FPRINT-92-04

52

FEB 04 1992

LIBRARY

ROUTE TO

NAME	LOCATION

RETURN TO FE...

INCOMPRESSIBLE QUANTUM FLUIDS, GAUGE-INVARIANCE, AND CURRENT ALGEBRA

Jürg Fröhlich and Urban M. Studer

Theoretische Physik, ETH – Hönggerberg
CH–8093 Zürich

1. INTRODUCTION

This rather long paper is a tale of non-relativistic quantum theory summarizing research that has been conducted during the last one and a half years, and the main results of which have been sketched in two lectures presented at the Cargèse summer school of 1991, as well as in lectures at several other institutions. Coworkers in our endeavor have been, or are, Thomas Kerler, Pieralberto Marchetti and Tony Zee. Basic help and guidance were generously provided by Rudolf Morf. We are deeply grateful to these colleagues without whom our enterprise would have suffered premature shipwreck. We also thank J. Avron and G. Felder for very helpful discussions.

After some basic ideas underlying our approach had been developed during a students seminar on the quantum Hall effect at ETH organized by Rudolf Morf and J.F., we became aware of independent, but slightly prior work of X.G. Wen [1,2] that bears much resemblance with ours [3,4,5]. A 1982 paper of B.I. Halperin [6], supplemented by more recent results on current algebra [7,8,9] and on Chern-Simons gauge theory [10,11,12], has been instrumental in triggering the work in [2,3]. J.F. should also like to acknowledge some very stimulating discussions with Paul Wiegmann, in spring of 1989, whose remarks turned out to be much to the point.

Work vaguely or closely related to Wen's and ours has been carried out by several people and can be found in [13], and refs. indicated therein.

The task assigned to J.F. at the Cargèse school was to lecture on low-dimensional quantum theory with braid statistics and quantum symmetries. This task could have been fulfilled by lecturing on the beautiful *mathematics* of braid statistics and quantum symmetries that involves operator algebra theory, quantum groups and their subtle representation theory, holomorphic vector bundles over Riemann surfaces, and, perhaps most importantly, the theory of tensor categories. However, as physicists,

we may have a feeling of loosing ground in this world of mathematics. In any event, other people essentially took over that task, and it appeared desirable to lecture about *physical systems* with braid statistics and quantum symmetires. Fortunately, such systems exist in nature! A two-dimensional electron gas in a strong transverse magnetic field can exhibit quasi-particle excitations of *fractional charge* and *fractional (abelian braid) statistics*, the famous Laughlin vortices. One can imagine two-dimensional systems of condensed matter which will actually exhibit quasi-particle excitations with *non-abelian* braid statistics and quantum symmetries; see e.g. [14]. But it is likely that such systems have not been realized in the laboratory, yet. [Candidate systems are 2D systems with broken reflection – and time reversal invariance made of particles of spin ≥ 1.]

The phenomena of braid statistics and quantum symmetries in a two-dimensional quantum system appear to be intimately related to the property of local gauge invariance of the system. One of the key ideas underlying the work described in this paper is that one can acquire a surprisingly rich amount of information on a system of non-relativistic matter by studying how it reacts when coupled to external gauge fields. In Sect. 2, we therefore study how systems of non-relativistic quantum mechanical particles with spin interact with external electromagnetic fields, with "tidal gauge fields" providing a quantum-mechanical description of Coriolis forces and spin precession in moving coordinates, and to a variable metric on space. Our formalism can be applied to systems in one, two, and three space dimensions. It reveals a basic $U(1)_{\text{em}} \times SU(2)_{\text{spin}}$-gauge-invariance of non-relativistic quantum theory which gives rise to powerful *Ward identities.*

In Sect. 3, we review and "explain" a number of classic effects in non-relativistic quantum theory from the point of view of its $U(1)_{\text{em}} \times SU(2)_{\text{spin}}$ gauge invariance, (supplemented by certain assumptions concerning the structure of states that minimize the energy – , or free energy density). Included are the Aharonov-Bohm effect and its $SU(2)_{\text{spin}}$-variant, the Aharonov-Casher effect, flux quantization in superconductors and vorticity quantization in superfluids, the London equation for the supercurrent density in a superconductor and the related Anderson-Higgs mechanism, and different variants of the Einstein-de Haas (-Barnett) effect.

It turns out that the celebrated quantum Hall effect (and the related quantum Hall effect for spin currents [5]) encountered in two-dimensional electron gases (realized, for example, in heterojunctures) subject to a strong, transverse, external magnetic field is yet another phenomenon reflecting the $U(1) \times SU(2)$-gauge-invariance of non-relativistic quantum theory. In Sect. 4, we therefore study two-dimensional, *incompressible* electron fluids in external electromagnetic fields. The notion of incompressibility that we are using is the following: A system at zero temperature (but positive density) is incompressible if the energy of all physical states describing extended (as opposed to localized) excitations of the groundstate is *strictly* above the

ground state energy. Incompressible systems are free of dissipation, and therefore the longitudinal resistance vanishes. Experimentally, this is found to be the case when the Hall conductivity is on a plateau [15].

By using $U(1)\times SU(2)$-Ward identities we show that two-dimensional, incompressible quantum fluids have *universal* properties. For example, their effective action as a functional of small perturbations in the external electromagnetic field has a universal form which we determine explicitly. The notion of universality that emerges here is very much the same as the one encountered in the theory of critical phenomena associated with continuous phase transitions.

Our results on the effective action, summarized in Sect. 4, imply the general equations describing the Hall effects for the electric charge – and current density and for the spin – and spin-current density in systems with vanishing longitudinal resistances. Moreover, they yield a proof of the Goldstone theorem for non-abelian symmetries.

In Sect. 4, we also use our expression for the effective action to find the spectrum of charge-, flux- and spin-carrying excitations of an incompressible quantum fluid, and we discuss the possible values of their electric charge and spin, and their statistics. Our analysis provides first insights into why the Hall conductivity and various other quantities characterizing the system, e.g., its magnetic susceptibility, are quantized. But our reasoning is somewhat heuristic, mathematically.

In order to bring more rigour into that analysis, we derive and discuss, in Sect. 5, algebras of chiral currents circulating in an incompressible quantum fluid along domain boundaries across which the value of the Hall conductivity jumps, in particular along its edges. The electric edge currents form chiral $U(1)$-current algebras, the edge spin-currents form $SU(2)$-Kac-Moody algebras. These results can be derived from $U(1)\times SU(2)$-gauge-invariance by using well known results on the $(1+1)$-dimensional chiral gauge anomalies and their relation to $(2+1)$-dimensional Chern-Simons theory [16]. [An alternative derivation of the existence of algebras of chiral edge currents in incompressible Hall fluids from quantized Chern-Simons theory, based on results in [10,11,12], is given in [3].]

The well known representation theory of chiral current algebras, combined with some physically natural requirements, then leads us to find *discrete* sets of possible values of the Hall conductivity, (certain rational multiples of $\frac{e^2}{h}$), of the fractional charges of excitations, and of other interesting quantities, which are compatible with the incompressibility of the Hall fluid. Our results can be viewed as "*gap-labelling theorems*": The energy spectrum of a two-dimensional electron fluid in an external magnetic field can have a *positive gap* above the groundstate energy (reflecting its *incompressibility*) only if its Hall conductivity belongs to a certain discrete set.

We also find the statistics of fractionally charged excitations (Laughlin vortices) from the representation theory of the algebras of chiral edge currents.

A complete discussion of edge spin-currents and currents associated with internal symmetries of the system would take too much space and is therefore deferred to another paper [17]. However, a few basic ideas are provided in Sect. 5.

Most of this paper was written during a two-weeks' stay of J.F. at I.H.É.S, Bures-sur-Yvette. J.F. thanks the director of I.H.É.S, M. Berger, his colleages and the staff at the Institut for their very friendly hospitality during a period that was quite hectic for him.

2. NON-RELATIVISTIC QUANTUM MECHANICS OF SPINNING PARTICLES COUPLED TO EXTERNAL METRICS AND ELECTROMAGNETIC FIELDS.

In this section we recall the formulation of non-relativistic quantum mechanics in general, including moving, coordinates on a Riemannian space. We consider systems of spinning particles coupled to the space metric and to external electromagnetic fields. [For mathematical background see, e.g. [18].] Since we are interested in time-dependent many-particle systems, it will be convenient to use a second-quantized Lagrangian formalism [19].

Physical space is a two-, or three-dimensional manifold, M, with possibly time-dependent metric, space-time is given by $N := \mathsf{R} \times M$. The system is confined to the interior of a space-time cyclinder $\Lambda \subset N$. The intersection of Λ with a fixed-time slice is denoted by Ω_t where t is time. In local coordinates, points in M are denoted by $\mathbf{x}, \mathbf{y}, \cdots$, points in N by $x = (t, \mathbf{x}), y = (t, \mathbf{y}), \cdots$. The Riemannian metric on M is denoted by $g_{ij}(t, \mathbf{x})$; space-time N carries the metric $\eta_{\mu\nu}(x)$, where $\eta_{00}(x) = 1$, $\eta_{0i}(x) = \eta_{i0}(x) = 0, \eta_{ij}(x) = -g_{ij}(t, \mathbf{x})$. In the tangent space at a point $\mathbf{x} \in M$ we also have the flat, Cartesian metric, δ_{AB}. [Similarly, in the tangent space at a space-time point $x \in N$ we have the usual Lorentz metric $\eta^0_{\alpha\beta}$.]

If the dimension of M is two we imagine that M is a surface in a three-dimensional Riemannian manifold L with metric also denoted by g_{ij}, and the metric on M is the induced metric. In physical applications L will usually be three-dimensional Euclidean space E^3, and M will be some surface in E^3.

So far, time is merely a parameter, and we temporarily omit it from our notations. In the cotangent bundle to L we choose local sections of orthonormal frames $(e^A(\mathbf{x}))_{A=1}^3$. The components of $e^A(\mathbf{x})$ in the basis $(dx^j)_{j=1}^3$ of $T^*_{\mathbf{x}}(L)$ are denoted by $e^A{}_i(\mathbf{x})$ and are called *"dreibein* (fields)". If $\dim M = 2$ we choose $(e^A(\mathbf{x}))_{A=1}^3$ such that, for $\mathbf{x} \in M \subset L$, $e^3(\mathbf{x})$ is orthogonal to $T^*_{\mathbf{x}}(M)$ in the metric of $T^*_{\mathbf{x}}(L)$. The metric on L can be expressed in terms of the dreibein as follows:

$$g_{ij}(\mathbf{x}) = \delta_{AB} e^A{}_i(\mathbf{x}) e^B{}_j(\mathbf{x}) \,. \tag{2.1}$$

If $\dim M = 2$ we choose local coordinates on L in a neighborhood of M such that the

metric on M at a point $\mathbf{x}$ is given by

$$g_{ij}(\mathbf{x}) = \sum_{A,B=1}^{2} \delta_{AB} e^A{}_i(\mathbf{x}) e^B{}_j(\mathbf{x}) \,, \quad i,j = 1,2, \tag{2.2}$$

i.e., the coordinate x^3 is transversal to M.

The inverse of the dreibein $e^A{}_i$ is given by

$$\mathcal{E}_A{}^i(\mathbf{x}) = \delta_{AB} g^{ij}(\mathbf{x}) e^B{}_j(\mathbf{x}) \,, \tag{2.3}$$

where (g^{ij}) is the inverse of (g_{ij}). Clearly

$$\mathcal{E}_A{}^i e^B{}_i = \delta_A^B \,, \quad \delta^{AB} \mathcal{E}_A{}^i \mathcal{E}_B{}^j = g^{ij} \,. \tag{2.4}$$

The dreibein $e^A{}_i$ is the matrix which transforms the coordinate basis (dx^i) of $T^*_{\mathbf{x}}(L)$ to an orthonormal basis, $(e^A(\mathbf{x}))$, of $T^*_{\mathbf{x}}(L)$,

$$e^A(\mathbf{x}) = e^A{}_i(\mathbf{x}) dx^i \,. \tag{2.5}$$

Similarly, $\mathcal{E}_A{}^i$ transforms the basis $\left(\frac{\partial}{\partial x^i}\right)$ of $T_{\mathbf{x}}(L)$ to an orthonormal basis, $(\mathcal{E}_A(\mathbf{x}))$, of $T_{\mathbf{x}}(L)$,

$$\mathcal{E}_A(\mathbf{x}) = \mathcal{E}_A{}^i(\mathbf{x}) \frac{\partial}{\partial x^i} \,. \tag{2.6}$$

On every cotangent space $T^*_{\mathbf{x}}(L)$, $\mathbf{x} \in L$, we have a three-dimensional (spin-1) representation, $(R(\mathbf{x}) \in SO(3))$ of the rotation group, acting on the dreibein $e^A{}_i$ as follows

$${}^R e^A{}_i(\mathbf{x}) = R(\mathbf{x})^A{}_B e^B{}_i(\mathbf{x}) \,. \tag{2.7}$$

We require that parallel transport on L be given by the Levi-Cività connection $\Gamma^i{}_{jl}$, so that the torsion, T, vanishes. Then we may define Cartan's *spin connection* $\lambda^A{}_B$ through Cartan's first structure equation

$$de^A + \lambda^A{}_B \wedge e^B \equiv T^A = 0 \,. \tag{2.8}$$

These equations enable us to express $\lambda_i{}^A{}_B$ in terms of the dreibeins $e^A{}_i$, their derivatives, and their inverses $\mathcal{E}_A{}^i$; (see [18]).

The curvature 2-form $\mathcal{R}^A{}_B$ of L is defined by Cartan's second structure equation

$$\mathcal{R}^A{}_B = d\lambda^A{}_B + \lambda^A{}_C \wedge \lambda^C{}_B \,. \tag{2.9}$$

It is easy to deduce from (2.8) and (2.9) how λ and R transform under the "gauge-transformations" (2.7) of the dreibein:

$$\begin{aligned} {}^R\lambda(\mathbf{x}) &= R(\mathbf{x})\lambda(\mathbf{x})R^T(\mathbf{x}) + R(\mathbf{x})dR^T(\mathbf{x}) \,, \\ {}^R\mathcal{R}(\mathbf{x}) &= R(\mathbf{x})\mathcal{R}(\mathbf{x})R^T(\mathbf{x}) \,. \end{aligned} \tag{2.10}$$

We now assume that the manifold L admit a spin structure. Then we may introduce *spinor bundles* over L. Let $s = 0, 1/2, 1, \cdots$ denote the spin, i.e., $2s+1$ is the

dimension of an irreducible representation of $SU(2) = \widetilde{SO}(3)$ with spin s. The fibre of the spin-s spinor bundle, $E^{(s)}$, over L is isomorphic to the $(2s+1)$-dimensional Hilbert space, $\mathcal{D}^{(s)}$, carrying the spin-s representation of $SU(2)$. Sections of the spin-s spinor bundle are denoted by $\psi^{(s)}(\mathbf{x})$. From now on, we choose the gauge transformations $(R(\mathbf{x}))$ to be $SU(2)$-valued. The action of these gauge transformations on the cotangent bundle is given by their adjoint (spin-1) representation, usually also denoted by $R(\mathbf{x})$[1]. Under a gauge transformation $(R(\mathbf{x}))$, a section $\psi^{(s)}$ of $E^{(s)}$ transforms as follows:

$$\psi^{(s)}(\mathbf{x}) \mapsto {}^{R}\psi^{(s)}(\mathbf{x}) := U^{(s)}(R(\mathbf{x}))\psi^{(s)}(\mathbf{x}) , \tag{2.11}$$

where $U^{(s)}$ is the spin-s representation of $SU(2)$. The transition functions of the spin-s spinor bundle are inherited from the transition functions of the cotangent bundle, $T^*(L)$, by lifting them to the spin-s representation of $SU(2)$. [Since we have assumed that L have a spin structure this is possible even if s is half-integer.]

Physically, what is meant by "spin up" or "spin down" is now a *local notion*, depending on the point $\mathbf{x} \in L$ at which the spin is located and determined by the frame $(e^A(\mathbf{x}))_{A=1}^3$.

We intend to develop non-relativistic quantum mechanics on Hilbert spaces of sections of these spinor bundles. In non-relativistic quantum mechanics, wave functions are complex-valued. We therefore tensor the fibre space $\mathcal{D}^{(s)}$-real when s is integer – by $\mathbf{C}$. The structure group of the resulting bundle, still denoted by $E^{(s)}$, is then $U(1) \times SU(2)$. The factor $U(1)$ (phase transformations of $\psi^{(s)}$) is connected to electromagnetism, as recognized by Weyl more than sixty years ago.

In order to keep our notations simple, it is advantageous to formulate quantum mechanics by using the language of second quantization. The sections $\psi^{(s)}(\mathbf{x})$ of $E^{(s)}$ are then interpreted as operator-valued distributions acting on Fock space and subject to *equal-time canonical (anti-) commutation relations*

$$\left[\psi_\alpha^{(s)\#}(\mathbf{x}), \psi_\beta^{(s)\#}(\mathbf{y})\right]_\pm = 0 , \tag{2.12}$$
$$\left[\psi_\alpha^{(s)}(\mathbf{x}), \psi_\beta^{(s)*}(\mathbf{y})\right]_\pm = \frac{1}{\sqrt{g(\mathbf{x})}}\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}) , \quad \alpha, \beta = 1, \cdots, 2s+1 ,$$

where $[\ ,\ ]_+$ denotes the anti-commutator and $[\ ,\ ]_-$ the usual commutator, $\psi^{(s)\#} = \psi^{(s)}$ or $\psi^{(s)*}$; $\psi^{(s)*}$, the *creation operator*, is the adjoint (on Fock space) of $\psi^{(s)}$, the *annihilation operator*; $g(\mathbf{x})$ denotes the determinant of $(g_{ij}(\mathbf{x}))$. The usual connection between spin and statistics is to choose anti-commutators in (2.12), corresponding to Fermi statistics, when s is half-integer, and commutators, corresponding to Bose statistics, when s is an integer.

Our purpose is now to specify some nonrelativistic dynamical laws for the operators $\psi^{(s)\#}$ in the Heisenberg picture. Let $\psi^{(s)\#}(x) = \psi^{(s)\#}(t, \mathbf{x})$ denote the Heisenberg

[1]There is little danger of confusion.

- picture creation - and annihilation operators with initial conditions $\psi^{(s)\#}(0,\mathbf{x}) = \psi^{(s)\#}(\mathbf{x})$. In order to formulate local dynamical laws for $\psi^{(s)\#}(x)$, we need to be able to differentiate these fields in t and $\mathbf{x}$. This necessitates introducing a notion of parallel displacement in $E^{(s)}$. Parallel displacement in $E^{(s)}$ is defined with the help of a $U(1) \times SU(2)$-connection, (a vector potential with values in $\mathbf{R} \oplus su(2)$, where $su(2)$ is the Lie algebra of $SU(2)$). Once such a connection is fixed, derivatives of sections $\psi^{(s)\#}$ are defined as *covariant derivatives*. Setting $x^0 := ct, (x^\mu) := (x^0, \mathbf{x})$, the covariant derivative in the μ-direction is given by

$$D_\mu = \frac{\partial}{\partial x^\mu} + ia_\mu(x) + w_\mu^{(s)}(x) \,, \tag{2.13}$$

where $a(x) := a_j(x)dx^j$ is the $U(1)$-connection (i.e., $a_j(x)$ is the jth component of a real-valued vector potential), and $a_0(x)$ is the scalar potential, $w^{(s)}(x) := w_j^{(s)}(x)dx^j$ is the $SU(2)$-connection, and $w_0^{(s)}(x)$ is the "Zeeman potential" in the spin-s representation of $su(2)$, i.e.,

$$w_\mu^{(s)}(x) = i \sum_{A=1}^{3} w_{\mu A}(x) L_A^{(s)} \,, \tag{2.14}$$

where $(L_A^{(s)})_{A=1}^3$ are Hermitian - we are physicists - generators of $su(2)$ in the spin-s representation, normalized such that $L_A^{(1/2)} = \sigma_A$, where σ_1, σ_2 and σ_3 are the usual Pauli matrices. We shall see that we should identify a with the electromagnetic vector potential, up to multiplication by a constant of nature. What about $w^{(s)}$? Clearly the spin connection $\lambda^A{}_B$, introduced in (2.8), must enter the definition of $w^{(s)}$. But we can add to λ a one-form, ρ, transforming under the adjoint representation of the $SU(2)$-gauge group. The sum is then still an $SU(2)$-connection. Hence

$$w_\mu^{(s)}(x) = \lambda_\mu^{(s)}(x) + \rho_\mu^{(s)}(x) \,, \tag{2.15}$$

where

$$\lambda_\mu^{(s)}(x) = \frac{i}{2} \sum_{A,B,C=1}^{3} \varepsilon_A{}^{BC} \lambda_\mu{}^A{}_B(x) L_C^{(s)} \,, \tag{2.16}$$

$\varepsilon_A{}^{BC} = \varepsilon^{ABC}$ is the sign of the permutation (ABC) of (1 2 3), and where

$$\rho_\mu{}^{(s)}(x) = i \sum_{A=1}^{3} \rho_{\mu A}(x) L_A^{(s)} \,. \tag{2.16'}$$

Under an $SU(2)$-gauge-transformation of the cotangent bundle, $\rho_\mu^{(s)}$ transforms as follows:

$$\rho_\mu{}^{(s)}(x) \mapsto {}^R\rho_\mu{}^{(s)}(x) = U^{(s)}(R(x))\rho_\mu{}^{(s)}(x)U^{(s)}(R(x))^* \,. \tag{2.17}$$

The transformation law of $\lambda_\mu{}^{(s)}$ can be inferred from (2.10).

