrections to (63).

16. THE HIGH-FREQUENCY LIMIT OF MACROVACUUM

The macroscopic equations (57a) state a Ricci non-flat character of macrovacuum, the macrovacuum source being defined in terms of the correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma}$, $Q_{\alpha\beta} = -Z^{\varepsilon}{}_{\alpha\sigma}{}^{\sigma}{}_{\varepsilon\beta}$. It seems somewhat puzzling and requires one to clear up the physical meaning of $Q_{\alpha\beta}$.

Let us take the high-frequency limit of the equation (57a), using the results of Section 15 up to a^2 for the metric (59). The tensor $Q_{\alpha\beta}$ can be shown to have the form

$$Q^{HF}_{\alpha\beta} = -\langle\delta\gamma^{\varepsilon}{}_{\alpha\sigma}\delta\gamma^{\sigma}{}_{\varepsilon\beta} - \delta\gamma^{\varepsilon}{}_{\alpha\beta}\delta\gamma^{\sigma}{}_{\varepsilon\sigma}\rangle \quad , \tag{69}$$

with

$$\delta\gamma^{\alpha}{}_{\beta\gamma} \equiv \gamma^{\alpha}{}_{\beta\gamma} - \bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}G^{\alpha\delta}(h_{\beta\delta|\gamma} + h_{\gamma\delta|\beta} - h_{\beta\gamma|\delta}) \quad . \tag{70}$$

After calculation of $Q^{HF}_{\alpha\beta}$ in the Lorentz gauge $h^{\mu\nu}{}_{|\nu} = 0$ and $h^{\alpha}_{\alpha} = 0$ by using (70) and the averaging rules [16,17] (note that up to a^2 there is no distinction between the Brill-Hartle averaging and the one adopted here), one can derive the remarkable result

$$Q^{HF}_{\alpha\beta} = -\tfrac{1}{4}\langle h^{\mu\nu}{}_{|\alpha} h_{\mu\nu|\beta}\rangle = -\varkappa T^{(GW)}_{\alpha\beta} \tag{71}$$

where $T^{(GW)}_{\alpha\beta}$ is Isaacson's stress tensor [16,17] for gravitational waves. Thus, the object $Q_{\alpha\beta}$ built up from the correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma}$ is seen to represent the gravitational stress tensor for macrovacuum. The tensor Ricci $M_{\alpha\beta}$ has been shown in preceding section to describe the background and within accuracy of the approximation [see (66d)] one has $Q_{\alpha\beta}G^{\alpha\beta} = 0$ from the contracted version of (57b) and $M_{\alpha\beta}G^{\alpha\beta} = 0$ from (57a). Then, the macrovacuum equations (57a) in the high-frequency limit are found to become

$$M_{\alpha\beta} = -\varkappa T^{(GW)}_{\alpha\beta} \tag{72}$$

which are exactly the equations derived by Isaacson - see eqs. (3.1b) of Ref. 16 and (35.60) of Ref. 17. The averaged contracted Bianchi identities (54) take the form $G^{\alpha\epsilon}M_{\alpha\beta|\epsilon} = 0$ and ensure the conservation of $T^{(GW)}_{\alpha\beta}$.

The fact of the macrovacuum equations (57a) becoming Isaacson's ones in the high-frequency limit is of vital importance for the theory, for such a limit establishes the correspondence principle [4] for any full theory of macroscopic gravity, playing the role similar to Newton's limit in any consistent microscopic gravitation theory, for example, in general relativity theory. The macroscopic field equations (52) derived within the approach proposed have been shown to have the

correct high-frequency limit (72) for the case of macrovacuum.

