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CRYSTAL BASE AND q–VERTEX OPERATORS

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ERRATA to the paper CRYSTAL BASE AND q-VERTEX OPERATORS

page 2, ℓ.15–16: Delete the following sentence.

'In the derivation of the second inversion relation we need to assume further that the dual module $V^{*\iota}$ (see Sect. 3.4) also has a global crystal base.'

page 4, eq.(2.3.2): Replace

$$\widetilde{\Phi}(z) \ : \ V(\lambda) \ \longrightarrow \ V(\mu) \otimes V[z, z^{-1}] \otimes \mathbb{Q}(q)((z)).$$

by the following two lines:

$$\widetilde{\Phi}(z) \ : \ V(\lambda) \ \longrightarrow \ V(\mu) \widehat{\otimes} V[z, z^{-1}].$$

The right hand side means $\oplus_\nu \prod_\xi V(\mu)_\xi \otimes V[z, z^{-1}]_{\nu-\xi}$.

page 7, eq.(2.5.2): $\mathrm{wt}\, v \neq \lambda$ should read $\mathrm{wt}\, v \neq \mu$.

page 11: Replace ℓ.7–21 ('Following [6] define $\cdots$ representation V_l of $U_q'(\widehat{\mathfrak{sl}}_2)$.') by the following.

Following [6] define the pairing $\langle\ ,\ \rangle_\iota$ of $V^{*\iota}$ and V by

$$\langle v^*, v\rangle_\iota = (-1)^{ht(\lambda_0-\lambda)} q^{(\alpha f(\lambda_0), \alpha f(\lambda_0)) - (\alpha f(\lambda), \alpha f(\lambda))} \langle v^*, v\rangle.$$

Here $\lambda = \mathrm{wt}\, v$, $\langle\ ,\ \rangle$ is the canonical pairing, and we set $ht(\xi) = \sum_{i\neq 0} n_i$ for $\xi = \sum_{i\neq 0} n_i\alpha_i$. If (L, B) is a crystal base of V then $(L^{*\iota}, B^{*\iota})$ is a crystal base of $V^{*\iota}$ where

$$L^{*\iota} = \{v^* \in V^* \mid \langle v^*, L\rangle_\iota \subset A\}$$

and $B^{*\iota}$ is the dual base of B via the pairing $\langle\ ,\ \rangle_\iota$. Moreover, from Lemma 2.2.3 in [9] it follows that if V has a global base then so does $V^{*\iota}$.

Remark. Finite dimensional representations having crystal pseudo-base (L, B) with perfect B were studied in detail in [7]. We do not know a general criterion to find modules satisfying the more stringent condition (2.2.3). It holds true for 'simple' representations such as the vector representation of $U_q'(\widehat{\mathfrak{sl}}_n)$ and the spin $l/2$ representation of $U_q'(\widehat{\mathfrak{sl}}_2)$.

page 11, ℓ.14 from the bottom: $\langle b, b^*\rangle_\iota = \pm 1$ should read $\langle b, b^*\rangle_\iota = 1$.

page 11, ℓ.6 from the bottom: Delete '$(-1)^{\langle h_i, \lambda\rangle + 1}$'.

page 12, ℓ.12: Delete 'up to signs'.

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ABSTRACT

The q-deformed vertex operators of Frenkel and Reshetikhin is studied in the limit $q \longrightarrow 0$. The calculation of the one point functions for the associated elliptic RSOS models is incorporated in the framework of crystal base theory.

1. Introduction

The integrable RSOS models of Andrews-Baxter-Forrester (ABF) [1] and their generalizations [2][3][4] are built upon elliptic solutions of the Yang-Baxter equation (YBE) in the interaction-round-a-face (IRF) formulation [5]. The one point functions in these models are known to be given in terms of branching functions for some coset pair of affine Lie algebras. (To be precise, this is so in one region of the parameter space of the model, called 'regime III'.) Similar results hold also for the vertex models corresponding to trigonometric solutions of YBE. As shown by Kang et al.[6][7], the theory of crystal base [8][9] affords in the latter case a powerful and systematic method for computing one point functions on the combinatorial level (i.e. assuming the validity of the corner transfer matrix method [5]).