If the dreibein $(e^A{}_i)$ is time-independent λ_0 vanishes, but after a time-dependent $SU(2)$-gauge - transformation λ_0 may be different form zero. In general, ρ_0 will be different from zero.

We shall see that, physically, (ρ_0, ρ) describes *Zeeman* - and *spin-orbit couplings* of the magnetic moments carried by the particles to the electromagnetic field. Geometrically, the part (ρ_0, ρ) of the $SU(2)$-connection w yields non-trivial *torsion.*

Having introduced a $U(1)\times SU(2)$-connection and defined covariant differentiation of $\psi^{(s)\#}$, we are now in a position to formulate local dynamical laws. It is convenient to use the *Lagrangian formalism*, but we could also work in the Hamiltonian formalism; see [3]. Let us consider a system of non-relativistic particles of fixed spin s and, to simplify our notations, drop the superscript $^{(s)}$. Our ansatz for the action of the system is ($dx = dt\, d\mathbf{x}$)

$$S_\Lambda(\psi^*, \psi; a, w, g) := \int_\Lambda \sqrt{g(t,\mathbf{x})}dx\Big[i\hbar c(\psi^* D_0\psi)(x) - \frac{g^{kl}(t,\mathbf{x})}{2m}(-i\hbar D_k\psi)^*(x)(-i\hbar D_l\psi)(x) - U(\psi^*,\psi)(x)\Big] , \tag{2.18}$$

where the covariant derivatives are given in (2.13), m is the effective mass of the particles, and $U(\psi^*, \psi)$ is a $U(1) \times SU(2)$-invariant functional of ψ^* and ψ, e.g.,

$$\begin{aligned} U(\psi^*,\psi)(x) &= \int \frac{1}{2}\sqrt{g(t,\mathbf{y})}d\mathbf{y} : (\psi^*(t,\mathbf{x})\psi(t,\mathbf{x}) - n)V(\mathbf{x}-\mathbf{y}) \\ &\times (\psi^*(t,\mathbf{y})\psi(t,\mathbf{y}) - n) : + v(t,\mathbf{x})\psi^*(t,\mathbf{x})\psi(t,\mathbf{x}) . \end{aligned} \tag{2.19}$$

The double colons indicate Wick ordering, V is some repulsive pair potential, n is the background density of the system, and $v(t, \mathbf{x})$ is a possibly time-dependent one-body (background) potential.

We recall that $\Lambda \subset \mathbf{R} \times M$ is a cylindrical region to which the system is confined. At fixed time t we impose Dirichlet boundary conditions at the boundary, $\partial\Omega_t$, of the region Ω_t to which the system is confined.

The field equations (or Euler-Lagrange equations) for $\psi(x)$ and $\psi^*(x)$ follow by setting the variation of S_Λ with respect to $\psi^*(x)$ and $\psi(x)$, respectively, to zero. The resulting equations are reminiscent of the *Pauli equations* for ψ and ψ^*.

In order to interpret these equations physically we start with a simple situation: We choose space M to be given by $\mathbf{E}^2$ (the $x-y$ plane in $L = \mathbf{E}^3$) or by $\mathbf{E}^3; g_{ij}(t,\mathbf{x}) = \delta_{ij}$, for all times t and all $\mathbf{x} \in M$, $\Lambda = \mathbf{R} \times \Omega$, where Ω is some time-independent open set in M. The field equation for $\psi(x)$ obtained by varying the action S_Λ defined in (2.18) with respect to $\psi^*(x)$ then essentially reduces to the Pauli equation found in standard text books of quantum mechanics [20], with a minor modification of order $1/m(m_0c)^2$ discussed in [5], *provided* we identify the $U(1)$-connection a with the electromagnetic vector potential

$$a_j(x) = \frac{e}{\hbar c}A_j(x) , \quad \text{and} \quad a_o(x) = -\frac{e}{\hbar c}\phi(x) , \tag{2.20}$$

where $-e$ is the charge of the particle, ϕ is the electrostatic potential, and the coefficients of the $su(2)$-valued components ρ_μ are expressed in terms of the electromagnetic field $(\vec{E}, \vec{B})$ as follows:

$$\rho_{oA}(x) = -\frac{\mu}{2c}B_A(x) , \tag{2.21}$$

where $B_A(x)$ is the A-component of the magnetic field $\vec{B}(x)$ in the basis $(e^1(x), e^2(x), e^3(x))$, and

$$\rho_{kA}(x) = -\frac{\mu}{4c}\sum_{C=1}^{3}\varepsilon_{kAC}E_C(x)\,, \tag{2.22}$$

with E_C the C-component of the electric field $\vec{E}$. In these equations μ is the magnetic moment of the particles, (up to a factor $\frac{\hbar}{2}$). For electrons, $\mu \approx -\frac{e}{m_0 c}$, where m_o is the electron mass in empty space. [In standard situations of solid state physics, the effective mass m can be considerably smaller than m_0.] The symbol ε_{kAC} is defined by

$$\varepsilon_{kAC} = \varepsilon_{kAC}(x) := e^D{}_k(x)\varepsilon_{DAC}\,, \tag{2.23}$$

where ε_{DAC} is the sign of the permutation (DAC) of (1 2 3). Of course, in the present case $e^D{}_k(x) = \delta^D_k$, but formula (2.23) is valid in general. Formulas (2.20)-(2.22) have been derived in [5] by comparing the Euler-Lagrange equations corresponding to the action S_Λ with the usual Pauli equation, including the Zeeman term and spin-orbit couplings.

It is now straightforward to find the correct physical interpretations of the connections a and w for spaces M which are arbitrary Riemannian spin manifolds. The $U(1)$-connection a is still expressed in terms of the electromagnetic vector potential $A = (-\phi, \vec{A})$ by formula (2.20). The $SU(2)$-connection w is given by

$$w_\mu = \lambda_\mu + \rho_\mu\,, \tag{2.24}$$

where λ_μ is the affine spin connection corresponding to the dreibein $e^A{}_i(x)$, see (2.8), and the coefficients of ρ_μ are given by

$$\rho_{oA}(x) = -\frac{\mu}{2c}\mathcal{E}_A{}^l(x)B_l(x) \equiv -\frac{\mu}{2c}B_A(x) \tag{2.25}$$

where $(\mathcal{E}_A{}^l(x))$ is the inverse of the dreibein $(e^A{}_i(x))$ and B_l is the l-component of $\vec{B}$ in the basis (dx^1, dx^2, dx^3); moreover

$$\begin{aligned}\rho_{jA}(x) &= -\frac{\mu}{4c}e^D{}_j(x)\varepsilon_{DAC}\mathcal{E}_C{}^l(x)E_l(x)\\ &\equiv -\frac{\mu}{4c}\varepsilon_{jA}{}^l(x)E_l(x)\,,\end{aligned} \tag{2.26}$$

where E_l is the l-component of $\vec{E}$ in local coordinates. Note that (2.25) and (2.26) are consistent with the transformation law (2.17) of ρ_μ under $SU(2)$-gauge-transformations.

We recall that the potential V in (2.19) is a pair potential (e.g. Coulomb, for charged particles, or van der Waals, for neutral atoms or molecules), and v is a potential created by the background in which the particles are moving; (v might depend on the scalar curvature of M).

We now suppose that the background of the system is moving according to some *classical flow* $\phi(t,\cdot)$. Here $\phi(t,\mathbf{y})$ is the position in M of a point particle at time t

starting at position $\mathbf{y}$ at time 0. Then, in the x-coordinates, the one-body potential $v(x)$ and the magnetic and electric fields $\vec{B}_c(x)$ and $\vec{E}_c(x)$, created by the background are *time-dependent*. This implies that, in the time-independent $\mathbf{x}$-coordinates on M, the Hamiltonian of the system is time-dependent which complicates the mathematical analysis of the system, in particular the analysis of its *thermal equilibrium properties*. It is quite clear, physically, that thermal equilibrium in such a system will be reached locally in regions moving with the background, (according to the flow $\phi(t,\cdot)$). Thus, we ought to formulate quantum mechanics in "*moving coordinates*", (y^1, y^2, y^3), where

$$\mathbf{x} = \phi(t,\mathbf{y}) \,, \quad \text{i.e.,} \quad \mathbf{y} = \phi^{-1}(t,\mathbf{x}) \,. \tag{2.27}$$

Time will not be transformed. In the new coordinates (y^1, y^2, y^3), the one-body potential $v(t,\mathbf{y})$ and the background fields $\vec{B}_c(t,\mathbf{y})$ and $\vec{E}_c(t,\mathbf{y})$ might now be *time-independent*. In this case, the Hamiltonian for spinless particles ($s = 0$) will be time-independent, and we can apply the rules of Gibbsian statistical mechanics to study thermal equilibrium.

Unfortunately, for spinning particles ($s = 1/2, 1, \cdots$), the situation is not quite as neat, because, in the y-coordinates, the dreibeins $\tilde{e}^A{}_i(y)$ are now *time-dependent*:

$$\tilde{e}^A{}_i(y) = e^A{}_j(\phi(t,\mathbf{y}))\frac{\partial \phi^j(t,\mathbf{y})}{\partial y^i} \,. \tag{2.28}$$

In order to eliminate as much of this undesirable time-dependence as possible, we may try to perform a suitable $SU(2)$-gauge transformation on the new dreibeins $\tilde{e}^A{}_i(y)$. What is the optimal choice? The answer is, perhaps, somewhat ambiguous, in general. But the following choice tends to be quite optimal: Let $(f^j(t,\mathbf{x}))$ be the vector (velocity) field generating the flow $\phi(t,\cdot)$, i.e.,

$$\frac{\partial}{\partial t}\phi(t,\mathbf{y}) = f(t,\phi(t,\mathbf{y})) \,. \tag{2.29}$$

Let

$$f^A(t,\mathbf{x}) := e^A{}_j(\mathbf{x}) f^j(t,\mathbf{x}) \,.$$

Then the infinitesimal rotation of an orthonormal frame carried along by the flow ϕ, at the point $\mathbf{x}$ and at time t, is given by

$$\Omega^A{}_B(t,\mathbf{x}) = \frac{1}{2}\{(\partial_B f^A)(t,\mathbf{x}) - (\partial_A f^B)(t,\mathbf{x})\} \,, \tag{2.30}$$

where $\partial_A = \mathcal{E}^i_A(\mathbf{x})\frac{\partial}{\partial x^i}$; see (2.6). The vector $\vec{\Omega}(t,\mathbf{x})$ dual to the antisymmetric matrix $(\Omega^A{}_B(t,\mathbf{x}))$ is called the *vorticity* of the vector field f and is the local angular velocity of the rotation induced by ϕ of a frame at the point $\mathbf{x}$, at time t.

We define a rotation matrix $R(t,\mathbf{x})^A{}_B$ by setting

$$R(t,\mathbf{x})^A{}_B := T\left[\exp\int_0^t dt'\Omega(t',\mathbf{x})\right]^A{}_B \,, \tag{2.31}$$

where T denotes time ordering. [The r.h.s. of (2.31) can be defined, for example, by a convergent Dyson series if $\vec{\Omega}(t,\mathbf{x})$ is uniformly bounded in t.] We now define

$$\hat{e}^A{}_i(y) := {}^R\tilde{e}^A{}_i(y) = R(t,\phi(t,\mathbf{y}))^A{}_B\tilde{e}^B{}_i(y)\,, \tag{2.32}$$

where $\tilde{e}^B{}_i(y)$ is given by (2.28). We also define the following transformed quantities:

$$\begin{aligned}
\hat{g}^{kl}(t,\mathbf{y}) &:= \frac{\partial y^k}{\partial x^m}\frac{\partial y^l}{\partial x^n}g^{mn}(t,\phi(t,\mathbf{y}))\,,\\
\hat{\psi}(t,\mathbf{y}) &:= U^{(s)}(t,\mathbf{y})\psi(t,\phi(t,\mathbf{y}))\,,\\
\hat{w}_k^{(s)}(t,\mathbf{y}) &:= \frac{\partial x^l}{\partial y^k}\Big\{U^{(s)}(t,\mathbf{y})w_l^{(s)}(t,\phi(t,\mathbf{y}))U^{(s)}(t,\mathbf{y})^*\\
&\qquad +U^{(s)}(t,\mathbf{y})\left(\frac{\partial}{\partial x^l}U^{(s)}\right)^*(t,\mathbf{y})\Big\}\,,\\
\hat{w}_0^{(s)}(t,\mathbf{y}) &:= U^{(s)}(t,\mathbf{y})\left[w_0^{(s)}(t,\phi(t,\mathbf{y}))+\frac{\partial x^l}{\partial y^0}w_l^{(s)}(t,\phi(t,\mathbf{y}))\right]U^{(s)}(t,\mathbf{y})\\
&+ U^{(s)}(t,\mathbf{y})\frac{1}{c}\frac{\partial}{\partial t}U^{(s)}(t,\mathbf{y})^*\,,
\end{aligned} \tag{2.33}$$

where

$$U^{(s)}(t,\mathbf{y}) := U^{(s)}(R(t,\phi(t,\mathbf{y})))\,,$$

and, for $l=1,2,3$,

$$\left(\frac{\partial}{\partial x^l}U^{(s)}\right)(t,\mathbf{y}) := U^{(s)}\left(\frac{\partial}{\partial x^l}R(t,\mathbf{x})\right)\Big|_{\mathbf{x}=\phi(t,\mathbf{y})}\,. \tag{2.34}$$

Finally, we have

$$\hat{a}_k(t,\mathbf{y}) := \frac{\partial x^l}{\partial y^k}a_l(t,\phi(t,\mathbf{y}))\,,$$

and

$$\hat{a}_0(t,\mathbf{y}) := a_0(t,\phi(t,\mathbf{y}))+\frac{\partial x^l}{\partial y^0}a_l(t,\phi(t,y))\,. \tag{2.35}$$

Our aim is now to rewrite the action S_Λ introduced in (2.18), (2.19) in the moving $\mathbf{y}$-coordinates, using the transformations (2.32)–(2.35). By (2.33), (2.34),

$$\psi(t,\mathbf{x}) = U^{(s)}(R(t,\mathbf{x}))^*\hat{\psi}(t,\phi^{-1}(t,\mathbf{x}))\,. \tag{2.36}$$

Hence

$$\begin{aligned}
U^{(s)}(R(t,\mathbf{x}))\frac{\partial}{\partial t}\psi(t,\mathbf{x}) &= \frac{\partial}{\partial t}\hat{\psi}(t,\mathbf{y})-\hat{f}^j(t,\mathbf{y})\frac{\partial}{\partial y^j}\hat{\psi}(t,\mathbf{y})\\
&-\frac{i}{4}\sum_{A,B,C}\varepsilon_A{}^{BC}\hat{\Omega}^A{}_B(t,\mathbf{y})L_C^{(s)}\hat{\psi}(t,\mathbf{y})\,,
\end{aligned} \tag{2.37}$$

where $-\hat{f}^j(t,\mathbf{y})$ is the jth component of the vector field generating $\phi^{-1}(t,\cdot)$ in $\mathbf{y}$-coordinates, and $\hat{\Omega}^A{}_B(t,\mathbf{y})$ is the vorticity of f in $\mathbf{y}$-coordinates with respect to the

dreibein ($\hat{e}^A(t,\mathbf{y})$). By comparing (2.37) with the last equation in (2.33) and with (2.34) and (2.35) we see that

$$U^{(s)}(R(t,\mathbf{x}))\left(\frac{1}{c}\frac{\partial}{\partial t}+ia_0(x)+w_0^{(s)}(x)\right)\psi(x) \tag{2.38}$$
$$=\left(\frac{1}{c}\frac{\partial}{\partial t}+i\hat{a}_0(y)+\hat{w}_0^{(s)}(y)\right)\hat{\psi}(y)-\frac{1}{c}\hat{f}^j(y)\left(\frac{\partial}{\partial y^j}+i\hat{a}_j(y)+\hat{w}_j(y)\right)\hat{\psi}(y)\,.$$

We now define the new covariant derivatives

$$\begin{aligned}\hat{D}_0 &:= \frac{1}{c}\frac{\partial}{\partial t}+i\hat{a}_0(y)+\hat{w}_0{}^{(s)}(y)\\ \hat{D}_j &:= \frac{\partial}{\partial y^j}+i\hat{a}_j(y)-i\frac{m}{\hbar}\hat{f}_j(y)+\hat{w}_j(y)\end{aligned} \tag{2.39}$$

where $\hat{f}_j=\hat{g}_{jl}\hat{f}^l$, and the new one-body potential

$$\begin{aligned}\hat{v}(t,\mathbf{y}) &:= v(t,\phi(t,\mathbf{y}))-\frac{m}{2}\hat{f}^j(y)\hat{f}_j(y)\\ &\quad -i\frac{\hbar}{2}\frac{1}{\sqrt{\hat{g}}}\frac{\partial}{\partial y^j}\left(\sqrt{\hat{g}}\hat{f}^j\right)(y)\,,\end{aligned} \tag{2.40}$$

as well as the two-body potential

$$\hat{V}(t,\mathbf{y}-\mathbf{y}'):=V(\phi(t,\mathbf{y})-\phi(t,\mathbf{y}'))\,. \tag{2.41}$$

After these preparations, one verifies easily that

$$\begin{aligned}&S_\Lambda(\psi^*,\psi;a,w,g)\\ &\quad=\hat{S}_{\hat{\Lambda}}(\hat{\psi}^*,\hat{\psi};\hat{a},\hat{w},\hat{f},\hat{g})\\ &\quad=\int_{\hat{\Lambda}}\sqrt{\hat{g}(t,\mathbf{y})}dy\,\Big[i\hbar c(\hat{\psi}^*\hat{D}_0\hat{\psi})(y)\\ &\quad-\frac{\hat{g}^{kl}(t,\mathbf{y})}{2m}(-i\hbar\hat{D}_k\hat{\psi})^*(y)(-i\hbar\hat{D}_l\hat{\psi})(y)\\ &\quad-\hat{U}(\hat{\psi}^*,\hat{\psi})(y)\Big]\,,\end{aligned} \tag{2.42}$$

where in the definition of $\hat{U}$ the potentials $\hat{v}$ and $\hat{V}$ of (2.40) and (2.41) are used, and $\hat{\Lambda}:=\{(t,\mathbf{y}):(t,\mathbf{x}=\phi(t,\mathbf{y}))\in\Lambda\}$. To prove (2.42), one expands the r.h.s. of (2.42) in powers of $\hat{f}$, integrates by part, and compares the resulting expression to (2.38), using (2.40), (2.39) and the fact that $(U^{(s)}\psi)^*(U^{(s)}\psi)=\psi^*\psi$.

Let us pause to interpret the result (2.42). By (2.39), $-\frac{m}{\hbar}\hat{f}_j$ enters the action $\hat{S}$ as a contribution to the $U(1)$-connection. By (2.20), $-m\hat{f}_j$ and $\frac{e}{c}\hat{A}_j$ play analogous roles, i.e.,

$$-m\vec{f}\leftrightarrow\frac{e}{c}\vec{A}\,. \tag{2.43}$$

The vector potential $\vec{A}$ gives rise to the *Lorentz force* in the classical limit. The Lorentz force has the same form as the *Coriolis force* if one replaces $\frac{e}{c}\vec{B}$ by $-2m\vec{\Omega}$, where $\vec{\Omega}$ is the local angular velocity which is precisely half the curl of the vector field

$\vec{f}$. Thus, $\vec{f}$ is the vector potential that gives rise to the Coriolis force in the classical limit. By (2.37) and (2.38), the new action $\hat{S}$ contains a term

$$\hat{\psi}^* \left(\hat{\vec{\Omega}} \cdot \frac{\hbar}{2} \vec{L}^{(s)} \right) \hat{\psi} \,, \tag{2.44}$$

where $\hat{\vec{\Omega}}$ is the curl of $\hat{\vec{f}}$. This has the form of the *Zeeman term*

$$-\mu \hat{\psi}^* \left(\hat{\vec{B}} \cdot \frac{\hbar}{2} \vec{L}^{(s)} \right) \hat{\psi} \tag{2.45}$$

which, by (2.25), (2.24) and (2.39), also appears in $\hat{S}$. Of course, $\mu \hat{\vec{B}}$ is precisely the angular velocity of spin precession in a magnetic field.