17. CONCLUSIONS

This paper presents a self-consistent procedure to average out Cartan's structure equations of a Riemannian space-time and the Einstein equations of general relativity. A system of the structure equations (24) - (31), (35), (36), (38) and (40) together with the relations (46a,b), (47) and (50a,b) describes the geometric structure of the macroscopic space-time. This system, being supplemented by some conditions, for example (42a-c) - (44), and the averaged Einstein equations (52) having the form of microscopic Einsteinian ones with the source (53), can be taken as a system of differential equations to treat macroscopic gravity.

The simplest system of equations to be solved are the eqs. (52) and (42a-c) to find the macroscopic metric tensor $G_{\alpha\beta}$ and the correlation tensor $Z^{\alpha}{}_{\beta\gamma}{}^{\mu}{}_{\nu\sigma}$. That will enable one to calculate the induction tensor $M^{\alpha}{}_{\beta\gamma\delta}$. In order to find the average field $R^{\alpha}{}_{\beta\gamma\delta}$ it is necessary to solve the equations (43) and (44) (the last ones can be written as the algebraic equations; see Ref. 3), or to solve directly the equations (29) in the case (a) for the affine deformation tensor $A^{\alpha}{}_{\beta\gamma}$. A cosmological solution to the system, as well as the results concerning the problem of finding the constitutive relations within the theory and more geometrical and physical interpretations of the objects present in the theory, will be given elsewhere.

EFERENCES

1. Shirokov, M.F., and Fisher, I.Z. (1962). *Astron. Zh.* 39, 899 (in Russian).
2. Ellis, G.F.R. (1984). In *General Relativity and Gravitation*, Bertotti, B., de Felici, F., and Pascolini, A., eds. (Reidel, Dordrecht).
3. Zalaletdinov, R.M. (1992). "Averaging out the Einstein equations", *Gen. Rel. Grav.* (to be published).
4. Zalaletdinov, R.M. (1992). "Macroscopic gravity: A new setting of the problem", submitted to *Phys. Lett. A*.
5. Zalaletdinov, R.M. (1992). "Averaging out the Einstein equations and macroscopic space-time geometry", talk given at 5th Regional Conference on Mathematical Physics, Trakya University, Edirne, Turkey, 15-22 December 1991, to appear in the *Proceedings* (World Scientific).
6. Zalaletdinov, R.M. (1992). "Macroscopic gravity: The geometry and field equations", submitted to *Phys. Lett. A*.
7. Arifov, L.Ya., Shein, A.V., and Zalaletdinov, R.M. (1990). "Averaging out tensor fields on Riemannian manifolds according to Lorentz. II. Commutation formulae for the averaging and derivation (absolute and covariant)." Preprint, Institute of Nuclear Physics, Uzbek Academy of Sciences, No. R-12-480 (Tashkent), 1 (in Russian).
8. Arifov, L.Ya., Shein, A.V., and Zalaletdinov, R.M. (1990). "Averaging out tensor fields on Riemannian manifolds according to Lorentz. III. Analysis of the averages." Preprint, Institute of Nuclear Physics, Uzbek Academy of Sciences, No. R-12-499. (Tashkent), 1 (in Russian).

9. Synge, J.L. (1960). *Relativity: The General Theory* (North-Holland, Amsterdam).
10. Zalaletdinov, R.M. (1992). "A bilocal exterior calculus", to be submitted.
11. Schutz, B.F. (1982). *Geometrical Methods of Mathematical Physics* (Cambridge University Press, Cambridge).
12. Christensen, S.M. (1976). *Phys. Rev.* D14, 2450.
13. Schouten, J.A., and Struik, D.J. (1935). *Einführung in die neueren Methoden der Differentialgeometrie.* (Noordhoff, Groningen-Batavia), vol. I.
14. Kramer, D., Stefani, H., MacCallum, M., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin).
15. Isaacson, R.A. (1968). *Phys. Rev.* 166, 1263.
16. Isaacson, R.A. (1968). *Phys. Rev.* 166, 1272.
17. Misner, C.W., Thorne, K.S., and Wheeler, J.A. (1973). *Gravitation* (Freeman, San Francisco).