In a recent work [10] Frenkel and Reshetikhin studied the q-deformation of the vertex operators à la Tsuchiya-Kanie [11] in conformal field theory. They showed that the correlation functions satisfy a q-difference analog of the Knizhnik-Zamolodchikov equation, and that the resulting connection matrices give rise to elliptic solutions of YBE of IRF type. It seems quite likely that the previously known models mentioned above are special cases of their construction. This has been confirmed in [10] in the simplest case including the ABF model.

The purpose of the present note is to study the q-vertex operators of [10] in the framework of the crystal base theory [6][8]. We show in particular that the computation of the one point functions in the elliptic RSOS models can be treated in much the same way as is done in [6]. Apart from technical details, the results of the present paper are basically simple consequences of the constructions of [6][8] and [10]. We are reporting them hereby in the hope that they might help as a step toward better understanding of integrable models with quantum group symmetry.

Throughout this note we follow the formulation and notations in [6]. The vertex operators we consider are of the form $\Phi : V(\lambda) \longrightarrow \widehat{V}(\mu) \otimes V$, where $V(\lambda)$ is

an integrable highest weight module, $\widehat{V}(\mu)$ is a completion of $V(\mu)$, and V is a finite dimensional module of the quantized enveloping algebra $U_q(\mathfrak{g})$. (This is one equivalent form of the vertex operator $\Phi(z)$ in the formulation of [10]). Our basic observation is that, provided V has a crystal base, the vertex operators preserve the crystal lattice (section 2, Proposition 2.3). Assuming that V has a global crystal base [6] we are led to a natural basis of the space of vertex operators labeled by 'admissible triples' (Proposition 2.4). We prove also that the composition of vertex operators $\Phi(z_1) \circ \Psi(z_2)$ is well defined at $z_1 = z_2$. As shown in [10] the Boltzmann weights of the elliptic RSOS models arise as connection (or braiding) matrices for the composition of vertex operators. From the observation above it follows immediately that these Boltzmann weights share the same energy function with the associated vertex model (section 3, eq.(3.2.1)). We shall prove the initial condition (Proposition 3.1) and the second inversion relation (Proposition 3.7) for the connection matrices, which are necessary in order to apply the corner transfer matrix method. In the derivation of the second inversion relation we need to assume further that the dual module V^{*t} (see Sect. 3.4) also has a global crystal base. In section 4 we show that the highest weight vectors in the tensor product $V(\xi) \otimes V(\eta)$ are labeled by 'restricted paths' (cf. [12]). Finally we relate these facts to the one point functions of the lattice model defined by the connection matrices.

2. Vertex operators

2.1. *Notations* We fix an affine Lie algebra $\mathfrak{g}$. Let Λ_i, $h_i = \alpha_i^\vee$, α_i, $\delta = \sum_{i=0}^{l} a_i \alpha_i$ and d have the same meaning as in [13], except that for the type $A_{2l}^{(2)}$ we reverse the ordering of vertices from [13]. Thus we have $a_0 = 1$ in all cases. The canonical central element will be denoted by $c = \sum_{i=0}^{l} a_i^\vee h_i$. Set $I = \{0, 1, \cdots, l\}$, $i_0 = 0 \in I$, $P = \mathbf{Z}\Lambda_0 \oplus \cdots \oplus \mathbf{Z}\Lambda_l \oplus \mathbf{Z}\delta$, $P^* = \mathbf{Z}h_0 \oplus \cdots \oplus \mathbf{Z}h_l \oplus \mathbf{Z}d$. We normalize the invariant form on P so that $(\alpha_i, \alpha_i) = 1$ for a short simple root α_i. It is related with the normalized form $(\ |\ )$ in [13] via $(\lambda, \mu) = r(\lambda|\mu)/2$, where the number r is such that the dual algebra $\mathfrak{g}^\vee$ (the one obtained by reversing arrows of the Dynkin diagram of $\mathfrak{g}$) is of type $X_l^{(r)}$. Setting $\rho = \sum_{i=0}^{l} \Lambda_i$ we have $2(\rho, \delta) = rh^\vee$, $h^\vee = \sum_{i=0}^{l} a_i^\vee$ being the dual Coxeter number.