Next, we must analyze the one-body potential $\hat{v}$ in moving coordinates. By (2.40), $\hat{v}$ is *complex-valued, unless*

$$\frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial y^j} \left(\sqrt{\hat{g}} \hat{f}^j \right) = 0 \,, \tag{2.46}$$

i.e., *unless the vector field* $\hat{\vec{f}}$ *is divergence-free.* A divergence-free vector field generates a *volume-preserving* flow ϕ, hence

$$\hat{g}(t, \mathbf{y}) \equiv \det(\hat{g}_{kl}(t, \mathbf{y})) = g(t, \phi(t, \mathbf{y})) \,. \tag{2.47}$$

Thus, for volume-preserving (i.e., *incompressible*) flows, and *only* for such flows, $\hat{v}$ is again real-valued. [This is, because if volume is preserved by ϕ then, by (2.47), the quantum mechanical time-evolution in the moving coordinate system preserves probabilities with respect to the volume element $\sqrt{g(t, \phi(t, \mathbf{y}))} d\mathbf{y}$, and hence is generated by a *Hermitian* (selfadjoint) Hamiltonian!] But $\hat{v}$ contains an additional term, $-\frac{m}{2} \hat{f}^j(y) \hat{f}_j(y)$, that was not present in the original one-body potential. What does it correspond to physically? It is the potential of the *centrifugal force*, (because $\frac{m}{2} \frac{\partial}{\partial y^i} (\hat{f}^j(t, \mathbf{y}) \hat{f}_j(t, \mathbf{y}))$ is precisely the i-component of the centrifugal force at the point $\mathbf{y}$, at time t; note, incidentally, that $\frac{m}{2} \hat{\vec{f}} \cdot \hat{\vec{f}}$ is the classical kinetic energy of the particle in the rest frame which must be subtracted in the $\mathbf{y}$-frame).

In conclusion, we find that quantum mechanics in moving coordinates is Hamiltonian, with a *Hermitian* (but possibly still time-dependent) Hamiltonian operator, iff the flow ϕ defining the moving coordinate system is *volume-preserving*, or *incompressible.* Henceforth this property is usually required. It is worthwhile recalling that in *two* space dimensions, incompressible flows are automatically symplectic (Hamiltonian) flows, because the vector fields generating them are divergence-free and hence are dual to the gradient of some (scalar) Hamiltonian function.

Let us consider, as an example, a system of particles of charge $-e$ and magnetic moment $\mu = -\frac{e}{m_0 c}$, with $m_0 = m$, (e.g., electrons, neglecting their anomalous magnetic moment). For such a system, we can eliminate, to order $\max(\vec{B}^2, |\partial_j \vec{B}|)$ the effect of an external magnetic field $\vec{B}$ by choosing moving coordinates with vorticity field $2\vec{\Omega} = -\mu \vec{B}$ and velocity field $\vec{f} = -\mu \vec{A} = \frac{e\vec{A}}{mc}$, where the electromagnetic

vector potential $\vec{A}$ is chosen in the Coulomb gauge, div $\vec{A} = 0$, in order for $\vec{f}$ to be divergence-free, up to a modification of the one-body potential v by the potential $-\frac{m}{2}\vec{f}\cdot\vec{f}$ of the centrifugal force and additional spin-orbit couplings, proportional to derivatives of $\vec{B}$ (if $\vec{B}$ is not homogeneous). This theorem follows directly from (2.43) – (2.45). It is the quantum-mechanical version of *Larmor's theorem.* (This theorem can be generalized, in order to take anomalous magnetic moments into account, by suitably changing the definition of $R(t,\mathbf{x})$ in eq. (2.31).)

Before we turn to some applications of the formalism presented in this section, we wish to emphasize once more that it applies equally well to (one-), two- and three-dimensional systems. It often happens in solid state physics, e.g. in two-dimensional heterojunctures used in measurements of the quantized Hall effect, that the system exhibits an approximate internal symmetry described by some compact group G. The spinors $\psi^{(s)}$ then transform according to some non-trivial representation, π, of G. A breaking of G might be described as the effect of coupling $\psi^{(s)\#}$ to an external gauge field with values in the representation $d\pi$ of the Lie algebra of G. Let us denote this gauge field by Z. By modifying the covariant derivatives,

$$D_\mu \mapsto D'_\mu := D_\mu + d\pi(Z_\mu)\,, \tag{2.48}$$

we may easily extend the entire formalism developed in this section to systems with gauged internal symmetries. This is important in applications, (e.g., to the quantum Hall effect).

Note that the action S_Λ introduced in eq. (2.18) is $U(1)_{\mathrm{em}} \times SU(2)_{\mathrm{spin}} \times G_{\mathrm{internal}}$ gauge-invariant: It does not change if, for an arbitrary real-valued function χ, an $SU(2)$-valued function R and a G-valued function g, the following substitutions are made:

$$\begin{aligned} \psi^{(s)} &\mapsto e^{i\chi}U^{(s)}(R)\otimes\pi(g)\psi^{(s)}\,, \\ a_\mu &\mapsto a_\mu - \partial_\mu\chi\,, \\ w_\mu &\mapsto Rw_\mu R^* + R\partial_\mu R^*\,, \end{aligned} \tag{2.49}$$

and

$$Z_\mu \mapsto gZ_\mu g^{-1} + g\partial_\mu g^{-1}\,.$$

Thus, barring gauge anomalies, (which actually cannot appear in systems of finitely many non-relativistic particles), the non-relativistic quantum mechanics of such systems is $U(1)_{\mathrm{em}} \times SU(2)_{\mathrm{spin}} \times G_{\mathrm{internal}}$ gauge-invariant. Ward identities expressing this gauge-invariance turn out to play an important role in establishing certain universal properties of such systems; see [5] and Sect. 4.

3. SOME KEY EFFECTS RELATED TO THE $U(1)_{em} \times SU(2)_{spin}$-GAUGE-INVARIANCE OF NON- RELATIVISTIC QUANTUM MECHANICS

Before we turn to our main topic, the analysis of two-dimensionl, incompressible quantum fluids and their relation to one-dimensional chiral current algebras, we wish, in this section, to sketch some effects in quantum mechanics related to its $U(1)_{em} \times SU(2)_{spin}(\times G_{internal})$-gauge invariance. Most of the material reviewed here is well known, but our perspective, emphasizing gauge-invariance, may be somewhat novel in a few instances.

(1) The Aharonov-Bohm effect [21].

A key effect reflecting Weyl's $U(1)_{em}$-gauge principle realized in quantum theory is the *Aharonov-Bohm effect*: Consider the scattering of quantum mechanical particles at a magnetic solenoid; (the wave functions of the particles are required to vanish inside the solenoid). Then the diffraction pattern seen on a screen depends non-trivially on the magnetic flux, Φ, through the solenoid in a periodic fashion, with period $\frac{h}{q}$ (or $\frac{hc}{q}$, in the units used in Sect. 2), where q is the charge of the particles. This is a consequence of the fact that the vector potential $\vec{A}$ outside the solenoid cannot be gauged away, globally, in spite of the fact that there is no electromagnetic field, thus leading to non-integrable $U(1)$-phases of quantum-mechanical wave functions which change the diffraction pattern.

The Aharonov-Bohm effect explains the possibility of fractional (or θ–, or abelian braid–) statistics of anyons [22] in two-dimensional systems: Anyons are particles carrying electric charge q and magnetic flux Φ ($= \sigma_H^{-1} q$, where σ_H is a "Hall conductivity") and hence give rise to Aharonov-Bohm phases which one can interpret as statistical phases.

After what we have learned in Sect. 2 on the $U(1)$-vector potential of Coriolis forces, it is clear that there should also exist a *"tidal" Aharonov-Bohm effect:* Consider a mass-current conducting superfluid in a large container penerated by some straight cylindrical tube that excludes the quantum fluid. Now set the fluid in circular motion around the axis of the tube with velocity field $\vec{f}$, where $|\vec{f}(r)| = \frac{V}{2\pi r}$ at a distance r from the axis of the tube, and V is a quantity of dimension cm^2/sec, the total vorticity. [We note that $V = \frac{2\pi \bar{L}_z}{NM}$, where M is the mass of the particles constituting the quantum fluid, $\bar{L}_z$ is the expectation value of the component of the total angular momentum operator parallel to the tube in the state of the system, and N is the particle number.] Small mass-currents excited in this system, scattered at the tube, will exhibit an Aharonov-Bohm effect depending periodically on V, with period $\frac{h}{m}$, where m is the mass of the particles constituting the current; see (2.43).

While this effect may be somewhat difficult to test experimentally, it is important theoretically: Consider a superfluid film with manifestly (e.g., by rotating it) or

spontaneously broken time reversal and reflection-in-lines invariance. Such a two-dimensional superfluid will, in general, exhibit *vortex excitations* of vorticity $V = n\frac{h}{M}$, $n = 0, \pm 1, \pm 2, \cdots$, where M is the mass of the constituent particles in the superfluid, and *fractional mass* (rather than fractional charge) $\sigma_H^{-1} V$, where $\sigma_H = \frac{M^2}{h}\sigma$ is the "tidal" Hall conductivity. Such excitations give rise to Aharonov-Bohm phases and hence are anyons if σ is not an integer, i.e., if the superfluid shows a fractional "tidal" Hall effect. The presence of such excitations may be tested experimentally by measuring fluctuations in the longitudinal resistance of superfluid current conduction; (see [23] for an analogous experiment).

In superfluids of particles with magnetic moments there are mixed "tidal" and electromagnetic effects (e.g., binding electric charge or magnetization to vorticity). See also [24] for a discussion of various effects encountered in superfluids.

(2) Flux quantization [25].

A superconductor exhibits the *Meissner effect*: A magnetic field cannot penetrate into the bulk of a superconducting material. However, in a type II superconductor, thin magnetic field tubes can thread through the bulk. They have the property that they carry a magnetic flux Φ which is an integer multiple of $\frac{h}{q}$, where q is the charge of the particles in the condensate, (e.g., $q = -2e$, for BCS pairs of electrons). These tubes are called *Abrikosov vortices*. The quantization of Φ is explained by requiring that outside an Abrikosov vortex the quantum mechanical properties of the system, in particular its superconducting nature, remain unchanged. From what we have said about the Aharonov-Bohm effect it follws that this requirement is fulfilled precisely if Φ is an integer multiple of $\frac{h}{q}$.

The formalism developed in Sect. 2 makes it clear that the Meissner effect and flux quantization for Abrikosov vortices have their partners in the theory of superfluidity: Consider a superfluid in some container. Now set the container in uniform rotation. The superfluid inside the container abhors angular velocity which would destroy the superfluidity and does, therefore, *not* follow the rotation of the container's walls. However, just like there can be Abrikosov vortices in a type II superconductor, the superfluid can eventually be set in motion, and the motion is generated by a velocity field $\vec{f}$, whose curl, $2\vec{\Omega}$, is localized along thin tubes. The tidal variant of the Aharonov-Bohm effect then predicts that the total vorticity in such a tube is quantized to be an integer multiple of $\frac{h}{M}$, where M is the mass of the particles (e.g. ^3He-pairs) constituting the superfluid. [This can also be understood by appealing to the quantization of orbital angular momentum.] If, in such a superfluid, one can excite mass-currents of quantum mechanical particles of mass $m < M$ one may be able to test the tidal Aharonov-Bohm effect.

Our conclusions survive a more detailed theoretical anaysis (see e.g. [26]) and are apparently tested experimentally. The phenomena described here may also be

relevant in the astrophysics of neutron stars which are apparently superfluid.

(3) The Aharonov-Casher effect [27].

Consider a system of quantum mechanical particles with spin s, electric charge 0, but with a magnetic moment $\mu \neq 0$, in a plane or in three-dimensional space. [The paritcles could be neutrons, or neutral atoms,....] Following Aharonov and Casher, we would like to study the influence of an external electric field on the dynamics of such particles. As a consequence of *relativistic effects* rapidly moving particles will, in their rest frame, feel a magnetic field that interacts with their magnetic moment.

In the formalism of Sect. 2, this effect should be described as follows: We choose the dreibein $(e^A(\mathbf{x}))_{A=1}^3$ to be the obvious one, namely $e^A{}_i(\mathbf{x}) = \delta_i^A$, for all $\mathbf{x}$, (with e^3 perpendicular to the plane of the system, in the case of a two-dimensional system). By equations (2.15), (2.21) and (2.22), the $SU(2)$-connection w on the spin-s spinor bundle $E^{(s)}$ is given by

$$\begin{aligned} w_\mu^{(s)}(\mathbf{x}) &= i\sum_{A=1}^{3} w_{\mu A}(\mathbf{x}) L_A^{(s)}\ , \quad \text{with} \\ w_{oA}(\mathbf{x}) &= -\frac{\mu}{2c} B_A(\mathbf{x}) = 0\ , \quad \text{and} \\ w_{iA}(\mathbf{x}) &= -\frac{\mu}{4c}\varepsilon_{iAD} E_D(\mathbf{x})\ . \end{aligned}$$

For general electic fields, the curvature, $dw(\mathbf{x}) + (w \wedge w)(\mathbf{x})$, of the $SU(2)$-connection w will not vanish on full-measure sets of space, and so we are not surprised to find that the electric field $\vec{E}(\mathbf{x})$ gives rise to non-trivial spin-orbit interactions. However, if we consider a system of particles confined to the $x-y$ plane in E^3 moving in the electric field of a charged wire placed along the z-axis with constant charge Q per unit of length we encounter an $SU(2)$-version of the Aharonov-Bohm effect: The electric field $\vec{E}(\mathbf{x})$ is then given by $\vec{E}(\mathbf{x}) = \frac{Q}{2\pi r^2}(x, y, 0)$, where $r = \sqrt{x^2+y^2}$. The coefficients of the $SU(2)$-connection w are given by

$$w_{13}(\mathbf{x}) = \frac{\mu}{4c} E_2(\mathbf{x}) = \frac{\mu Q}{8\pi c r^2} y\ , \tag{3.1}$$

$$w_{23}(\mathbf{x}) = -\frac{\mu}{4c} E_1(\mathbf{x}) = -\frac{\mu Q}{8\pi c r^2} x\ , \tag{3.2}$$

$w_{i1} = w_{i2} \equiv 0$, for $i = 1, 2$, and $w_{3i}(\mathbf{x})$ – which does *not* vanish – does not enter the dynamics of a system confined to the $x-y$ plane. One then checks easily that, for the two-dimensional system in the $x-y$ plane,

$$dw(\mathbf{x}) + (w \wedge w)(\mathbf{x}) = -\frac{\mu Q}{4c}\delta(\mathbf{x})\ , \tag{3.3}$$

i.e., w is *flat* outside the wire.

The quantum mechanics of this system is described by the action S_Λ introduced in (2.18), with $\Lambda = \mathsf{R} \times (\mathsf{E}^2 \backslash \{0\})$, $a_\mu = 0$, and $w_\mu^{(s)} = i w_{\mu 3} L_3^{(s)}$, with $w_{o3} = 0$ and

w_{i3} as given in (3.1) and (3.2). The point is that the scattering of the particles at the charged wire depends on its charge per unit of length, Q, because, although w is flat except at the origin, it cannot be gauged away globally! Therefore, w gives rise to "non-integrable $SU(2)$-phase factors" in the wave functions of the particles which affect their interference patterns. These patterns are periodic in Q with a period given by $\frac{4hc}{\mu}$, as follows easily from (3.1), (3.2) and (3.3).

The effect described here was first described by Aharonov and Casher [27] in a somewhat more classical language.

Next, let us consider a *two-dimensional* system on a cone with tip at $\mathbf{x} = 0$. The system consists of particles with non-zero spin. Then the spin connection $\lambda_k{}^A{}_B$, although flat for $\mathbf{x} \neq 0$, cannot be gauged away globally, although $\rho_\mu = 0$ if there are no electromagnetic fields. The $SU(2)$-connection w has the same form as in the previous example, but Q is now given by the defect angle. Scattering of particles at the tip of the cone now yields interference patterns depending on the defect angle Q. This is the "geometrical version" of the Aharonov-Casher effect which is presumably better known than its electromagnetic cousin, see e.g. [28]. What might be more surprising is that we could consider spinning particles on a two-dimensional crystal lattice with disclinations. The scattering at a disclination should also display a "geometrical Aharonov-Casher effect".

Do spinless particles "see" the tip of the cone, or is spin important? The answer depends on our choice of a quantum-mechanical state space: We must impose some "boundary conditions" on the wave functions: $\psi(r, \varphi + 2\pi - Q) = e^{i\theta}\psi(r, \varphi)$, where φ is the polar angle, and θ is some phase to be specified; besides some boundary condition at $r = 0$. But no matter how we choose θ, we can make the tip of the cone *"invisible"* to spinless particles by threading a magnetic flux through $\mathbf{x} = 0$. If the particles have spin *and* a non-zero magnetic moment then, in addition, we would have to put a charge at $\mathbf{x} = 0$, in order to make the tip invisible.

Recall that the Aharonov-Bohm effect explains why two-dimensional quantum theory can describe anyons with fractional statistics, namely particles carrying charge and flux (or mass and vorticity, ...). It is natural to ask whether the Aharonov-Casher effect also has something to do with exotic statistics in two-dimensional quantum theory. The answer is yes! The Aharonov-Casher effect is closely related to the existence of particles in two-dimensional quantum theory with *non-abelian braid statistics* [29]. Such particles can have topological interactions that can be described by some $SU(2)$-Knizhnik-Zamolodchikov connection [30]. Consider, for example, a two-dimensional chiral spin liquid made of particles with spin $s_0 \geq 1$ – if such systems exist. An incompressible chiral spin liquid of this type will most likely exhibit excitations of arbitrary spin $s = 1/2, \cdots, s_0$. The claim is that an excitation of non-zero spin $s < s_0$ will exhibit non-abelian braid statistics, as pointed out in [14]. This will be discussed again in the following section.

We would like to finally remark that there is also an analogue of the Aharonov-Casher effect where $SU(2)_{\text{spin}}$ is replaced by a gauged internal symmetry group G. This effect can, perhaps, be tested in inhomogeneous heterojunctures. It is related, physically and mathematically, to the existence of particles in two-dimensional quantum theory with topological pair interactions described by a G-Knizhnik-Zamolodchikov connection that, just as in the case of $SU(2)_{\text{spin}}$, may give rise to non-abelian braid statistics.

(4) Einstein-de Haas (-Barnett) effect [31].

Consider a cylinder of iron or some other ferromagnetic material suspended at a wire in such a way that it can freely rotate around its axis. Let us suppose that, initially, it is demagnetized and at rest. Now, imagine that the cylinder is set into rapid rotation around its axis. As explained in Sect. 2, the quantum mechanics of the electrons in this material should now be described in a uniformly rotating coordinate system fixed to the background. In this coordinate system, the electronic Hamiltonian will be time-independent, but it now contains a Zeeman term

$$\vec{\Omega} \cdot \frac{\hbar}{2}\vec{\sigma}\,, \quad (\vec{\Omega} = \text{ angular velocity})\,, \tag{3.4}$$

a tidal vector potential $\vec{f} = \vec{\Omega} \wedge \vec{x}$, and a potential $-\frac{m}{2}|\vec{\Omega} \wedge \vec{x}|^2$ of centrifugal forces; see (2.44), (2.39) and (2.30), and (2.40), respectively. These terms can be combined into $\vec{\Omega} \cdot \vec{J}$, where $\vec{J}$ is the total angular momentum operator [32]. The centrifugal forces will be balanced by the chemical potential of the background. Thus the Hamiltonian is essentially equivalent to the one for the cylinder at rest in a magnetic field $\vec{B} = -\mu^{-1}\vec{\Omega}$. The result is, in both cases, that the cylinder is *magnetized*, because the spins will be aligned with $-\vec{\Omega}, \pm\vec{B}$, respectively. Conversely, if one turns on a magnetic field, $\vec{B}$, antiparallel to the spontaneous magnetization of a magnetized piece of iron, thereby *increasing* the *free energy* of the system, the system reacts by starting to rotate around the axis of the external magnetic field so as to offset the effect of $\vec{B}$ on the electrons by rotation. It thereby returns to a state corresponding to a local minimum of the free energy. A similar effect is observed when one tries to magnetize a paramagnet. It would appear interesting to test a *local version* of this effect in a "ferro-fluid". If the magnetic field acting on a highly mobile ferro-fluid, locally in thermal equilibrium, is modified locally the fluid reacts by starting to flow with a velocity field that optimally offsets the change in the magnetic field so as to restore local equilibrium. The particle – and magnetic current densities induced are given by $n\vec{f}$ and $\vec{M} \otimes \vec{f}$, respectively, where $\vec{f}$ is the velocity field, n the particle density and $\vec{M}$ the magnetization density. A somewhat analogous effect for quantum Hall fluids will be discussed in the next section.

There is another variant [32] of the Einstein-de Haas effect: consider a beam of non-relativistic particles, e.g. heavy ions, with spin, rotating in a storage ring with

some mean angular velocity $\vec{\Omega}$. Then they experience a tidal Zeeman energy, given in (2.44), in addition to the usual magnetic Zeeman energy (2.45). After relaxation to a steady state, the tidal Zeeman energy obviously affects the ratio of "spin-up" to "spin-down" ions in the beam!