Throughout this paper we shall follow the notations of [6] unless otherwise stated. Set $P_{cl} = P/\mathbf{Z}\delta$, $(P_{cl})^* = \oplus \mathbf{Z}h_i \subset P^*$ and let $cl : P \to P_{cl}$ denote the canonical map. We fix $af : P_{cl} \to P$ by $af(cl(\alpha_i)) = \alpha_i$ $(i \neq 0)$ and $af(cl(\Lambda_0)) = \Lambda_0$ so that $cl \circ af = \mathrm{id}$ and $af(cl(\alpha_0)) = \alpha_0 - \delta$. With the data $\mathfrak{g}$, P, I above is associated the quantized affine algebra $U = U_q(\mathfrak{g}; P, I)$ defined over $\mathbf{Q}(q)$ (q an indeterminate). Its presentation, the comultiplication Δ and the antipode a are those given in (2.1.7)–(2.1.12) in [6]. The subalgebra of U generated by e_i, f_i $(i \in I)$ and q^h $(h \in (P_{cl})^*)$ is denoted by $U' = U'_q(\mathfrak{g}; P_{cl}, I)$. By crystal base we will mean the *lower* ctystal base. Hence if V_l denotes the $l+1$ dimensional irreducible module of $U_q(\mathfrak{sl}_2)$ with highest weight vector v_0^l, then its crystal lattice is given by $L_l = \oplus_{k=0}^{l} A v_k^l$, $v_k^l = f_1^{(k)} v_0^l$. Here

$$A = \{f \in \mathbf{Q}(q) \mid f \text{ has no pole at } q = 0\}.$$

For a positive integer k we set $(P_+)_k = \{\lambda \in P \mid \langle h_i, \lambda\rangle \in \mathbf{Z}_{\geq 0}\ \forall i \in I,\ \langle c, \lambda\rangle = k\}$, $(P_+^0)_k = \{\lambda \in (P_+)_k \mid \langle \lambda, d\rangle = 0\}$, and likewise for $(P_{cl+})_k$. As in [6] $V(\lambda) = \oplus_{\nu\in\lambda-Q_+} V(\lambda)_\nu$ denotes the irreducible highest weight module with highest weight λ and its weight decomposition. We fix a nonzero highest weight vector u_λ of $V(\lambda)$ throughout.

Let V be a U-module. We shall regard its linear dual $V^* = \mathrm{Hom}_{\mathbf{Q}(q)}(V, \mathbf{Q}(q))$ as equipped with a U-module structure via the antipode a:

$$\langle xv^*, v\rangle = \langle v^*, a(x)v\rangle \quad v^* \in V^*,\ v \in V,\ x \in U \tag{2.1.1}$$

where $\langle\ ,\ \rangle$ denotes the canonical pairing of V^* and V. When confusion may arise, the dual module structure via the antipode a is denoted by V^{*a}, but normally we omit writing a. Similar convention is used also for U'-modules.

2.2 *Finite dimensional modules and R-matrices* Let $V \in \mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ be a finite dimensional U'-module. We equip $V[z, z^{-1}] = \mathbf{Q}(q)[z, z^{-1}] \otimes V$ with a U-module structure as follows.

$$\begin{aligned} &e_i(z^n \otimes v) = z^{\delta_{i0}+n} \otimes e_i v, \quad f_i(z^n \otimes v) = z^{-\delta_{i0}+n} \otimes f_i v, \\ &\mathrm{wt}\,(z^n \otimes v) = n\delta + af(\mathrm{wt}\, v), \end{aligned}$$

where $n \in \mathbf{Z}$ and $v \in V$ is a weight vector. We shall often write $z^n \otimes v$ as vz^n.