Similar considerations are important e.g. in the study of electronic spectra of rotating molecules in the Born-Oppenheimer approximation; see [32].

(5) Supercurrents [25].

Consider a superconducting condensate of charged bosons, e.g. electron pairs, of charge q and mass M, in equilibrium. Imagine that a magnetic field, $\vec{B}$, is turned on inside the bulk of this system. Since the superconducting state minimizes the free energy of the system, the condensate reacts to turning on $\vec{B}$ by developing a *flow* with velocity field $\vec{f}$ in such a way as to offset the effect of $\vec{B}$. Neglecting the centrifugal potential, $-\frac{m}{2}\vec{f}\cdot\vec{f}$, and the magnetic field created by the resulting current, it follows from eqs. (2.42), (2.43) and (2.46) that the optimal velocity field $\vec{f}$ is given by

$$\vec{f} = \frac{q}{Mc}\vec{A}^{\,T},$$

where $\vec{A}^{\,T}$ is the vector potential of $\vec{B}$ in the Coulomb gauge (i.e., div $\vec{A}^{\,T} = 0$). Thus the system exhibits a supercurrent density, $\vec{j}_s$, given in our approximation by

$$\vec{j}_s = qn\vec{f} = \frac{q^2 n}{Mc}\vec{A}^{\,T}, \tag{3.5}$$

where n is the density of the condensate. This is the London equation for type II superconductors. Recalling that $\vec{j}(x) = \delta S_{\text{eff.}}(\vec{A})/\delta\vec{A}(x)$ – see also Sect. 4 – one may proceed from eq. (3.5), fairly easily, to the Anderson-Higgs mechanism. Note that, by eq. (3.5), a supercurrent $\vec{j}_s$ is really a sign for the presence of a *vector potential*, $\vec{A}^T$, and thus can be used for experimental tests of the Aharonov-Bohm effect![2]

There is an $SU(2)_{\text{spin}}$-analogue of these effects in condensates of *neutral* bosons with *magnetic moments*. For example, in principle, one encounters an "$SU(2)$-Anderson-Higgs mechanism" and, for spin-polarized condensates, spin supercurrents induced by electric fields.

We have already alluded to the *Hall effect* earlier in this section. Just as the Aharonov-Bohm effect reflects the $U(1)_{\text{em}}$-gauge-invariance of quantum theory, so does the Hall effect for the electric current, as emphasized by Laughlin [33]. Actually one might view the Hall effect as a time-dependent version of the Aharonov-Bohm effect [21]. In the same vein, both the Aharonov-Casher effect and the Hall effect for the *spin current* reflect the $SU(2)_{\text{spin}}$-gauge-invariance of non-relativistic quantum theory, as emphasized in [5]. In the next section, we attempt to unravel the universal aspects of the quantum Hall effect in two-dimensional, incompressible electron fluids with broken parity and time reversal invariance.

[2] N. Byers and C.N. Yang, Phys. Rev. Lett. 7, 46 (1961).

4. "SCALING LIMIT" OF THE EFFECTIVE ACTION OF A TWO-DIMENSIONAL, INCOMPRESSIBLE QUANTUM FLUID.

In this section we study the generating ("partition") function of two-dimensional non-relativistic quantum systems coupled to electromagnetic fields:

$$Z_\Lambda(a,w) := \int \mathcal{D}\psi^* \mathcal{D}\psi e^{iS_\Lambda(\psi^*,\psi;a,w)/\hbar} , \tag{4.1}$$

where the gauge potentials a and w have been introduced in (2.13)–(2.16), and S_Λ is the action of the system given in (2.18); see also (2.39), (2.40) and (2.42). The integration variables ψ^* and ψ are Grassmann variables (anti-commuting c-numbers) for Fermi statistics, and complex c-number fields for Bose statistics.

We have not displayed the metric, g_{ij}, of space explicitly, since it will be kept fixed, and usually $M = \mathsf{E}^2$ with $g_{ij} = \delta_{ij}$, for simplicity. We realize that, for the study of the *stress tensor*, pressure – and density fluctuations and curvature effects, we would have to choose a variable external metric (or, at least, a variable conformal factor in g_{ij}). This would be important for an understanding of density waves, in particular surface density waves (which are interesting in two-dimensional quantum fluids), and of critical phenomena. But, unfortunately, we cannot cover everything that is interesting; confer e.g. to [17]. We note, however, that curvature effects can be studied by analyzing the dependence of $Z_\Lambda(a,w)$ on w which contains the spin connection, λ; see (2.15).

We define the electric charge – and current densities, j^0 and $\vec{j}$, by

$$\begin{aligned} j^0(x) &= \psi^*(x)\psi(x) , \\ j^k(x) &= -\frac{i\hbar}{2mc} g^{kl}(x) \left[(D_l\psi)^*(x)\psi(x) - \psi^*(x)(D_l\psi)(x) \right] , \end{aligned} \tag{4.2}$$

and the spin – and spin current densities, $\vec{s}^\mu(x)$, by

$$\begin{aligned} \vec{s}^{\,0}(x) &= \psi^*(x)\vec{L}^{(s)}\psi(x) , \\ \vec{s}^{\,k}(x) &= -\frac{i\hbar}{2mc} g^{kl}(x) \left[(D_l\psi)^*(x)\vec{L}^{(s)}\psi(x) - \psi^*(x)\vec{L}^{(s)}(D_l\psi)(x) \right] , \end{aligned} \tag{4.3}$$

where $(L_1^{(s)}, L_2^{(s)}, L_3^{(s)})$ are the generators of the spin-s representation of $su(2)$. Similarly, we can define the currents associated with internal symmetries; but, for simplicity, we shall not consider them here. The electric current is conserved (continuity equation holds), but the spin current is, in general, *not* conserved, because it couples to a non-abelian vector potential. It is, however, *covariantly conserved*; see (4.11).

It is straightforward to infer from (4.1), (2.18), (4.2) and (4.3) that the *time-ordered current Green functions* of the system are given, *at non-coinciding arguments*, by

$$\begin{aligned} &\left\langle T \left[\prod_{i=1}^{n} j^{\mu_i}(x_i) \prod_{l=1}^{m} s^{\nu_l}_{A_l}(y_l) \right] \right\rangle^c_{a,w} \\ &= i^{n+m} \prod_{i=1}^{n} \frac{\delta}{\delta a_{\mu_i}(x_i)} \prod_{l=1}^{m} \frac{\delta}{\delta w_{\nu_l A_l}(y_l)} \ln Z_\Lambda(a,w) , \end{aligned} \tag{4.4}$$

where $\langle(\cdot)\rangle^c_{a,w}$ denotes the connected expectation functional of the system in an external gauge field configuration, (a,w), (with "ground state asymptotic conditions", as $t \to \pm\infty$, to be specific), and T indicates time-ordering. At *coinciding arguments*, eq. (4.4) is modified by *Schwinger terms*, (but this will not be very important).

We define the effective gauge field action by

$$S^{\text{eff}}_\Lambda(a,w) := \frac{\hbar}{i} \ln Z_\Lambda(a,w) \,. \tag{4.5}$$

The idea is to try to calculate the "leading terms" in $S^{\text{eff}}_\Lambda(a,w)$ which, via (4.4), will provide us with information on the current Green functions. By "leading terms" we mean those terms which dominate at large distance scales and low frequencies. The calculation of the leading terms in S^{eff}_Λ may look like a fairly vast problem. Actually, making a single assumption on the excitation spectrum of the system, "*incompressibility*", and using the $U(1)_{\text{em}} \times SU(2)_{\text{spin}}$-*gauge-invariance* of the system, that calculation can be carried out, [5].

Let χ be a real-valued function and R an $SU(2)$-valued function on space-time $N = \mathbf{R} \times M$. Consider the gauge transformations in eq. (2.49), i.e.,

$$a \mapsto {}^\chi a \,, \quad {}^\chi a_\mu := a_\mu - \partial_\mu \chi \,, \tag{4.6}$$

and

$$w \mapsto {}^R w \,, \quad {}^R w_\mu := R w_\mu R^* + R \partial_\mu R^* \,. \tag{4.7}$$

Changing integration variables,

$$\psi \mapsto {}^{\chi,R}\psi := e^{i\chi} U^{(s)}(R)\psi \tag{4.8}$$

in the functional integral (4.1), and using the gauge-invariance of S_Λ under the transformations (4.6) – (4.8) and the fact that the Jacobian of (4.8) is unity, we find the *Ward identity*

$$S^{\text{eff}}_\Lambda(a,w) = S^{\text{eff}}_\Lambda({}^\chi a, {}^R w) \,, \tag{4.9}$$

for all χ and R. [For a system of finitely many particles in a bounded region of space, (4.9) can be proven rigorously. This identity is stable under passing to limits, for χ's and dR's of compact support.]

By differentiating (4.9) in χ or R and setting $\chi = 0, R = 1$, we find, using (4.4) (for $n + m = 1$), that

$$\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \langle j^\mu \rangle_{a,w}\right) = 0 \,, \tag{4.10}$$

and

$$\frac{1}{\sqrt{g}} D_\mu \left(\sqrt{g} \langle s^\mu \rangle_{a,w}\right)_A = 0 \,, \quad A = 1,2,3 \,,$$

or

$$\begin{aligned} &\frac{1}{\sqrt{g(x)}} \partial_\mu \left(\sqrt{g(x)} \langle \vec{s}^\mu(x) \rangle_{a,w}\right) \\ &\qquad = 2\vec{w}_\mu(x) \wedge \langle \vec{s}^\mu(x) \rangle_{a,w} \,, \end{aligned} \tag{4.11}$$

for *arbitrary* a and w. These "infinitesimal" Ward identities play an important role in determining the general form of S_Λ^{eff}. They can be generalized, in an obvious way, to systems with internal symmetries.

We now proceed to determine the form of S_Λ^{eff} in the scaling limit. We need to consider ever larger systems and ever slower variations in time. Let $1 \leq \theta < \infty$ be a scale parameter. We set

$$\begin{aligned} g_{ij}(x) &\equiv g_{ij}^{(\theta)}(x) := \gamma_{ij}\left(\frac{x}{\theta}\right), \quad \text{and} \\ \Lambda &\equiv \Lambda^{(\theta)} := \theta\Lambda_0 , \end{aligned} \tag{4.12}$$

where γ_{ij} is a fixed metric on M (e.g. $\gamma_{ij} = \delta_{ij}$), and Λ_0 is a fixed space-time cylinder;

$$x = (x^0, \mathbf{x}) = \theta(\xi^0, \boldsymbol{\xi}) = \theta\xi , \quad \xi \in \Lambda_0 . \tag{4.13}$$

Then

$$\frac{\partial}{\partial x^\mu} = \theta^{-1}\frac{\partial}{\partial \xi^\mu} . \tag{4.14}$$

We propose to study the reaction of the system to a small change in the external gauge potentials a and w. We choose fixed background potentials, $a_c(x)$ and $w_c(x)$, defined on all of space-time, and set

$$a_\mu{}^{(\theta)}(x) := a_{c,\mu}(x) + \theta^{-1}\tilde{a}_\mu\left(\frac{x}{\theta}\right) , \tag{4.15}$$

and

$$w_\mu{}^{(\theta)}(x) := w_{c,\mu}(x) + \theta^{-1}\tilde{w}_\mu\left(\frac{x}{\theta}\right) , \tag{4.16}$$

where $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi)$ are *fixed* functions defined on Λ_0. If m is the effective mass of the particles and μ their magnetic moment in physical $(t, \mathbf{x})$-coordinates then the mass $m^{(\theta)}$ and magnetic moment $\mu^{(\theta)}$ in *rescaled coordinates*, $\tau \equiv \frac{\xi^0}{c}, \boldsymbol{\xi}$, are given by

$$m^{(\theta)} = m \cdot \theta , \quad \text{and } \mu^{(\theta)} = \mu\theta^{-1} , \tag{4.17}$$

as follows from eqs. (2.18), (2.25) and (2.26), (i.e., in the *rescaled* system the particles are heavy and have small magnetic moments. Moreover, the range of the two-body potential, in the *rescaled* system, becomes shorter and shorter, as θ becomes large).

One *basic assumption* underlying our analysis is that $S_{\theta\Lambda_0}^{\text{eff}}(a^{(\theta)}, w^{(\theta)})$ is *four times continuously differentiable* in $\tilde{a}_\mu^{(\theta)}(x) \equiv \theta^{-1}\tilde{a}_\mu\left(\frac{x}{\theta}\right)$ and $\tilde{w}_\mu^{(\theta)}(x) \equiv \theta^{-1}\tilde{w}_\mu\left(\frac{x}{\theta}\right)$ at $\tilde{a}_\mu^{(\theta)} = \tilde{w}_\mu^{(\theta)} = 0$, for a *suitable choice of background potentials*, a_c and w_c, and for $\tilde{a}_\mu$ and $\tilde{w}_\mu$ constrained to belong to *suitable spaces*, $\mathcal{A}$ and $\mathcal{W}$, *of fluctuation potentials*, to be specified later. We may then expand $S_{\theta\Lambda_0}^{\text{eff}}$ to third order in $\tilde{a}^{(\theta)}$ and $\tilde{w}^{(\theta)}$, with a fourth order remainder term. Among the terms thus generated we shall only retain the leading terms in θ, namely those scaling with a *non-negative power* of θ which are commonly called *relevant* and *marginal* terms. The sum of these terms will be denoted by $S_{\Lambda_0}^*(\tilde{a}, \tilde{w})$, a functional that we call the *scaling limit of the effective action.*

Using identity (4.4) to find the Taylor coefficients of $S^{\rm eff}_{\theta\Lambda_0}(a^{(\theta)}, w^{(\theta)})$, plugging (4.15) and (4.16) into the resulting expressions, and finally passing to $(\xi^0, \boldsymbol{\xi})$-coordinates, we find that the coefficient of the term of $n^{\rm th}$ order in $\tilde{a}$ and of $m^{\rm th}$ order in $\tilde{w}$ in $S^{\rm eff}_{\theta\Lambda_0}$ is given by a distribution

$$\varphi_\theta^{\mu_1\cdots\nu_1\cdots}{}_{A_1\cdots A_m}(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_m)$$

which, at non-coinciding arguments, is given by

$$(-i)^{n+m}\left\langle T\left[\prod_{i=1}^{n}(\theta^2 j^{\mu_i}(\theta\xi_i))\prod_{l=1}^{m}(\theta^2 s^{\nu_l}_{A_l}(\theta\eta_l))\right]\right\rangle^c_{a_c, w_c}, \tag{4.18}$$

in accordance with the circumstance that, in *three* space-time dimensions, the scaling dimension of currents is 2!

We may now formulate our *basic assumption of incompressibility*: We imagine that, for *certain choices of the background potentials* a_c *and* w_c, the excitation spectrum of the system above its groundstate (energy) is such that connected Green functions of its *currents* have *"good" cluster properties* (better than in a system with Goldstone bosons), in such a way that the limits of the distributions $\varphi_\theta^{\mu_1\cdots\nu_1\cdots}{}_{A_1\cdots A_m}$, as $\theta \to \infty$, are *local distributions*, i.e., sums of products of derivatives of δ-functions.

This incompressiblity assumption is by no means a mild or minor assumption. It tends to be a really hard analytical problem of many-body theory to show that, for a concrete system, it is satisfied. [For some recent ideas about how to establish it for quantum Hall fluids at certain filling factors see [34,35,14].] What we propose to do here is to use it to calculate the general form of the action $S^*_{\Lambda_0}$ in the scaling limit. We only sketch some ideas; for the details see [5].

Our calculation is based on the following four principles:

(A) *Incompressibility*: $\varphi_\theta^{\mu_1\cdots\mu_n\nu_1\cdots\nu_m}{}_{A_1\cdots A_m}$ converge to local distributions, as $\theta \to \infty$, for all n and m.

(B) $U(1)_{\rm em} \times SU(2)_{\rm spin}$-*gauge-invariance*: Ward identities (4.9) – (4.11).

(C) *Only relevant and marginal terms are kept in* $S^*_{\Lambda_0}$.

(D) *Extra symmetries of the system*, e.g., for $a_{c0} = 0$, $w_{c\mu A} = \delta_{A3} w_{c\mu 3}$, global rotations around the 3-axis in spin space are a continuous, global symmetry of the system with an associated *conserved* Noether current $s_{\mu 3}(x)$; or translation invariance in the scaling limit $(\theta \to \infty)\cdots$, *are exploited to reduce the number of terms.*

From (A) and eqs. (4.15) and (4.16) it immediately follows that all terms contributing to $S^{\rm eff}_{\theta\Lambda_0}$ of order 4 or higher in $\tilde{a}$ and $\tilde{w}$ are *irrelevant*, (scaling like $\theta^{-D}, D >$

0). In particular, a fourth-order remainder term does *not* contribute to $S^*_{\Lambda_0}$, (principle (C)). We now present the final result, in the special case of systems which are incompressible for a choice of $w_{c\mu}$ satisfying

$$w_{c\mu A}(x) = \delta_{A3} w_{c\mu 3}(x) \,, \tag{4.19}$$

or, in view of eqs. (2.25) and (2.26), for a background electromagnetic field $(\vec{E}_c, \vec{B}_c)$ with

$$\vec{B}_c(x) = (0, 0, B_c(x)) \,, \; \vec{E}_c(x) = (E_1(x), E_2(x), 0) \,, \tag{4.20}$$

and a spin connection

$$\left(\lambda_\mu{}^A{}_B\right) = \begin{pmatrix} 0 & \lambda_\mu & 0 \\ -\lambda_\mu & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{4.21}$$

in the coordinate system $(e^1(x), e^2(x), e^3(x))$. In this situation, the scaling limit of the effective action is given by

$$\begin{aligned} -\frac{1}{\hbar} S^*_{\Lambda_0}(\tilde{a}, \tilde{w}) &= \int_{\Lambda_0} j^\mu_c \tilde{a}_\mu dv + \int_{\Lambda_0} m^\mu_3 \tilde{w}_{\mu 3} dv \\ &+ \sum_{A=1}^{2} \int_{\Lambda_0} \tau_1^{\mu\nu} \tilde{w}_{\mu A} \tilde{w}_{\nu A} dv + \sum_{A,B=1}^{2} \int_{\Lambda_0} \tau_2^{\mu\nu} \varepsilon_{AB} \tilde{w}_{\mu A} \tilde{w}_{\nu B} dv \\ &+ \sum_{A,B,C=1}^{3} \int_{\Lambda_0} \eta^{\mu\nu\rho}_{ABC} \tilde{w}_{\mu A} \tilde{w}_{\nu B} \tilde{w}_{\rho C} dv \\ &+ \frac{\sigma}{4\pi} \int_{\Lambda_0} \tilde{a} \wedge d\tilde{a} + \frac{\chi}{2\pi} \int_{\Lambda_0} \tilde{a} \wedge d\tilde{w}_3 + \frac{\sigma_s}{4\pi} \int_{\Lambda_0} \tilde{w}_3 \wedge d\tilde{w}_3 \\ &+ \frac{k}{4\pi} \int_{\Lambda_0} \mathrm{tr}\left(w \wedge dw + \frac{2}{3} w \wedge w \wedge w\right) + B.T. \end{aligned} \tag{4.22}$$

where j^μ_c is an electric – and m^μ_3 a magnetic supercurrent circulating in the system when $a = a_c$, $w = w_c$; $\tau_1^{\mu\nu}$ is a function symmetric in μ and ν, while $\tau_2^{\mu\nu}$ is antisymmetric in μ and ν; the function $\eta^{\mu\nu\rho}_{ABC}$ is symmetric under interchanges of $(\mu A), (\nu B)$ and (ρC) and vanishes if two or more of the indices A, B, C are equal to 3; $dv = \sqrt{\gamma(\xi)} d\xi$ is the volume element on space-time; σ, χ, σ_s and k are real constants, whose possible values will be studied in Sect. 5; $w(\xi) = w_c^{(\theta)}(\xi) + \tilde{w}(\xi)$ is the total $SU(2)$ connection, with $w_c^{(\theta)}(\xi) = \theta w_c(\theta\xi)$, by (4.16); and "B.T." are boundary terms only depending on $a|_{\partial\Lambda_0}$, $w|_{\partial\Lambda_0}$ which will be studied in Sect. 5. In the last four terms on the r.h.s. of (4.22) we are using a *new notation*:

$$\begin{aligned} \tilde{a} &= \sum_{\mu=0}^{2} \tilde{a}_\mu d\xi^\mu \,, \quad d\tilde{a} = \sum_{\mu,\nu=0}^{2} \partial_\mu \tilde{a}_\nu \, d\xi^\mu \wedge d\xi^\nu \\ \tilde{w}_3 &= \sum_{\mu=0}^{2} \tilde{w}_{\mu 3} d\xi^\mu \,, \quad w = \sum_{\mu=0}^{2} \sum_{A=1}^{3} w_{\mu A} \sigma_A \, d\xi^\mu \,. \end{aligned} \tag{4.23}$$

See [5] for more details.