Analogously, for an invertible element $x \in \mathbf{Q}(q)$ we let V_x denote the U'-module whose underlying space is V, equipped with the structure map $\pi_x : U' \longrightarrow \mathrm{End}(V)$

$$\pi_x(e_i) = x^{\delta_{i0}}\pi(e_i), \quad \pi_x(f_i) = x^{-\delta_{i0}}\pi(f_i), \quad \pi_x(q^h) = \pi(q^h),$$

where π signifies the original structure map. (In the notation of [6], $V_x = \Phi_x(V)$; it is not to be confused with the weight space of V.)

In the following sections we shall assume that

$$V \text{ has a crystal base } (L, B), \tag{2.2.1}$$

$$B \text{ is perfect of level } N, \tag{2.2.2}$$

$$V \text{ has a global base.} \tag{2.2.3}$$

The conditions (2.2.1), (2.2.2) imply in particular that

$$\mathrm{wt}\, V \subset \lambda_0 - \sum_{i\neq 0} \mathbf{Z}_{\geq 0}\alpha_i, \quad \dim V_{\lambda_0} = 1 \text{ for some } \lambda_0 \in P_{cl}. \tag{2.2.4}$$

Let $V, W \in \mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ satisfy (2.2.1),(2.2.2). Then there exists an intertwiner of U-modules

$$\check{R}_{VW}(z_1/z_2) \ :\ V[z_1, z_1^{-1}] \otimes W[z_2, z_2^{-1}] \longrightarrow W[z_2, z_2^{-1}] \otimes V[z_1, z_1^{-1}],$$

which commutes with the multiplication by z_1, z_2 and depends rationally on $z = z_1/z_2$. Set $R_{VW}(z) = P\check{R}_{VW}(z)$, $Pw \otimes v = v \otimes w$. Take a nonzero vector $v_0 \in V_{\lambda_0}$ where λ_0 is as in (2.2.4), and let $w_0 \in W_{\mu_0}$, μ_0 be the counterpart for W. We normalize $R_{VW}(z)$ by

$$R_{VW}(z)v_0 \otimes w_0 = v_0 \otimes w_0. \tag{2.2.5}$$

We have then (cf. [10])

$$\check{R}_{VW}(z)\check{R}_{WV}(z^{-1}) = 1, \tag{2.2.6}$$

$$\left(R_{VW}(z)^{-1}\right)^{t_1} = \beta_{VW}(z)R_{V^*W}(z), \tag{2.2.7}$$

with some rational function $\beta_{VW}(z) \in \mathbf{Q}(q)(z)$.

2.3. *Vertex operators* Let $\lambda \in (P_+^0)_k$. We define a completion of $V(\lambda)$ by $\widehat{V}(\lambda) = \prod_{\nu\in\lambda-Q_+} V(\lambda)_\nu$. Its dual

$$V^*(\lambda) = \left(\widehat{V}(\lambda)\right)^{*a} = \oplus_{\nu\in\lambda-Q_+}\left(V(\lambda)_\nu\right)^*$$

is the irreducible lowest weight module with lowest weight $-\lambda$.

Now fix $\lambda, \mu \in (P_+^0)_k$ and $V \in \mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ satisfying (2.2.1)–(2.2.3). We take a weight basis $\{v_j\} \subset V$ such that $v_j \bmod qL \in B$. In [10] Frenkel and Reshetikhin studied the vertex operators

$$\Phi(z) = z^{\Delta_\mu - \Delta_\lambda}\widetilde{\Phi}(z), \quad \Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{r(k + h^\vee)}, \tag{2.3.1}$$

where by definition $\widetilde{\Phi}(z)$ is an intertwiner of U-modules

$$\widetilde{\Phi}(z) \ : \ V(\lambda) \longrightarrow V(\mu) \otimes V[z, z^{-1}] \otimes \mathbf{Q}(q)((z)). \tag{2.3.2}$$

Define its weight components Φ_{jn} by

$$\widetilde{\Phi}(z)v = \sum_j \sum_{n\in\mathbf{Z}} \Phi_{jn}v \otimes v_j z^{-n},$$

$$\Phi_{jn} \ : \ V(\lambda)_\nu \longrightarrow V(\mu)_{\nu - af(wtv_j) + n\delta}.$$

Then $\Phi = \sum_j \left(\sum_{n\in\mathbf{Z}} \Phi_{jn}\right) \otimes v_j$ gives rise to an intertwiner of U'-modules

$$\Phi \ : \ V(\lambda) \longrightarrow \widehat{V}(\mu) \otimes V. \tag{2.3.3}$$

It is easy to see that the correspondence $\Phi(z) \leftrightarrow \Phi$ is bijective. We shall use both pictures (2.3.1)–(2.3.2) and (2.3.3).