In Sect. 5, we shall use results on $U(1)$ – and $SU(2)$ chiral current algebra to determine the possible values of σ, χ, σ_s and k and find some relations between them.

Here we wish to point out that the functions $j_c^\mu, m_3^\mu, \tau_1^{\mu\nu}, \tau_2^{\mu\nu}$ and $\eta_{ABC}^{\mu\nu\rho}$ are not all independent, but are constrained by the infinitesimal Ward identities (4.10) and (4.11): By (4.4)

$$\langle j^\mu(\xi)\rangle_{a^{(\theta)},w^{(\theta)}} = \frac{\delta S^*_{\Lambda_0}(\tilde{a},\tilde{w})}{\delta \tilde{a}_\mu(\xi)} + \cdots , \tag{4.24}$$

$$\langle s_A^\mu(\xi)\rangle_{a^{(\theta)},w^{(\theta)}} = \frac{\delta S^*_{\Lambda_0}(\tilde{a},\tilde{w})}{\delta \tilde{w}_{\mu A}(\xi)} + \cdots . \tag{4.25}$$

The dots stand for contributions from irrelevant terms in the effective action. We calculate the r.h.s. of these equations by using (4.22) and plug the result into eqs. (4.10) and (4.11). As a result we obtain the following constraints (see [5]).

$$\begin{aligned}
&(a) \qquad \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}j_c^\mu) = 0 . \\
&(b) \qquad \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}m_3^\mu) = 0 . \\
&(c) \qquad \sum_{B=1}^{2}\varepsilon_{AB}\left\{m_3^\mu - 2\tau_1^{0\mu}w_{c03}^{(\theta)}\right\}\tilde{w}_{\mu B} \\
&\qquad +2w_{c03}^{(\theta)}\sum_{j=1}^{2}\tau_2^{0j}\tilde{w}_{jA} = 0 , \quad A=1,2 . \\
&(d) \qquad \frac{1}{\sqrt{g}}\partial_\mu\left\{\sqrt{g}\tau_1^{\mu\nu}\tilde{w}_{\nu A} + \sqrt{g}\sum_{B=1}^{2}\varepsilon_{AB}\tau_2^{\mu\nu}\tilde{w}_{\nu B}\right\} \\
&\qquad = -2\left\{\sum_{B=1}^{2}\varepsilon_{AB}\tau_1^{\mu\nu}\tilde{w}_{\nu B} - \tau_2^{\mu\nu}\tilde{w}_{\nu A}\right\}\tilde{w}_{\mu 3} \\
&\qquad -3\sum_{B=1}^{2}\varepsilon_{AB}\sum_{C,D=1}^{3}\eta_{BCD}^{0\nu\rho}w_{c03}^{(\theta)}\tilde{w}_{\nu C}\tilde{w}_{\rho D} , \quad A=1,2 .
\end{aligned} \tag{4.26}$$

Constraints (a) and (b) just express the conservation of the supercurrents j_c^μ and m_3^μ when $\tilde{a} = \tilde{w} = 0$.

If we impose (4.26)-(c) and (d), for *arbitrary* smooth fluctuation potentials $\tilde{w}$, then it follows that

$$m_3^\mu = \tau_1^{\mu\nu} = \tau_2^{\mu\nu} = 0 , \quad \text{for all } \mu \text{ and } \nu , \tag{4.27}$$

in particular, the system *cannot* be *magnetized* ($m_3^0 = 0$) and cannot support persistent spin currents. This may seem rather strange, because we would expect that if $w_{c03} = -\frac{\mu}{2c}B_c$, for some large magnetic field $\vec{B}_c = (0,0,B_c)$, then the system would be magnetized in the 3-direction. What has gone wrong? The point is that the assumed properties that S_Λ^{eff} is four times continuously differentiable in $\tilde{a}$ and $\tilde{w}$ and that the system remains incompressible in an *arbitrary* function-space neighborhood of (a_c, w_c) of sufficiently small diameter must fail for magnetized systems! The reason is that an *arbitrarily small* fluctuation field $\tilde{w}$ which oscillates rapidly in time can destroy the incompressibility of the system, and hence our estimate on the fourth order remainder in the Taylor expansion of S_Λ^{eff} breaks down.

We thus assume, for example, that, for a *time-independent* background field w_c, the system remains incompressible and $S_\Lambda^{\rm eff}$ is four times continuously differentiable in $(\tilde{a}, \tilde{w})$ on the function-space sets

$$\begin{aligned} \mathcal{A} &= \{\tilde{a}_\mu \in \mathcal{S}\} \ , \\ \mathcal{W} &= \{\tilde{w}_{\mu A} \in \mathcal{S} : \tilde{w}_{\mu A} \text{ is time-independent}\} \ , \end{aligned} \tag{4.28}$$

where $\mathcal{S}$ is some Schwartz space neighbourhood of 0. Then constraints (c) and (d) of (4.26) imply that

$$\tau_1^{00}(\xi) = \frac{m_3^0(\xi)}{2w_{c03}(\xi)} \ , \quad \tau_1^{0i} = \tau_1^{ij} = 0 \ , \tag{4.29}$$

$$\begin{aligned} \tau_2^{\mu\nu} &= 0 \ , \quad \eta_{AA3}^{000}(\xi) = -\frac{\tau_1^{00}(\xi)}{3w_{c03}(\xi)} \ , \quad \text{for } A = 1,2; \\ &\qquad \text{all other } \eta_{AB3}^{0\mu\nu} \text{ vanish.} \end{aligned} \tag{4.30}$$

Hence $(m_3^\mu) = (m_3^0, \mathbf{0})$. Under somewhat more restrictive assumptions on $\mathcal{W}$, imposing e.g. relations (2.25) and (2.26) on $\tilde{w}$ which couple $\tilde{w}$ to $\tilde{a}$, a non-zero spin current $\mathbf{m}_3 = (m_3^1, m_3^2)$ is possible, too. For a more detailed discussion see [5].

A corollary of our derivation of $S_{\Lambda_0}^*$, in particular of (4.29), using gauge invariance and incompressibility, is the *Goldstone theorem*, [36]: If the magnetization, $M = \mu\frac{\hbar}{2}m_3^0$, does *not* tend to 0, as $\vec{B}_c = (0,0,B_c)$ tends to 0 (with $w_{c03} = -\frac{\mu}{2c}B_c$) then the system *cannot* be incompressible at $\vec{B}_c = 0$, i.e., there are gapless extended modes, the *Goldstone bosons*, coupled to the groundstate by the spin current; see [5]. Our proof also works for systems with continuous non-abelian internal symmetries.

Next, let us briefly discuss the *linear response equations* (4.24) and (4.25) that follow from our expression (4.22) for the effective action $S_{\Lambda_0}^*$ in the scaling limit, for systems characterized by conditions (4.28) – (4.30). It is a simple exercise to verify that

$$\begin{aligned} \sqrt{g(\xi)}\,\langle j^\mu(\xi)\rangle_{a,w} &= \sqrt{g(\xi)}j_c^\mu(\xi) + \frac{\sigma}{2\pi}\varepsilon^{\mu\nu\rho}\left(\partial_\nu\tilde{a}_\rho\right)(\xi) \\ &\quad + \frac{\chi}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\nu\tilde{w}_{\rho 3})(\xi) + \cdots \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} \sqrt{g(\xi)}\,\langle s_A^\mu(\xi)\rangle_{a,w} &= \sqrt{g(\xi)}\delta_{A3}\delta_0^\mu m_3^0(\xi) + \delta_{A3}\frac{\chi}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\nu\tilde{a}_\rho)(\xi) \\ &+ \delta_{A3}\frac{\sigma_s}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\nu\tilde{w}_{\rho 3})(\xi) \\ &- \frac{k}{\pi}\varepsilon^{\mu\nu\rho}\left\{(\partial_\nu w_{\rho A})(\xi) - \varepsilon_{ABC}w_{\nu B}(\xi)w_{\rho C}(\xi)\right\} \\ &+ \sqrt{g(\xi)}2(1-\delta_{A3})\delta_0^\mu\tau_1^{00}(\xi)\tilde{w}_{0A}(\xi) + \cdots \ , \end{aligned} \tag{4.32}$$

where the dots stand for terms coming from irrelevant terms in the effective action, or from terms of order two in $\tilde{w}$ (e.g. a term proportional to η_{ABC}^{000}) which are of little interest in linear response theory. Furthermore, $w_{\mu A} = w_{c\mu A}^{(\theta)} + \tilde{w}_{\mu A}$.

In order to understand the physical contents of these equations, we must remind ourselves of the physical meaning of the connections a and w elucidated in Sect. 2: From eqs. (2.20), (2.39) and (2.43) we know that

$$a_j(x) = \frac{e}{\hbar c} A_j(x) - \frac{m}{\hbar} f_j(x) \,, \tag{4.33}$$

where $\vec{A}$ is the electromagnetic vector potential, $-e$ is the charge and m the effective mass of the particles in the quantum fluid, and $\vec{f}$ is a divergence-free velocity field generating some incompressible superfluid flow. Furthermore, by (2.20),

$$a_0(x) = -\frac{e}{\hbar c}\phi(x) \,, \tag{4.34}$$

where ϕ is the electrostatic potential.

Since we are studying two-dimensional incompressible quantum fluids on a surface M imbedded in E^3, it is natural to choose an $SU(2)_{\text{spin}}$-gauge with the property that $e^3(t,\mathbf{x})$ is orthogonal to the tangent space of M at $\mathbf{x}$, for all times t, as discussed at the beginning of Sect. 2. Then the $SU(2)$-spin connection $\lambda^{(1/2)}$ has the form

$$\lambda_j^{(1/2)} = i\lambda_j\sigma_3 \,,\; j = 1,2; \quad \text{and } \lambda_{0\,B}^{\,A} = \frac{1}{2}\left[\mathfrak{E}^l_A \partial_0 e_l^B - \mathfrak{E}^l_B \partial_0 e_l^A\right] . \tag{4.35}$$

It then follows from (2.24), (2.25) and (2.39), (2.44) that

$$w_{0A}(x) = -\frac{\mu}{2c} B_A(x) + \delta_{A3}\Omega(x) + \lambda_{0A} \tag{4.36}$$

where $\vec{\Omega}(x) = (0,0,\Omega(x))$ is the curl of $\vec{f}$ in the dreibein basis $(e^A(x))_{A=1}^3$, and μ is the magnetic moment of the particles. Finally, by (2.15), (4.35), (2.26) and (2.33),

$$w_{jA}(x) = \delta_{A3}(\lambda_j(x) + \cdots) - \frac{\mu}{4c}\sum_{C=1}^{3} \varepsilon_{jAC}(x) E_C(x) \,, \tag{4.37}$$

where the dots correspond to terms proportional to derivatives of $\Omega(t',\mathbf{x}), t' \leq t$, (and are generated by the $SU(2)_{\text{spin}}$-gauge-transformation defined in (2.31)).

Finally, we define the charge density operator, in physical units

$$\rho(\xi) := e\sqrt{g(\xi)}\, j^0(\xi) \,, \tag{4.38}$$

the electric current density by

$$\mathbf{J}(\xi) = ec\sqrt{g(\xi)}\,\mathbf{j}(\xi) \,, \tag{4.39}$$

the spin density by

$$\vec{S}^0(\xi) = \frac{\hbar}{2}\sqrt{g(\xi)}\,\vec{s}^{\,0}(\xi) \,, \tag{4.40}$$

and the spin current density by

$$\vec{S}^j(\xi) = \frac{\hbar c}{2}\sqrt{g(\xi)}\,\vec{s}^{\,j}(\xi) \,. \tag{4.41}$$

Then equation (4.31) for the 0-component reads

$$\begin{aligned}\langle\rho(\xi)\rangle_{a,w} &= \rho_c(\xi) + \frac{\sigma_H}{c}\tilde{B}_3(\xi) - 2\sigma\frac{em}{h}\tilde{\Omega}(\xi) \\ &- \frac{\chi}{2\pi}\left(\frac{e\mu}{4c}\nabla\cdot\tilde{\boldsymbol{E}}(\xi) - e\mathcal{R}(\xi)\right) + \cdots ,\end{aligned} \tag{4.42}$$

where $\sigma_H = \frac{e^2}{h}\sigma$ is the Hall conductivity, $\nabla\cdot(\quad)$ denotes the divergence , $\mathcal{R}(\xi) = \text{curl}\,\boldsymbol{\lambda}(\xi)$ is the scalar curvature of M at ξ, and the dots stand for contributions from irrelevant terms. It will turn out that

$$\chi_\perp := \frac{e\mu}{4\pi c}\chi \tag{4.43}$$

is the magnetic susceptibility of the system in the 3-direction normal to the surface. In (4.42) and the following formulas the tildes ˜ indicate contributions from $\tilde{a}$ and $\tilde{w}$; (we have absorbed the spin connection λ into $\tilde{w}$, but without decorating it with a ˜). Next, one verifies that

$$\begin{aligned}\left\langle J^i(\xi)\right\rangle_{a,w} &= J^i_c(\xi) + \sigma_H\varepsilon^{ij}\tilde{E}_j(\xi) \\ &+ \frac{\sigma em}{h}\varepsilon^{ij}\frac{\partial}{\partial\tau}\tilde{f}_j(\xi) \\ &- \frac{\chi}{2\pi}\left(\frac{e\mu}{2}\varepsilon^{ij}\partial_j\tilde{B}_3(\xi) - e\varepsilon_{ij}\partial_j\tilde{\Omega}(\xi)\right) \\ &+ \frac{\chi}{2\pi}\left(\frac{e\mu}{4c}\frac{\partial}{\partial\tau}\tilde{E}^i(\xi) - e\varepsilon^{ij}\frac{\partial}{\partial\tau}\lambda_j(\xi)\right) \\ &+ \cdots ,\end{aligned} \tag{4.44}$$

where $\tau = \xi^0/c$ is the rescaled time variable.

From (4.32) we find, for example, that

$$\begin{aligned}\mu\langle S^0_3(\xi)\rangle_{a,w} &= M(\xi) + \sigma_H^{\text{spin}}\left(\frac{\mu}{2c}\nabla\cdot\tilde{\boldsymbol{E}}(\xi) - 2\mathcal{R}(\xi)\right) + k\frac{\mu^2\hbar}{8\pi c}\nabla\cdot\mathbf{E}_c(\xi) \\ &+ \chi_\perp\left(\tilde{B}_3(\xi) - \frac{2c}{\mu}\tilde{\Omega}(\xi)\right) + \cdots ,\end{aligned} \tag{4.45}$$

where M is the magnetization at (a_c, w_c), $\chi_\perp$ is the magnetic susceptibility at (a_c, w_c) given in (4.43), and

$$\sigma_H^{\text{spin}} = \mu\hbar\frac{k}{4\pi} - \mu\hbar\frac{\sigma_s}{8\pi} \tag{4.46}$$

is the *Hall conductivity for the spin current.* As eq. (4.45) shows, σ_H^{spin} is a *pseudoscalar.* Next

$$\begin{aligned}\langle S^i_3(\xi)\rangle_{a,w} &= \sigma_H^{\text{spin}}\left(\varepsilon^{ij}\partial_j\tilde{B}_3(\xi) - \frac{2c}{\mu}\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi) - \frac{1}{2c}\frac{\partial}{\partial\tau}\tilde{E}^i(\xi)\right. \\ &+ \left.\frac{2}{\mu}\frac{\partial}{\partial\tau}\lambda^i(\xi)\right) + k\frac{\mu\hbar}{4\pi}\varepsilon^{ij}\partial_j B_c(\xi) \\ &+ \chi_\perp\left(c\mu^{-1}\varepsilon^{ij}\tilde{E}_j(\xi) + j\frac{mc}{\mu e}\frac{\partial}{\partial\tau}\tilde{f}^i(\xi)\right) + \cdots\end{aligned} \tag{4.47}$$

where the dots stand for terms proportional to λ_0 and further irrelevant and higher-order terms. A similar story could be told for $\langle S_A^\mu(\xi)\rangle_{a,w}$, but we refrain from telling it and refer the reader to his drawing board, or to [5]. [We do not guarantee all signs and factors of 2π in our formulas!]

We encourage the reader to notice how neatly our formulas summarize the laws of the Hall effect, including effects due to *tidal forces* coming from *superfluid flow* and due to the *curvature* of the sample. [We believe that the tidal terms might be relevant in the study of the transition from one plateau of σ_H to the next one in very pure samples.]

Our next topic concerns the analysis of some quasi-particle excitations above the groundstate in a two-dimensional, incompressible quantum fluid, whose effective action in the scaling limit is given by the action $S^*_{\Lambda_0}$ computed above; see (4.22). For simplicity, we start by considering a flat, two-dimensional system of charged fermions with vanishing magnetic moment, so that the $SU(2)_{\text{spin}}$-connection w vanishes identically in an appropriate $SU(2)$-gauge, $(e^1(x), e^2(x), e^3(x)$ are chosen to be

time-independent, so that there is no tidal Zeeman term; see Sect. 2). We suppose that, in a small neighborhood of a suitably chosen background potential a_c – typically $a_{c0} = 0, b_c = da_c$ constant and of suitable magnitude – the system is incompressible. Then the action in the scaling limit is given by

$$-\frac{1}{\hbar}S^*_{\Lambda_0}(\tilde{a}) = \int_{\Lambda_0} j_c^\mu \tilde{a}_\mu dv + \frac{\sigma}{4\pi}\int_{\Lambda_0} \tilde{a}\wedge d\tilde{a}\ , \tag{4.48}$$

up to boundary terms. The first term on the r.h.s. is unimportant in the following discussion, and we set $j_c^\mu = 0$.

Let us produce a *"Laughlin vortex"* [37] in this system by turning on a magnetic field $\tilde{b}(\xi) = \partial_1\tilde{a}_2(\xi) - \partial_2\tilde{a}_1(\xi)$ in a small disc. [Actually, $\tilde{b}(\xi)$ could be a vorticity field of a superfluid flow if, instead of a quantum Hall fluid, we consider a superfluid film. We shall nevertheless use "magnetic language" in the following discussion.] From our discussion of the Aharonov-Bohm effect in Sect. 3–(1) we know that this excitation only disturbs the system locally, and thus may have a finite energy difference to the groundstate energy, if

$$\frac{1}{2\pi}\int \tilde{b}(t,\boldsymbol{\xi})d^2\boldsymbol{\xi} = n\ , \quad n\in \mathbf{Z}\ . \tag{4.49}$$

By eq. (4.31) for $\mu = 0$, we have that

$$\langle j^0(\xi)\rangle_{a,w} = \frac{\sigma}{2\pi}\tilde{b}(\xi)\ ,$$

and hence the charge of the excitation (background charge normalized to 0) is given by

$$q = \int \langle j^0(t,\boldsymbol{\xi})\rangle_{a,w} d^2\boldsymbol{\xi} = \sigma n\ . \tag{4.50}$$

If σ is not an integer then q will be fractional, in general. Now, consider two such excitations localized in two disjoint small disks and interchange them along some

paths oriented anti-clock-wise. According to Sect. 3–(1), the Aharonov-Bohm phase picked up in this process is given by

$$e^{2\pi i\theta} := e^{i\pi q n} = e^{i\pi\sigma n^2} , \tag{4.51}$$

where we have normalized the statistical phase θ such that $\theta = 1/2$ corresponds to *Fermi statistics*; $\theta = 0$ corresponds to *bosons*, and $\theta \neq 0, 1/2 \pmod 1$ to *anyons* [22]. Thus, Laughlin vortices are anyons, unless σn^2 is an integer.

Among the excitations that one can produce in this fashion there should be the particles constituting the system. Let us suppose that the state of the system is fully spin-polarized, (as is the case for filling factors $\nu \overset{\text{e.g.}}{=} \frac{1}{3}, \frac{1}{5}$ in quantum Hall fluids). Suppose a magnetic flux of n_0 produces a state of N electrons. From (4.50) we then infer that

$$\sigma = \frac{N}{n_0} . \tag{4.52}$$

If N is *odd* this state is composed of N fermions and hence describes a fermion, so that, by (4.51),

$$e^{i\pi N n_0} = -1 . \tag{4.53}$$

Thus n_0 must be *odd*, too. In fact, one may show that if N and n_0 have no common divisor then n_0 is odd. In particular, for $N = 1$ we conclude that

$$\sigma = 1/n_0 , \text{ with } n_0 \text{ odd} . \tag{4.54}$$

This is the famous odd-denominator rule; see e.g. [38]. An excitation with vorticity 1 then has fractional charge $q = 1/n_0$ and is an anyon, for $n_0 > 1$.

Note that the vector potential, $\tilde{a}$, created by a pointlike excitation of charge q located at $\xi = 0$ is given by

$$\tilde{a}_i(\xi) = -\frac{q}{\sigma}\varepsilon_{ij}\frac{\xi^j}{|\xi|^2} , \tag{4.55}$$

as follows from (4.49) and (4.50) for $\langle j^0(\xi)\rangle_{a,w} = q\delta_0(\xi)$. This is the "$U(1)$-Knizhnik-Zamolodchikov connection".

Next, we consider another "in vitro" system, namely a "chiral spin liquid". [It is not entirely clear that such systems exist in nature.] A chiral spin liquid is a system of neutral particles of spin $s > 0$ and with non-zero magnetic moment ($\mu = 1$, in present units) having a spin-singlet groundstate for some non-zero, constant magnetic field, $\vec{B}_c$. It is assumed, here, to be incompressible and to exhibit breaking of parity and time reversal, but no spontaneous magnetization. In our formalism, the effective action of such a system in the scaling limit is given by

$$-\frac{1}{\hbar}S^*_{\Lambda_0}(w) = \frac{k}{4\pi}\int_{\Lambda_0} \mathrm{tr}(w \wedge dw + \frac{2}{3} w \wedge w \wedge w) , \tag{4.56}$$

up to boundary terms. Under reflections in lines, w_i transforms as a vector, w_0 as a pseudoscalar and k as a pseudoscalar. Let us consider an excitation created by

turning on an $SU(2)$-gauge field w with field strength, g, given by

$$g(\xi) = dw(\xi) + w(\xi) \wedge w(\xi) .$$

For example, we may choose g to be given by

$$\vec{g}_{12}(\xi) = -\vec{e} g_0(\xi) ,$$

where $\vec{e}$ is some unit vector in $\mathbf{R}^3$ and g_0 is time-independent, $\vec{g}_{0i}(\xi) = 0$. By eq. (4.32), the spin density of this excitation is given by

$$\langle \vec{s}^{\,0}(\xi) \rangle_w = \vec{e} \frac{k}{\pi} g_0(\xi) , \tag{4.57}$$

so that the expectation value of its total spin operator, $\vec{S}$, is given by

$$\langle \vec{S} \rangle_w = \vec{e} \frac{k}{\pi} \int g_0(\xi) d^2\xi .$$

Such an excitation is commonly called a "*spinon*". Quantum mechanically, spin is quantized: $\vec{S} \cdot \vec{S} = 4l(l+1)$, $l \in \frac{1}{2}\mathbf{Z}$. Consider a spinon of spin l located at the point $\xi = \xi_1$. Then eq. (4.57) says that $\vec{g}_{12}(\xi)$ is the solution of the equation

$$\langle \vec{L}^{(l)} \rangle_w \delta(\xi - \xi_1) = -\frac{k}{\pi} \vec{g}_{12}(\xi) , \tag{4.58}$$

where $\vec{L}^{(l)}$ is the spin operator, $\vec{S}$, in the spin-l representation; (see Sect. 2). A connection $\vec{w}$ for the field strength $\vec{g}$ satisfying (4.58), with $\vec{g}_{0i}(\xi) = 0$, is given by

$$\vec{w}_0(\xi) = 0 , \quad \vec{w}_i(\xi) = \frac{2}{k} \langle \vec{L}^{(l)} \rangle_w \varepsilon_{ij} \frac{\xi^j - \xi_1^j}{|\xi - \xi_1|^2} . \tag{4.59}$$

Suppose, we now create a second spinon of spin l' moving in the background gauge field $\vec{w}$ excited by the first spinon. Its dynamics is coupled to $\vec{w}$ through the covariant derivatives (see Sect. 2, eqs. (2.13), (2.14)):

$$D_\mu = \partial_\mu + i\vec{w}_\mu \cdot \vec{L}^{(l')} , \tag{4.60}$$

with $\vec{w}_\mu$ as in (4.59). Let us imagine that it makes sense to do "two-spinon quantum mechanics" on a Hilbert space $\mathcal{H}^{(l)} \otimes \mathcal{H}^{(l')}$, with

$$\mathcal{H}^{(l)} = \mathcal{D}^{(l)} \otimes L^2(M, dv) ,$$

where $\mathcal{D}^{(l)}$ carries the spin-l representation of $SU(2)$. By (4.59) and (4.60), the covariant derivatives on $\mathcal{H}^{(l)} \otimes \mathcal{H}^{(l')}$ are then given by

$$D_0^1 = \frac{\partial}{\partial \xi_1^0} , \quad D_j^1 = \frac{\partial}{\partial \xi_1^j} + \frac{2i}{k} \varepsilon_{jn} \frac{\xi_1^n - \xi_2^n}{|\xi_1 - \xi_2|^2} \sum_{A=1}^{3} L_A^{(l)} \otimes L_A^{(l')} ,$$

and

$$D_0^2 = \frac{\partial}{\partial \xi_2^0} , \quad D_j^2 = \frac{\partial}{\partial \xi_2^j} + \frac{2i}{k} \varepsilon_{jn} \frac{\xi_2^n - \xi_1^n}{|\xi_1 - \xi_2|^2} \sum_{A=1}^{3} L_A^{(l)} \otimes L_A^{(l')} , \tag{4.61}$$

These are the covariant derivatives associated with the celebrated *Knizhnik-Zamolodchikov connection*, [30]. For the "two-spinon quantum mechanics" with parallel transport given by (4.61) to be consistent with unitarity, it is necessary that

$$k = \pm(\kappa + 2)\,, \quad \kappa = 1, 2, \cdots\,. \tag{4.62}$$

This follows from results in [30,39]. Recalling what we have said in Sect. 3–(3) about the Aharonov-Casher effect, we observe that the "phase factor" arising in the parallel transport of a quantum mechanical spinon in the field excited by a classical spinon with spin orthogonal to the plane of the system is an Aharonov-Casher phase factor.

Let us now exchange the positions of two quantum mechanical, pointlike spinons along anti-clockwise oriented paths. Then the "Aharonov-Casher phase factor" multiplying the wave function is given by a matrix

$$R^{(\kappa)}_{ll'} : \mathcal{D}^{(l)} \otimes \mathcal{D}^{(l')} \to \mathcal{D}^{(l')} \otimes \mathcal{D}^{(l)}$$

which is the *braid matrix* for exchanging a chiral vertex of spin l with a chiral vertex of spin l' in the chiral *Wess-Zumino-Novikov-Witten model* [30] at level κ. It is given by

$$R^{(\kappa)}_{ll'} = T\pi_l \otimes \pi_{l'}(\mathcal{R}^{(\kappa)})\,, \tag{4.63}$$

where $\mathcal{R}^{(\kappa)}$ is the universal R-matrix of the quantum group $U_q(sl_2)$, with $q = \exp i\pi/(\kappa+2)$, and T is the flip (transposition of factors). All this can be extended to "n-spinon quantum mechanics". The matrices $R^{(\kappa)}_{ll'}$ determine an exotic quantum statistics related to non-abelian (for $\kappa > 1$, $l, l' < \frac{\kappa}{2}$) representations of the braid groups (more precisely, the groupoids of coloured braids) which is commonly called *non-abelian braid statistics* [40,29]. We wish to note that l and l' are forced to be $\leq \frac{\kappa}{2}$, i.e., there are no spinons of spin $> \frac{\kappa}{2}$. One might call this phenomenon *"spin screening"*. If the particles of spin s constituting the chiral spin liquid appear as spinon excitations above the groundstate then

$$\kappa \geq 2s\,, \tag{4.64}$$

since these particles carry spin s. One can argue that the statistics of these particles must be *abelian* braid statistics, i.e., they are anyons. In fact, it then follows that they are semions ($\theta = 1/4$). Now, for a given level κ, the matrices $R^{(\kappa)}_{ll}$ define an abelian representation of the braid groups if and only if $2l = \kappa$. It follows that, for a chiral spin liquid made of particles of spin s

$$\kappa = 2s\,. \tag{4.65}$$

Any spinon-excitation of spin $l < s$ then has *non-abelian* braid statistics!

The reader may feel that our "derivation" of "spinon quantum mechanics" from the effective action $S^*_{\Lambda_0}(w)$ given in (4.56) is based on idealizations – see (4.58) – and

jumps in the logics – reasoning between (4.60) and (4.61) – that might make it appear to be quite problematic. Actually, it turns out that our conclusions concerning spinon statistics, in particular eqs. (4.63) and (4.65), are perfectly correct. This follows from an analysis of the mysterious boundary terms, "B.T.", in the effective action; see [17], and Sect. 5 for the example of anyons.

In order to understand spin-singlet quantum Hall fluids, one must glue the Laughlin vortices described in (4.49) – (4.53) to the spinons discussed above. One checks that for $\sigma = 2/n_0$, n_0 odd, and $\kappa = 2s = 1$, a Laughlin vortex of vorticity $n = -\frac{n_0}{2}$ (!) glued to a spinon of spin $s = 1/2$ is an excitation of charge $q = -1$, spin $1/2$ and *Fermi statistics*, [3,17]. These are the properties of an electron. In an electronic quantum Hall fluid (*without* any *very exotic* internal symmetries) one does not find any excitations with non-abelian braid statistics. However, if one could manufacture a quantum Hall fluid made of charge carriers of spin $s = \frac{3}{2}, \frac{5}{2}, \cdots$, with a spin-singlet ground state it would display excitations with non-abelian braid statistics [14]. It may appear difficult to build such a system, in practice. But, perhaps, one can think of incompressible superfluid films of particles of higher spin, with broken parity and time reversal, which would also exhibit excitations with non-abelian braid statistics.

The analysis sketched above extends, in a straightforward way, to systems with continuous internal symmetries and corresponding gauge fields; see [17].

It may be worthwhile emphasizing that in quantum Hall fluids with non-vanishing magnetic susceptibility (spin-polarized Hall fluids) the fractional statistics of Laughlin vortices always appears as a consequence of a *combination* of the Aharonov-Bohm – *and* the Aharonov-Casher effect; (but notice that, for spin-polarized quantum Hall fluids, the Aharonov-Casher phase factors are automatically abelian). This is a consequence of the fact that electrons have a non-vanishing magnetic moment and follows from eq. (4.42).

Finally, we come to a brief comment concerning the relation of our definition of the Hall conductivity $\sigma_H = \frac{e^2}{h}\sigma$ as the coefficient of a Chern-Simons term, $\frac{\sigma}{4\pi}\int \tilde{a} \wedge d\tilde{a}$, in the effective gauge field action $S^*_{\Lambda_0}$, see (4.22), of an incompressible quantum Hall fluid to the more conventional definition via the *Kubo formula* [41]. It follows easily from eqs. (4.4), (4.5) and (4.22) that σ appears in the following *current sum rules:* For every choice of a permutation $(\mu\nu\rho)$ of (012),

$$i\frac{\sigma}{\pi} = \text{sign}\,(\mu\nu\rho) \int (x-y)^\mu \, \langle T[j^\nu(x) j^\rho(y)]\rangle^c_{a_c,w_c} \, d^3y \,. \tag{4.66}$$

These are *three* equations for one and the same quantity σ. The equation for $(\mu\nu\rho) = (012)$ is

$$i\frac{\sigma}{\pi} = \int (t-s) \left\langle T[j^1(t,\mathbf{x}) j^2(s,\mathbf{y})]\right\rangle^c_{a_c,w_c} \, ds \, d^2\mathbf{y} \tag{4.67}$$

which is just the Kubo formula (in "mathematical units", with no guarantee for signs and factors of π); compare e.g. to [41]. The other two equations are an automatic consequence of $U(1)_{\text{em}}$-gauge-invariance. See [5] for a more systematic study of current

sum rules and "proofs".

Thouless and coworkers [42], and followers [43], have derived from the Kubo formula that

$$\sigma = \frac{1}{n_0} c_1 \,, \tag{4.68}$$

where n_0 is the groundstate degeneracy and c_1 is the *first Chern number* of a vector bundle over a two-dimensional torus of magnetic fluxes (ϕ_1, ϕ_2). So, c_1 is an integer which, in formula (4.52), was called $N = \#$ of electrons created when one turns on a local magnetic field of total flux n_0. Does our formulation "know" that n_0 is the degeneracy of the groundstate? Yes, it does! This follows e.g. from the material in Sect. 5 and has been noted in [1]; (see also [17] for a more precise derivation).

Bellissard [44] and Avron, Seiler and Simon [45] have also given a definition of σ as an *index*. Their definition is equivalent to ours, too, and the proof follows from the material in Sect. 5; see Sect. 6 of ref. [3].

We finally note that σ_H^{spin} (for $k = 0$, i.e., spin-polarized quantum Hall fluids) can be shown to be given by a Kubo formula involving spin currents and can then be shown to be proportional to a first Chern number of a vector bundle over a two-dimensional torus of electric charges per unit length (Q_1, Q_2).

In a fairly precise sense one finds that the Hall effect for the electric current is a time-dependent form of the *Aharanov-Bohm effect*, while the Hall effect for the spin current corresponds to the time-dependent *Aharonov-Casher effect*.

5. ANOMALY CANCELLATION AND ALGEBRAS OF CHIRAL EDGE CURRENTS IN TWO-DIMENSIONAL, INCOMPRESSIBLE QUANTUM FLUIDS.

In this last section we outline some ideas on the origin of the quantization of the values of the constants σ, χ, σ_s and k which appear as the coefficients of the Chern-Simons terms in the effective action $S^*_{\Lambda_0}$ of incompressible quantum fluids in the scaling limit; see (4.22). This topic is intimately connected with the so far mysterious boundary terms, "B.T", on the r.h.s of eq. (4.22). Since this is a somewhat technical topic, we have to limit our review to a few basic aspects and refer the reader to [4,17] for more details.

Briefly, our analysis of the boundary terms in $S^*_{\Lambda_0}$ and of the quantization of the coefficients σ, χ, σ_s and k relies upon the following two *key ideas*:

(i) The "important" - more precisely the *anomalous* - part of the boundary terms in the action $S^*_{\Lambda_0}$ is completely determined by the Chern-Simons terms in $S^*_{\Lambda_0}$ by invoking $U(1)_{\text{em}} \times SU(2)_{\text{spin}} (\times G_{\text{internal}})$-gauge-invariance of the *total* effective action of non-relativistic quantum theory.

(ii) This anomalous part of the boundary terms of $S^*_{\Lambda_0}$ turns out to be the gen-

erating functional of the connected Green functions of chiral current operators which generate $U(1)$-, $SU(2)$- (and G-) current (Kac-Moody) algebras [9]. Some physical and mathematical principles concerning the representation theory of these current algebras then constrain the values of the coefficients σ, χ, σ_s and k to belong to certain discrete sets.

Remark on (ii). We already have found constraints on the values of σ, (χ, σ_s) and k in Sect. 4 by analyzing the statistics of Laughlin vortices and "spinons" and imposing the constraint that, among excitations composed of Laughlin vortices glued to spinons, one should find excited states of the particles constituting the incompressible quantum fluid – in the case of a quantum Hall fluid, the *electrons* or *holes*. In Sect. 4, it turned out that if one imposes the principle of unitarity on the quantum mechanics of spinons then k must be an integer. Our analysis of Laughlin vortices predicted σ to be a rational number (with an *odd* denominator for quantum Hall fluids composed of spinless, charged fermions).

Let us start our analysis by recalling a well known lemma that shows how $SU(2)$-gauge-invariance forces k to be an integer: Let g be an $SU(2)$-gauge-transformation with the property that

$$g(\tau, \xi) \to \mathbf{1}\,, \quad \text{continuously as } (\tau, \xi) \to \partial\Lambda_0\ , \tag{5.1}$$

or $\tau \to \pm\infty$. For Λ_0 a cylinder, the family of all such $SU(2)$-gauge transformations splits into disjoint homotopy classes labelled by an integer winding number, $n(g)$; (recall that $\pi_3(SU(2)) = \mathbf{Z}$!). Let g be a gauge transformation with winding number $n(g) \neq 0$. The gauge-transformed $SU(2)$-connection, ${}^g w$, is given by

$$w \mapsto {}^g w = gwg^{-1} + gdg^{-1}\ . \tag{5.2}$$

Let us study how the $SU(2)$-Chern-Simons term

$$S_{CS}(w) := \frac{k}{4\pi}\int_{\Lambda_0} tr(w \wedge dw + \frac{2}{3} w \wedge w \wedge w) \tag{5.3}$$

in $S^*_{\Lambda_0}$ transforms under the transformation (5.2). The well known answer is that

$$S_{CS}({}^g w) = S_{CS}(w) + 2\pi k n(g)\ . \tag{5.4}$$

Now, non-relativistic quantum theory is fully gauge-invariant under local $SU(2)_{\text{spin}}$-gauge-transformations, including time-dependant ones. Therefore, the generating (partition) function

$$Z_\Lambda(a^{(\theta)}, w^{(\theta)}) \equiv \exp\frac{i}{\hbar} S^{\text{eff}}_\Lambda(a^{(\theta)}, w^{(\theta)}) \sim \exp\frac{i}{\hbar} S^*_{(\Lambda/\theta)}(\tilde{a}, \tilde{w}) \tag{5.5}$$

must be invariant under the transformation (5.2). Asymptotically, as $\theta \to \infty$, the *only* gauge-variance of $Z_{\theta\Lambda_0}(a^{(\theta)}, w^{(\theta)})$ comes from the Chern-Simons term (5.3) in

$S^*_{\Lambda_0}(\tilde{a}, \tilde{w})$. Hence we must require that

$$\begin{aligned} \exp \frac{i}{\hbar} S^*_{\Lambda_0}({}^g w) &= \exp \frac{i}{\hbar} \left[S^*_{\Lambda_0}(w) - 2\pi k \hbar n(g) \right] \\ &\stackrel{!}{=} \exp \frac{i}{\hbar} S^*_{\Lambda_0}(w) , \end{aligned} \tag{5.6}$$

for arbitrary integers $n(g)$. Thus

$$k \in \mathbb{Z} . \tag{5.7}$$

The same result could have been deduced by considering the transformation properties of the Chern-Simons term $S_{CS}(w)$ under gauge transformations, g, not vanishing at the boundary $\partial\Lambda_0$. The non-invariance of $S_{CS}(w)$ under such gauge transformations actually determines one of the *boundary terms* in $S^*_{\Lambda_0}$, [2,17].

What about the values of σ, σ_s and χ? Consider, for example, the abelian Chern-Simons term

$$S'_{CS}(\tilde{a}) := \frac{\sigma}{4\pi} \int_{\Lambda_0} \tilde{a} \wedge d\tilde{a} \tag{5.8}$$

in the effective action $S^*_{\Lambda_0}$. As explained in [5], $S^*_{\Lambda_0}(\tilde{a}, \tilde{w})$ must be invariant under $U(1)_{\text{em}}$-gauge-transformations

$$\tilde{a} \mapsto {}^{\chi}\tilde{a} = \tilde{a} + d\chi , \tag{5.9}$$

(in spite of the fact that $\tilde{a}$ is only a fluctuation potential, i.e., $\tilde{a} = a - a_c$!) Since $\pi_3(U(1)) = 0$, the local $U(1)$-gauge-transformations on Λ_0 do not split into different homotopy classes, and hence there is no a-priori quantization of σ. However, by considering the transformation properties of $S'_{CS}(\tilde{a})$ under gauge transformations, χ, which do *not* vanish at the boundary $\partial\Lambda_0$, we shall be able to infer some constraints on the possible values of σ.

In order not to get lost in many technicalities, we refrain from studying a general quantum Hall fluid here; but see [3,17]. Rather, we shall confine our analysis of boundary terms and edge currents to idealized quantum Hall fluids of *spinless fermions*, so that $w = 0$, from now on. This is an important special case for coming to grips with the general case (which also involves $SU(2)_{\text{spin}}$ and, possibly, G_{internal}). But the general case would lead us into a little orgy of "*branching rules*" for representations of subalgebras of Kac-Moody algebras which is deferred to another paper, although, physically, the general case is important for understanding quantum Hall fluids with spin-singlet groundstates (e.g., for a filling factor $\nu = \frac{8}{5}$ [46]), or with internal symmetries, (e.g. certain hierarchy states of the electron fluid, or, perhaps, the fluid corresponding to $\nu = \frac{5}{2}$; [3,47]).

If the particles in a two-dimensional, incompressible quantum fluid are spinless fermions then $w = 0$, and its effective action in the scaling limit is given by

$$-S^*_{\Lambda_0}(\tilde{a}) = \int_{\Lambda_0} j^\mu_c(\xi) \tilde{a}_\mu(\xi) dv + S'_{CS}(\tilde{a}) + \text{B.T.}(\tilde{a}) , \tag{5.10}$$

where $S'_{CS}(\tilde{a})$ is given in (5.8), and B.T. stands for the celebrated boundary terms, and, for the rest of this section, $\hbar = 1$. Let us now perform a gauge transformation (5.9) on $\tilde{a}$, with χ not vanishing at $\partial\Lambda_0$. Then

$$\begin{aligned} S^*_{\Lambda_0}(\tilde{a} + d\chi) &= S^*_{\Lambda_0}(\tilde{a}) - \int_{\partial\Lambda_0} j_{c,n}\chi do \\ &+ \frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi \wedge \tilde{a} - \text{B.T.}(\tilde{a} + d\chi) + \text{B.T.}(\tilde{a}) \,, \end{aligned} \tag{5.11}$$

where $j_{c,n}$ is the component of $j_c{}^\mu$ normal to the boundary $\partial\Lambda_0$ of Λ_0, and do is the surface element. Note that $S^*_{\Lambda_0}(\tilde{a} + d\chi)$ would be equal to $S^*_{\Lambda_0}(\tilde{a})$, and we could set B.T.$= 0$, if

$$j_{c,n} \propto \text{dual of } d\tilde{a}\,|_{\partial\Lambda_0} \,.$$

However, j_c is the current supported by the quantum fluid when $a = a_c, \tilde{a} = 0$, and $\tilde{a}$ is an arbitrary fluctuation potential. Therefore such a relation between j_c and $d\tilde{a}$ does not make sense for arbitrary $\tilde{a}$. Experimentally, for the electron fluid in a heterojuncture, for example, $\tilde{a}$ can be tuned in a fairly arbitrary way, and the boundary $\partial\Lambda_0$ is such that there is no leakage of electric charge through $\partial\Lambda_0$, i.e.,

$$j_{c,n} = 0 \,. \tag{5.12}$$

In this case, the second term on the r.h.s. of (5.11) vanishes, but the third term is different from 0, for suitable choices of χ and $\tilde{a}$. Imposing gauge invariance of the effective action $S^*_{\Lambda_0}$ thus yields the following equation for the boundary terms:

$$\text{B.T.}(\tilde{a} + d\chi) - \text{B.T.}(\tilde{a}) = \frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi \wedge \tilde{a} \,, \tag{5.13}$$

for arbitrary $\tilde{a}$ and χ. This equation is well known from the study of the (1+1)-*dimensional chiral anomaly* [16]. To solve it, it is convenient to use light-cone coordinates on $\partial\Lambda_0$. We set

$$u_\pm = \frac{1}{\sqrt{2}}(v\tau \pm \theta) \tag{5.14}$$

where τ is a time-like and θ a space-like coordinate on $\partial\Lambda_0$, and v is some velocity. Since the term $\int_{\partial\Lambda_0} \tilde{a} \wedge d\chi$ is topological, eq. (5.13) does not impose any specific choice of τ, θ and v. Mathematically, it is convenient to set $v = 1$ and choose θ to be an angle ranging over the interval $[0, 2\pi]$. However, if τ and θ are measured in physical units then v would be the *propagation speed of surface charge density waves.* The value of this physically interesting quantity will not be determined by $S^*_{\Lambda_0}$. [It would only be computable from a more microscopic analysis of the system.] We now set

$$\tilde{a}\,|_{\partial\Lambda_0} = A_+ du_+ + A_- du_- \,, \tag{5.15}$$

where

$$A_\pm := \frac{1}{\sqrt{2}}\left(\tilde{a}_\tau\,|_{\partial\Lambda_0} \pm \tilde{a}_\theta|_{\partial\Lambda_0}\right) \,. \tag{5.16}$$

the boundary vector potential. An analysis due to Halperin [6] and elaborated upon in [3] shows that this describes precisely the physics of boundary degrees of freedom of an integer (non-interacting) quantum Hall fluid *with* $|\sigma|$ *filled Landau levels.* Actually, the logics can be turned around: If we consider a non-interacting quantum Hall fluid with N filled Landau bands coupled to a small fluctuation vector potential $\tilde{a}$ then those quantum mechanical degrees of freedom which are localized near the boundary of the system produce a $U(1)$-gauge-anomaly corresponding to the action $\frac{iN}{4\pi}\Delta_{L/R}(A)$, (where the choice of L or R depends on the sign of the external magnetic field). For this anomaly to be canceled – as required by the $U(1)$-gauge invariance of non-relativistic quantum theory – it is necessary that the effective gauge field action of the *bulk degrees of freedom* contain a Chern-Simons term $\pm\frac{iN}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}$. As shown in eqs. (4.42) and (4.44), this term reproduces the basic equations of the quantum Hall effect, with a quantized Hall conductivity $\sigma_H = \frac{e^2}{h}N$, $N = 0, 1, 2\cdots$.

So we understand the integral quantum Hall effect for non-interacting electrons pretty well – although there are actually still plenty of interesting analytical (*spectral*) problems for systems with a large amount of disorder and for systems of spinning electrons with spin-orbit interactions which should be studied more carefully!

But what if σ is *not* an integer? Then the $U(1)$-anomaly of the Chern-Simons term in the effective action is cancelled by the term $\pm\frac{i\sigma}{4\pi}\Delta_{L/R}(A)$, as shown above. Of course $\frac{i}{4\pi}\Delta_{L/R}(A)$ remains the generating functional of a chiral $U(1)$-current algebra of left- or right-moving currents. What kind of a system does the corresponding chiral $U(1)$- current, $J^{\pm} = j^{\mu=0}_{L/R}$, describe physically? Of course, it still describes chiral electric charge density waves circulating around the boundary edges of the system. But what are the basic charge carriers like? Here a little general culture on current algebra (see e.g. [9]) helps: Let us start by considering free, massless Dirac fermions in $1+1$ dimensions. By (5.22)

$$j^{\mu}_{L/R} = \frac{1}{2}(j^{\mu} \mp j^{\mu}_5)\,. \tag{5.23}$$

Let us first suppose that the external gauge field A is set to 0. Then j^{μ} and j^{μ}_5 are conserved currents, i.e.,

$$\partial_{\mu}j^{\mu} = \partial_{\mu}j^{\mu}_5 = 0\,. \tag{5.24}$$

The general solution of eqs. (5.24) is

$$j^{\mu} = \varepsilon^{\mu\nu}\partial_{\nu}\varphi\,, \quad j^{\mu}_5 = \varepsilon^{\mu\nu}\partial_{\nu}\varphi_5\,, \tag{5.25}$$

where φ and φ_5 are scalar fields of scaling dimension 0. However, in two space-time dimensions, $j^{\mu}_5 = -\varepsilon^{\mu\nu}j_{\nu}$, with $\varepsilon^{01} = -\varepsilon^{10} = 1$. Therefore $j^{\mu}_5 = \partial^{\mu}\varphi$, and (5.24) implies that

$$\partial_{\mu}\partial^{\mu}\varphi \equiv \Box\varphi = 0\,, \tag{5.26}$$

i.e., φ is a free, massless scalar field. Any solution of (5.26) has the form

$$\varphi = \sqrt{2}(\varphi_L(u_+) + \varphi_R(u_-))\,. \tag{5.27}$$

In light cone coordinates, the r.h.s. of eq. (5.13) is given by

$$\frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi\wedge\tilde{a} = \frac{\sigma}{4\pi}\int_{\partial\Lambda_0}(A_+\partial_-\chi - A_-\partial_+\chi)d^2u\ , \tag{5.17}$$

where

$$\partial_\pm\chi \equiv \frac{\partial}{\partial u_\pm}\chi\ .$$

We note that, in light-cone coordinates, the d'Alembertian, $\Box = \frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\theta^2}$, is given by

$$\Box = 2\partial_+\partial_-\ . \tag{5.18}$$

After these preparations, it is a simple exercise to verify that the solution of the functional equation (5.13) is given by

$$\begin{aligned} \text{B.T.}(\tilde{a}) &= -\frac{\sigma}{4\pi}\Delta_R(A) + W(A) \\ &= \frac{\sigma}{4\pi}\Delta_L(A) + W(A)\ , \end{aligned} \tag{5.19}$$

where

$$\Delta_{L/R}(A) = \int_{\partial\Lambda_0}\left\{A_\mp A_\pm - 2A_\mp\frac{\partial_\pm^2}{\Box}A_\mp\right\}d^2u\ , \tag{5.20}$$

and $W(A)$ is an arbitrary *gauge-invariant* functional of the boundary vector potential given by A_+ and A_-. Note that replacing σ by $-\sigma$ corresponds to replacing L (left) by R (right)! Readers, who still remember the basic formulas arising in the study of the $(1+1)$-dimensional, chiral $U(1)$-anomaly will recognize $\Delta_L(A)$ as the effective gauge field action of a chiral (left-moving) relativistic fermion minimally coupled to a $U(1)$-gauge field A, in two space-time dimensions. One checks that

$$\begin{aligned} \frac{i}{4\pi}\Delta_L(A) &= \ln\det\left[\not{\partial} + i\not{A}\left(\frac{1-\gamma_5}{2}\right)\right] \\ &= \ln\det(\not{\partial} + i\not{A})\Big|_{A_+=0} + \frac{i}{4\pi}\int A_+A_-d^2u\ . \end{aligned} \tag{5.21}$$

Let ψ be a $(1+1)$-dimensional two-component Dirac spinor, and $\bar{\psi} = \psi^*\gamma_0$ its conjugate. The expression for the left-moving current, j_L^μ, is given by

$$j_L^\mu = N\left(\bar{\psi}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi\right)\ , \tag{5.22}$$

where N indicates normal ordering. This current generates a chiral $U(1)$-current algebra. Comparing (5.22) to (5.21) and recalling the basics of Berezin integration, we observe that $\frac{i}{4\pi}\Delta_L(A)\Big|_{A_+=0}$ is the generating functional for the connected Green functions of the left moving current j_L^μ.

We conclude that *if* σ *we're an integer* we could cancel the anomaly, $\frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi\wedge\tilde{a}$, of the Chern-Simons term, $\frac{\sigma}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}$, in the effective action of the quantum fluid under a gauge transformation, $\tilde{a}\mapsto\tilde{a}+d\chi$, *by* $|\sigma|$ *bands of left- or right-moving* (depending on the sign of σ) *free, relativistic complex fermions* minimally coupled to

By (5.23),

$$J^+ = -\partial_- \varphi_R \,, \quad J^- = \partial_+ \varphi_L \,, \tag{5.28}$$

with $\partial_\pm J^\pm = 0$. These formulas hold at the level of quantized fields and are at the origin of abelian bosonization in two space-time dimensions. Now, any sum of free fields is again a free field. Thus, let us write, for fun,

$$\varphi = \frac{1}{2\pi}(\phi_1 + \cdots + \phi_N) \,, \tag{5.29}$$

where $\phi_1, \cdots, \phi_N$ are distinct, free, massless scalar fields. We set

$$\hat{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} . \tag{5.30}$$

For $A = 0$, the action of $\hat{\phi}$ is given by

$$S_{WZW}(\hat{\phi}) = \frac{1}{4\pi} \int_{\partial\Lambda_0} \partial_+ \hat{\phi} \cdot K \partial_- \hat{\phi} d^2u \,, \tag{5.31}$$

where K is a positive $N \times N$ matrix, and $\hat{a} \cdot \hat{b} := \sum_{i=1}^{N} a_i b_i$. If one wishes to describe chiral left- (or right-) moving free fields one supplements the action (5.31) by the constraints

$$\partial_- \hat{\phi} = 0 \quad (\partial_+ \hat{\phi} = 0, \text{ resp.}) \,. \tag{5.32}$$

The matrix K describes linear couplings between the fields $\phi_1, \cdots, \phi_N$ and fixes their normalization when one uses a standard path-integral quantization.

Let us now study what happens when one attempts to couple the fields $\phi_1, \cdots \phi_N$ to the vector potential A. The first problem one encounters is that expressions like $\partial_\pm \hat{\phi}$, for example the chiral constraint $\partial_- \hat{\phi} = 0$, are *not* invariant under $U(1)$-gauge-transformations. We must find out how $\hat{\phi}$ transforms under gauge transformations. For $N = 1$ and $K = 1$, it is well known and easy to check that the fermion operators ψ_L and ψ_R are given by vertex operators, $\psi_{L/R} =: e^{i\varphi_{L/R}} :$, where the double colons indicate Wick ordering. Hence φ and $\phi_1, \cdots, \phi_N$ transform like angular variables under $U(1)$-gauge transformations. An adequate ansatz is

$$\phi_j \mapsto {}^{\chi}\phi_j = \phi_j + \sum_{i=1}^{N} (K^{-1})_{ji} \chi \,. \tag{5.33}$$

A gauge-invariant form of the chiral constraint is then given by

$$\partial_- \phi_j - \sum_{i=1}^{N} (K^{-1})_{ji} A_- = 0 \,. \tag{5.34}$$

We set

$$\hat{\chi} := \begin{pmatrix} \chi \\ \vdots \\ \chi \end{pmatrix} , \quad \hat{A} = \begin{pmatrix} A \\ \vdots \\ A \end{pmatrix} , \tag{5.35}$$

with N components each. An action reducing to (5.31) for $A = 0$ is given by

$$\begin{aligned} S_{WZW}(\hat{\phi}, \hat{A}) &= \frac{1}{4\pi} \int_{\partial\Lambda_0} \partial_+ \hat{\phi} \cdot K \partial_- \hat{\phi} d^2u \\ &- \frac{1}{2\pi} \int_{\partial\Lambda_0} \left\{ \hat{A}_- \cdot \partial_+ \hat{\phi} - (\partial_- \hat{\phi} - K^{-1} \hat{A}_-) \cdot \hat{A}_+ \right\} d^2u \\ &+ \frac{k}{4\pi} \int_{\partial\Lambda_0} A_- A_+ d^2u \,, \end{aligned} \tag{5.36}$$

where

$$k := \sum_{i,j} (K^{-1})_{ij} \,. \tag{5.37}$$

Note that expression (5.36) is symmetric in "+" and "−". If we want to describe chiral fields we supplement the dynamics determined by the action (5.36) by the *gauge-invariant* chiral constraint (5.34). Let us now check how $S_{WZW}(\hat{\phi}, \hat{A})$ transforms under the $U(1)$-gauge-transformations (5.33) and $A_\pm \mapsto {}^\chi A_\pm = A_\pm + \partial_\pm \chi$. After a fairly brief calculation we find that

$$\begin{aligned} S_{WZW}({}^\chi\hat{\phi}, {}^\chi\hat{A}) &= S_{WZW}(\hat{\phi}, \hat{A}) \\ &+ \frac{k}{4\pi} \int_{\partial\Lambda_0} (A_+ \partial_- \chi - A_- \partial_+ \chi) d^2u \\ &+ \frac{1}{2\pi} \int_{\partial\Lambda_0} (\partial_- \hat{\phi} - K^{-1} \hat{A}_-) \cdot \partial_+ \hat{\chi} d^2u \,. \end{aligned} \tag{5.38}$$

The last term on the r.h.s. of (5.38) vanishes when the chiral constraint (5.34) is imposed. We observe that the second term on the r.h.s. of (5.38) is precisely the anomaly (5.17) of the Chern-Simons action if $k = \sigma$.

Let us also note that

$$\zeta_L(A) := \int \mathcal{D}\hat{\phi} e^{-iS_{WZW}(\hat{\phi}, \hat{A})} \delta(\partial_- \hat{\phi} - K^{-1} \hat{A}_-) \tag{5.39}$$

is, for $A_+ = 0$, the generating function for the current $\frac{1}{2\pi} \sum_{i=1}^N \partial_+ \phi_i = \partial_+ \varphi$ which, by eq. (5.28), is precisely the left-handed current J^-. Since the integration measure is gauge-invariant, i.e., $\mathcal{D}\hat{\phi} = \mathcal{D}^\chi \hat{\phi}$, it follows from (5.38) and (5.17), (5.20) that

$$\zeta_L(A) = \exp\left(-\frac{ik}{4\pi} \Delta_L(A) \right) \,. \tag{5.40}$$

Thus, we conclude that if the coefficient σ of the Chern-Simons term, $\frac{\sigma}{4\pi} \int_{\Lambda_0} \tilde{a} \wedge d\tilde{a}$, in the effective action $S^*_{\Lambda_0}$ of an incompressible quantum fluid satisfies

$$\sigma = k \equiv \sum_{ij=1}^N (K^{-1})_{ij} \tag{5.41}$$

then

$$\exp(-iS'_{CS}(\tilde{a})) \zeta_L(A = \tilde{a}|_{\partial\Lambda_0}) \tag{5.42}$$

is $U(1)$-gauge-invariant, i.e., *anomaly-free*. For $\sigma = -k$, the same holds if "+" and "−" and "left (L)" and "right (R)" are interchanged.

Next, we must investigate the physics of the $\hat{\phi}$-system on the boundary of an incompressible quantum fluid. In particular, we must find physical constraints on the matrix K. When the gauge field A is zero, the electric charge operator Q is given by

$$Q = \oint j^0 d\theta = \oint (\partial_\theta \varphi) d\theta \,, \tag{5.43}$$

by (5.25). By (5.14), (5.29) and (5.34), this yields

$$\begin{aligned} Q &= \sum_{j=1}^{N} Q_j \,, \quad \text{with} \\ Q_j &= \frac{1}{2\pi} \oint (\partial_+ \phi_j) d\theta \,. \end{aligned} \tag{5.44}$$

Unfortunately, these expressions are *not* $U(1)$-gauge-invariant. But it is clear how to render them gauge-invariant. The correct definition of the charge operator associated with ϕ_j is

$$Q_j = \frac{1}{2\pi} \oint (\partial_+ \phi_j - \sum_i (K^{-1})_{ji} A_+) d\theta \tag{5.45}$$

which is manifestly gauge invariant. Let us replace

$$\hat{A} = \begin{pmatrix} A \\ \vdots \\ A \end{pmatrix} \mapsto \hat{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \,, \tag{5.46}$$

where under a $U(1)$-gauge transformation χ

$$a_j \mapsto {}^{\chi}\alpha_j = \alpha_j + d\chi \,, \tag{5.47}$$

for $j = 1, \cdots, N$. We call α_j the "vector potential of the jth band", in accordance with the structure of the couplings $\mp \frac{1}{2\pi} \int_{\partial\Lambda_0} \hat{\alpha}_\mp \cdot \partial_\pm \hat{\phi}$ in the action $S_{WZW}(\hat{\phi}, \hat{\alpha})$ given in (5.36). Imagine that we now increase the magnetic flux inside the system by n_i units in the ith band, e.g. by creating n_i Laughlin vortices in the ith band, $i = 1, \cdots, N$. Then

$$\frac{1}{2\pi} \oint \alpha_{i\theta} d\theta = n_i \,, \tag{5.48}$$

and the charge in the jth band, Q_j, changes by an amount

$$\Delta Q_j = \sum_i (K^{-1})_{ji} n_i \,, \tag{5.49}$$

as follows from (5.45), with $\hat{A}$ replaced by $\hat{\alpha}$. In vector notation,

$$\Delta \hat{Q} = K^{-1} \hat{n} \,, \quad \text{or} \quad \hat{n} = K \Delta \hat{Q} \,. \tag{5.50}$$

We now imagine that every band admits excitations with the quantum numbers of an electron or hole, i.e., for every $j = 1, \cdots, N$, there are excitations changing the

total charges by $\Delta Q_i^{(j)} = \pm\delta_{ij}$ and having Fermi statistics. By formula (5.36) for the action S_{WZW}, such excitations are created by the vertex operators

$$\exp \frac{i}{2\pi} \int \hat{\alpha}_- \cdot \partial_+ \hat{\phi} d\theta \ , \tag{5.51}$$

with

$$-\frac{\sqrt{2}}{2\pi} \oint \hat{\alpha}_- d\theta = \frac{1}{2\pi} \oint \hat{\alpha}_\theta d\theta = \hat{n} \ , \tag{5.52}$$

and $\hat{n}$ is given by

$$n_i = \sum_{l=1}^{N} K_{il} \Delta Q_l^{(j)} = K_{ij} \ , \quad i = 1, \cdots, N \ ; \tag{5.53}$$

see (5.50). The statistics of the vertex operator (5.51) is described by the phase

$$\exp 2\pi i h(\hat{\alpha}) \ , \tag{5.54}$$

where $h(\hat{\alpha})$ is its conformal dimension. By (5.36), $h(\hat{\alpha})$ turns out to be given by

$$\begin{aligned} h(\hat{\alpha}) &= \frac{1}{2} \hat{n} \cdot K^{-1} \hat{n} \\ &= \frac{1}{2} \Delta \hat{Q} \cdot K \Delta \hat{Q} \ , \end{aligned} \tag{5.55}$$

and the second equation follows from (5.50). Thus, for an electron or hole in the jth band,

$$h(\hat{\alpha}) = \frac{1}{2} K_{jj} \ .$$

By (5.54), this excitation has Fermi statistics iff

$$K_{jj} = 2l^{(j)} + 1 \ , \quad l^{(j)} = 0, 1, 2, \cdots \ , \tag{5.56}$$

for $j = 1, \cdots, N$. Clearly, electrons and holes are excitations which are *relatively local* to each other, (meaning that microscopic electronic wave functions are *single-valued*). Hence a vertex operator creating an electron or hole in the ith band must commute or anti-commute with a vertex operator creating an electron or hole in the jth band, for all i and j. One readily checks that this will be the case iff

$$\exp 2\pi i \Delta \hat{Q}^{(i)} \cdot K \Delta \hat{Q}^{(j)} = \exp 2\pi i K_{ij} = 1 \ ,$$

hence

$$K_{ij} \in \mathbf{Z} \ , \quad \text{for all } i \text{ and } j \ . \tag{5.57}$$

Actually, if one assumes that two vertex operators creating electrons in *different* bands must commute – as one normally would – then it follows that

$$K_{ij} \in 2\mathbf{Z} \ , \quad \text{for} \quad i \neq j \ . \tag{5.57'}$$

Plugging results (5.56) and (5.57) into formula (5.41) one observes that

$$\sigma = \pm \sum_{i,j=1}^{N} (K^{-1})_{ij} \tag{5.58}$$

is a *rational* number, and hence the Hall conductivity $\sigma_H = \frac{e^2}{h}\sigma$ is a *rational* multiple of $\frac{e^2}{h}$, for every *incompressible* quantum fluid of scalar (spin-polarized) electrons! A similar conclusion holds if spin and internal symmetries are included, but one obtains different sets of rational numbers as the possible values of σ compatible with incompressibility; see [17].

We note that, for $\hat{\alpha} = \hat{A}$ (see (5.46)) and $\Delta Q = \sum_{j=1}^{N} \Delta Q_j$, eq. (5.49) implies that

$$\Delta Q = \frac{1}{2\pi} \sum_{j,i=1}^{N} (K^{-1})_{ji} \oint A_\theta d\theta = \frac{\sigma}{2\pi} \int \tilde{b}(t,\xi) d^2\xi \,, \tag{5.59}$$

by Stokes' theorem. This is an integrated form of eq. (4.31), see also (4.42) and (4.50), for a quantum fluid with vanishing magnetic susceptibility, as is the case for spinless electrons.

Clearly, for a given rational value of σ, formula (5.58), along with the constraints (5.56) and(5.57), does not determine the "band coupling matrix" K uniquely. This is an intrinsic weakness of our very general approach. A given rational value of σ corresponding to a plateau of the Hall conductivity can, in general, be reproduced by many different systems of chiral boundary currents corresponding to distinct K-matrices. In order to find out which K-matrix is the most likely candidate corresponding to a given plateau of σ, one must invoke additional information on the quantum Hall fluid, in particular stability properties against small perturbations, whose elucidation requires analytical or numerical work, or *symmetries*.

As a first step towards reducing the plethora of possible K-matrices we propose to study what kind of *invariant* information is coded into a matrix K. For this purpose one should try to find the full spectrum of charged excitations of a system corresponding to a given matrix K satisfying (5.56) and (5.57). For an excitation of a quantum Hall fluid to have a finite energy difference to the groundstate energy (called a finite-energy excitation), it should perturb the groundstate only *locally*. As a corollary of our discussion of the Aharonov-Bohm effect it follows that the magnetic flux $\hat{n}$ of a finite-energy excitation must be quantized, i.e.,

$$n_j \in \mathbf{Z} \,, \quad \text{for} \quad j = 1, \cdots, N \,.$$

[If an electron in the jth band is transported around such an excitation it picks up a statistical phase $\exp 2\pi i n_j$ which is unity if $n_j \in \mathbf{Z}$.] We conclude that finite-energy excitations of an incompressible quantum Hall fluid can be labelled, in part, by their magnetic flux quantum numbers $\hat{n}$ which are the sites of the lattice $\Phi := \mathbf{Z}^n$. Eq. (5.50) then says that the electric charges corresponding to an excitation with

magnetic flux $\hat{n}$ are given by $\Delta\hat{Q} = K^{-1}\hat{n}$ and form the sites of a lattice $\Gamma := K^{-1}\Phi$. The lattice Γ contains the sublattice $\mathbf{Z}^n$ of excitations with integer charge, i.e., of multi-electron, multi-hole excitations. The quotient space, $\Gamma/\mathbf{Z}^n$, is an abelian group with n generators. It tells us everything about the possible *fractional* charges of finite-energy excitations.

We now observe that what we are calling the "j^{th} band", $j = 1, \cdots N$, is based on a somewhat arbitrary convention of how the fields ϕ_j are coupled to the external electromagnetic vector potential A, (i.e., on the electric charges assigned to these fields). If S is some integral $N \times N$ matrix of determinant 1, i.e., $S \in SL(N, \mathbf{Z})$, then S leaves Φ invariant. Two systems corresponding to matrices K and K', with

$$K' = S^T K S \tag{5.60}$$

describe the same lattices (Φ and Γ) of excitations and correspond to *equivalent* quantum Hall fluids which only differ in the assignment of electric charges to the fields $\phi_1, \cdots, \phi_N$. This observation poses the problem of defining and then finding *normal forms* for the integral, positive quadratic forms K on the lattice Φ, (with respect to conjugation by $SL(N, \mathbf{Z})$); see [4]. This is known to be a subtle mathematical problem which is not solved in general; (see [48]).

But let us return to the problem of symmetries of quantum Hall fluids. A natural symmetry of such a fluid at small values of the filling factor ν is likely to be invariance under arbitrary permutations of the bands. This symmetry would imply that

$$K_{ij} = K_{\pi(i)\pi(j)} \ , \quad i, j = 1, \cdots N \ , \tag{5.61}$$

for arbitrary permutations, π, of $\{1, \cdots, N\}$. Together with conditions (5.56) and (5.57) eqs. (5.61) imply that

$$K_{ii} = 2l + 1 \ , \quad i = 1, \cdots, N \ , \tag{5.62}$$

for some $l = 0, 1, 2, \cdots$ *independent* of i, and

$$K_{ij} = n \in \mathbf{Z} \ , \quad \text{for } i \neq j \ . \tag{5.63}$$

Thus

$$K = (2l + 1 - n)1_N + nNP_N \ , \tag{5.64}$$

where P_N is the orthogonal projection on the unit vector in $\mathbf{R}^N$ all of whose components are given by $1/\sqrt{N}$. Hence

$$K^{-1} = (2l + 1 - n)^{-1} \left(1_N - \frac{nN}{2l + 1 + n(N-1)} P_N \right) \ ,$$

and this equation and (5.58) yield

$$\sigma \equiv \sigma_K = \pm \frac{N}{2l + 1 + n(N-1)} \ . \tag{5.65}$$

Imposing constraint (5.57') we must assume that n is an *even* integer. This reproduces the *odd-denominator rule.* [In general, the odd-denominator rule only holds for an odd number of bands! See also [4].]

A "second generation hierarchy state" of a quantum Hall fluid might be defined as a system with a coupling matrix K_h given by

$$K_h = \begin{pmatrix} K & & 0 \\ & \ddots & \\ 0 & & K \end{pmatrix} + m(pN)P_{pN} \,, \tag{5.66}$$

where K is an $N \times N$ matrix of the form (5.64), the first matrix on the r.h.s. of (5.66) is a $(pN) \times (pN)$ matrix build from p matrices K, and m is an (even) integer. For the Hall conductivity of this system one finds [4]

$$\sigma_{K_h} = \pm \frac{1}{m + (1/p|\sigma_k|)} = \pm \frac{p|\sigma_k|}{mp|\sigma_K| + 1} \,. \tag{5.67}$$

One can now go on and define "third generation hierarchy states" , etc.

Next, one might ask what form the matrix K must have if the system exhibits a full unitary group, $U(N)$, of symmetries permuting its N bands of edge current; (an obvious example of such a system is an integer quantum Hall fluid of non-interacting electrons with $\sigma = \pm N$). The algebra of edge currents must then contain a Kac-Moody subalgebra $\widehat{su}(N)$ (at level 1). This is a much larger symmetry than the permutation symmetry discussed above. Correspondingly, the K-matrices compatible with this larger symmetry are more constrained: They have the form

$$K_{ii} = 2l + 1 \,,\ i = 1, \cdots, N, \quad K_{ij} = 2l \,,\ \text{for } i \neq j \,, \tag{5.68}$$

for some $l = 0, 1, 2, \cdots$. The corresponding Hall conductivity is found to be

$$\sigma_K = \pm \frac{N}{2lN + 1} \,. \tag{5.69}$$

The proof of (5.68) (see [4]) involves showing that there is a matrix $S \in SL(N, \mathbf{Z})$ such that

$$\begin{aligned} S^T K S =: R \,, \quad & \text{with} \\ R_{NN} \;=\; 2l + 1 \,,\ R_{NN-1} = R_{N-1N} = -1 \,, & \end{aligned} \tag{5.70}$$

and $(R_{ij})_{i,j=1}^{N-1}$ is the Cartan matrix of $su(N)$. In connection with quantum Hall fluids the matrix R first appeared in [49].

We note that quantum Hall fluids with K-matrices as in (5.68) correspond to Jain's states [50].

It may be good to consider the simplest example of a fractional quantum Hall fluid covered by our theory: We set $N = 1$ and $K = 2l + 1$. For $l = 0$, this is an integer quantum Hall fluid with $\sigma = \pm 1$. For $l = 1$, corresponding to $\sigma = \pm 1/3$

we find Laughlin's fluid [34]. There are also quantum Hall fluids corresponding to $l = 2$ and 3, ($\sigma = \pm 1/5, \pm 1/7$, respectively). There are no known quantum Hall fluids corresponding to $l = 4, 5, \cdots$, since they would correspond to electron gases of so low a density that they form a Wigner crystal and thereby loose their incompressibility.

The charged excitations in a fluid with $K = 2l + 1$ have vorticity $n = 1, \cdots, 2l + 1$ and charge $n/(2l + 1)$. For $n < 2l + 1$, their charge is thus *fractional*, and by (5.54), (5.55), they are *anyons*.

At the end of Sect. 4 (see (4.68)) we mentioned that, in the conventional approach to the quantum Hall effect, the denominator $n_0 = 2l + 1$ of the Hall conductivity σ is interpreted as the degeneracy of the ground state of the quantum Hall fluid. In our approach this has a straightforward explanation: The algebra of a chiral edge current of a quantum Hall fluid with $\sigma = \frac{1}{2l+1}$ ($K = 2l+1, N = 1$) has $2l+1$ inequivalent representations labelled by fluxes $1, 2, \cdots, 2l + 1$ which correspond to charges $\frac{1}{2l+1}, \frac{2}{2l+1}, \cdots, 1$. Every one of these representations corresponds to a groundstate of the quantum Hall fluid with a one-component boundary, in the thermodynamic limit which is approached when the scale parameter θ tends to ∞. In this limit, the $2l + 1$ distinct groundstates have the *same* energy per electron.

It is shown in [1,3,4] that, in the scaling limit, the groundstates of such a quantum Hall fluid are described by the conformal blocks of the free, massless field at level $2l+1$. On a Riemann surface of genus g with n punctures there are thus $(2l+1)^{g+n}$ degenerate groundstates. These results are best understood by studying the topological Chern-Simons gauge theory associated to the chiral edge currents [10,12]. The quantized gauge potential of this theory turns out to be the *vector potential of the conserved electric current density* $(j^\mu)^2_{\mu=0}$; see [3].

It is worthwhile to observe that the conformal blocks of the massless free field at level $2l + 1$ on the plane with n punctures are the *Laughlin wave functions* for n quasi-particles (characterized by their magnetic flux) of a quantum Hall fluid with $\sigma = \pm\frac{1}{2l+1}$. [This "coincidence" may partially justify some ansätze for hierarchy constructions based on Laughlin-type wave functions for quasi-particles. By and large, it may however have played rather a misleading role.]

We wish to note, furthermore, that if a vortex of strength $n = 2l + 1$ is created in the bulk of a quantum Hall fluid with $\sigma = \pm\frac{1}{2l+1}$ and a one-component boundary then, in the thermodynamic limit ($\theta \to \infty$), the total charge of the fluid changes by $\pm K^{-1}(2l + 1) = \sigma(2l + 1) = \pm 1$, as shown in eq. (5.59). More precisely, a charge of ± 1 is transferred from the place where the vortex is created to the boundary of the system; see also Sect. 6 of [3] for more details. This result relates our definition of the Hall conductivity σ to one where σ is defined as an index, [44,45].

The results reviewed here for the simple example of a quantum Hall fluid with $N = 1$ and $K = 2l + 1$ have straightforward extensions to fluids corresponding to arbitrary N and general K-matrices as discussed above.

Finally, the material in this section can be generalized to incompressible quantum fluids of particles with spin and internal symmetries. These generalizations are important in understanding quantum Hall fluids with $\sigma = \frac{8}{5}$ (spin singlet state) or $\sigma = \frac{5}{2}$, for example. But this is another story.

Now that we have reached the end of this paper life would just start to become interesting. We could now continue our tale by studying the *domain structure* of incompressible quantum fluids, in particular of quantum Hall fluids, some aspects of the *transition* of a quantum Hall fluid from *one plateau* of σ to a *neighbouring plateau*, presumably closely related to the problem of domain structure and *domain wandering*, the *stability* of plateaux and the *role of disorder* in the stability problem, $\cdots\infty$. But, most importantly, we should now finally address the analytical problem of *proving* that certain quantum fluids are indeed *incompressible*.

But all this must await another occasion – quite apart from the fact that much further more analytical work is needed!

REFERENCES

1. X.G. Wen, Phys. Rev. B**40**, 7387 (1989): Int. J. Mod. Phys. B**2**, 239 (1990).
2. X.G. Wen, Phys. Rev. B**43**, 11025 (1991); B. Blok and X.G. Wen, Phys. Rev. B**42**, 8133, 8145 (1990).
3. J. Fröhlich and T. Kerler, Nucl. Phys. B**354**, 369 (1991).
4. J. Fröhlich and A. Zee, Nucl. Phys. B**364**, 517 (1991).
5. J. Fröhlich and U.M. Studer, "$U(1) \times SU(2)$-gauge invariance of non-relativistic quantum mechanics, and generalized Hall effects", preprint ETH-TH/91-13 (September 1991).
6. B.I. Halperin, Phys. Rev. B**25**, 2185 (1982).
7. A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov, Nucl. Phys. B**241**, 333 (1984).
8. V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B**247**, 83 (1984); R. Dijkgraaf, "A geometrical approach to two-dimensional conformal field theory", Ph.D. thesis, Utrecht, 1989.
9. P. Goddard and D. Olive, Int. J. Mod. Phys. **1**, 303 (1986).
10. E. Witten, Commun. Math. Phys. **121**, 351 (1989).
11. J. Fröhlich and P.-A. Marchetti, Lett. Math. Phys. **16**, 347 (1988); J. Fröhlich and C. King, Commun. Math. Phys. **126**, 167 (1989).

12. G. Moore and N. Seiberg, Phys. Lett. **B220**, 422 (1989); S. Elitzur, G. Moore, A. Schwimmer, and N. Seiberg, Nucl. Phys. **B326**, 108 (1989).

13. M. Büttiker, Phys. Rev. **B38**, 9375 (1988); A.H. MacDonald, Phys. Rev. Lett. **64**, 220 (1990); M. Stone, Int. J. Mod. Phys. **B5**, 509 (1991); A.V. Balatsky, Phys. Rev. **B43**, 1257 (1991).

14. S.C. Zhang, T.H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82, 980 (E) (1989); J. Fröhlich, T. Kerler, and P.-A. Marchetti, "Non-abelian bosonization in two-dimensional condensed matter pysics", preprint DFPD91/TH/16 (July 1991).

15. R.E. Prange and S.M. Girvin (eds.), *The quantum Hall effect*, Graduate Texts in Contemporary Physics (Springer, New York, 1987).

16. S.B. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current algebra and anomalies* (World Scientific, Singapore, 1985).

17. A.H. Chamsedine, J. Fröhlich, and U.M. Studer, to appear.

18. T. Eguchi, P.B. Gilkey, and A.J. Hanson, Phys. Rep. **66**, 213 (1980).

19. See e.g.: J.W. Negele and H. Orland, *Quantum many-particle systems*, Frontiers in Physics, vol. 68 (Addison-Wesley), New York, 1987).

20. G. Baym, *Lectures on quantum mechanics* (Benjamin/Cummings Publishing Company, Reading, Massachusetts, 1969); A. Messiah, *Quantum mechanics* (North-Holland Publishing Company, Amsterdam, 1961); C. Piron, *Mécanique quantique (base et applications)*, (Lausanne: Presses Polytechniques et Universitaires Romandes, 1990).

21. Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

22. J.M. Leinaas and J. Myrheim, Nuovo Cimento **37** B, 1 (1977); G.A. Goldin, R. Menikoff, and D.H. Sharp, J. Math. Phys. **21**, 650 (1980); **22**, 1664 (1981); Phys. Rev. Lett. **51**, 2246 (1983); F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982).

23. R.G. Clark et al., Phys. Rev. Lett. **60**, 1747 (1988); J.A. Simmons et al., Phys. Rev. Lett. **63**, 1731 (1989).

24. G.E. Volovik and V.M. Yakovenko, J. Phys. : Condens. Matter **1**, 5263 (1989); G.E. Volovik, "Exotic properties of superfluid ^3He", preprint 1991.

25. P.G. de Gennes, *Superconductivity of metals and alloys* (Benjamin, New York, 1966).

26. D. Pines and P. Nozières, *The theory of quantum liquids* (Addison-Wesley, Redwood City CA, 1989).

27. Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).

28. G. t'Hooft, Commun. Math. Phys. **117**, 685 (1988); S. Deser and R. Jackiw, Commun. Math. Phys. **118**, 495 (1988); B.S. Kay and U.M. Studer, Commun. Math. Phys. **139**, 103 (1991).

29. J. Fröhlich, F. Gabbiani, and P.-A Marchetti, in *The algebraic theory of superselection sectors: Introduction and recent results*, ed. D. Kastler (World Scientific, Singapore, 1990); J. Fröhlich and P.-A. Marchetti, Nucl. Phys. B **356**, 533 (1991).

30. V.G. Knizhnik and A.B. Zamolodchikov, in ref. [8]; A. Tsuchiya and Y. Kanie, Lett. Math. Phys. **13**, 303 (1987).

31. L.D. Landau and E.M. Lifshitz, *Electrodynamics of continuous media*, vol. 8 of Course of theoretical physics (Pergamon Press, Oxford, 1960).

32. A.K Kerman and N. Onishi, Nucl. Phys. A**361**, 179 (1981); J.S. Bell and J.M. Leinaas, Nucl. Phys. B**212**, 131 (1983); B**284**, 488 (1987).

33. R.B. Laughlin, Phys. Rev. B**23**, 5632 (1981); see also ref. [6].

34. R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983); B.I. Halperin, Helv. Phys. Acta **56**, 75 (1983); F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); F.D.M. Haldane and E.H. Rezayi, Phys. Rev. Lett. **54**, 237 (1985); R. Morf and B.I. Halperin, Phys. Rev. B**33**, 2221 (1986).

35. D.H. Lee and S.C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991); S.C. Zhang, "The Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect", preprint RJ 8364 (76087) (September 1991); N. Read, Phys. Rev. Lett. **62**, 86 (1989).

36. J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

37. B.I. Halperin, Phys. Rev. Lett. **52**, 1583, 2390(E) (1984); D. Arovas, J.R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984); S. Girvin and A.H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987); R.B. Laughlin, in *Two-dimensional strongly correlated electronic systems*, eds. Z.-Z. Gan and Z.-B. Su (Gordon and Breach, London, 1989).

38. R. Tao and Y.-S. Wu, Phys. Rev. B**31**, 6859 (1985).

39. T. Kohno, Ann. de l'Inst. Fourier de l'Univ. de Grenoble **37**, 139 (1987); in *Contemporary mathematics: Artin's braid group*, eds. J.S. Birman and A. Libgober (American Math. Soc. , Providence RI, 1988).

40. K. Fredenhagen, K.-H. Rehren, and B. Schroer, Commun. Math. Phys. **125**, 201 (1989).

41. E. Fradkin, *Field theories of condensed matter systems*, Frontiers in Physics, vol. 82 (Addison-Wesley, Redwood City, California, 1991).

42. D.J. Thouless, M. Kohmoto, M.P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982); M. Kohmoto, Ann. Phys. (N.Y.) **160**, 343 (1985); Q. Niu and D.J. Thouless, Phys. Rev. B**35**, 2188 (1987).

43. J.E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983); I. Dana, J.E. Avron, and J. Zak, J. Phys. C**18**, L679 (1985); J.E. Avron, R. Seiler, and L. Yaffe, Commun. Math. Phys. **110**, 33 (1987); H. Kunz, Commun. Math. Phys. **112**, 121 (1987).

44. J. Bellissard, in *Localization in disordered systems*, eds. W. Weller and P. Ziesche, Texts in Physics (Teubner, Leipzig, 1988).

45. J.E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **65**, 2185 (1990).

46. R.G. Clark et al., Phys. Rev. Lett. **62**, 1536 (1989); J.P. Eisenstein et al. , Phys. Rev. Lett. **62**, 1540 (1989); B.I. Halperin, in ref. [34].

47. J.P. Eisenstein et al., Phys. Rev. Lett. **61**, 997 (1988); T. Chakraborty and P. Pietiläinen, *The fractional quantum Hall effect: Properties of an incompressible quantum fluid* (Springer, Berlin Heidelberg New York, 1988); and refs. therein; M. Greiter, X.G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991).

48. J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7 (Springer, New York, 1973).

49. N. Read, Phys. Rev. Lett. **65**, 1502 (1990).

50. J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B**40**, 8079 (1989).