PROPOSITION 2.1. *For each $i \in I$ and $v \in V(\lambda)$ there exists an $M \in \mathbf{Z}_{>0}$ such that*

$$e_i^M \Phi_{jn} v = 0, \quad f_i^M \Phi_{jn} v = 0 \quad \text{for all } j, n.$$

In particular the dimensions of $\{U_q'(\mathfrak{g}_i)\Phi_{jn}v\}_{j,n}$ are bounded.

Proof. First note the following fact which can be verified easily. Let W be a $U_q(\mathfrak{sl}_2)$-module and $V_l = \oplus_{0 \le k \le l} \mathbf{Q}(q) v_k^l$ the $l+1$ dimensional irreducible module ($v_k^l = f_1^{(k)} v_0^l$, $e_1 v_0^l = 0$). If $u = \sum_{k=0}^l w_k \otimes v_k^l \in W \otimes V_l$ satisfies $\Delta(e_1^m)u = 0$, then $e_1^{m+l} w_k = 0$ for all k.

Since $V(\lambda)$ is integrable, we have $e_i^m v = 0$ for some m. From the remark above it follows that $e_i^{m+M_i}\Phi_{jn}v = 0$ where $M_i = \max \dim U_q'(\mathfrak{g}_i)v_j - 1$. On the other hand, if $v \in V(\lambda)_\nu$ then $\langle h_i, \mathrm{wt}\, \Phi_{jn} v\rangle = \langle h_i, \nu - \mathrm{wt}\, v_j\rangle$ is independent of n. Hence the assertion on f_i is also true. □

2.4. *Stability of crystal lattice* We shall show that the vertex operators preserve the crystal lattice. Recall that $V(\lambda)$ carries a unique nondegenerate symmetric bilinear form (,) with the properties [8] $(u_\lambda, u_\lambda) = 1$, $(xu, v) = (u, \psi(x)v)$ ($u, v \in V(\lambda)$, $x \in U$), where ψ is the anti-involution of U: $\psi(e_i) = q_i f_i t_i^{-1}$, $\psi(f_i) = q_i^{-1} t_i e_i$, $\psi(q^h) = q^h$. Let Φ^t be the element corresponding to Φ under the isomorphism

$$\mathrm{Hom}_{U'}(V(\lambda), \widehat{V}(\mu) \otimes V) \simeq \mathrm{Hom}_{U'}(V^*(\mu) \otimes V(\lambda), V),$$

namely we set $\Phi^t(u \otimes v) = \sum_j \sum_n \langle u, \Phi_{jn} v\rangle v_j$. Identifing $V(\mu) \simeq V^*(\mu)$ via (,), we regard Φ^t as a linear map $V(\mu) \otimes V(\lambda) \to V$ such that

$$\Phi^t(u \otimes e_i v) = e_i \Phi^t(u \otimes v) + q_i^{-1-\langle h_i, \mathrm{wt}\, v\rangle} \Phi^t(f_i u \otimes v), \tag{2.4.1}$$

$$\Phi^t(u \otimes f_i v) = q_i^{\langle h_i, \mathrm{wt}\, u\rangle} f_i \Phi^t(u \otimes v) + q_i^{1+\langle h_i, \mathrm{wt}\, u\rangle} \Phi^t(e_i u \otimes v), \tag{2.4.2}$$

$$\mathrm{wt}\, \Phi^t(u \otimes v) = cl(\mathrm{wt}\, v - \mathrm{wt}\, u). \tag{2.4.3}$$

LEMMA 2.2. *Let V be an integrable $U_q(\mathfrak{sl}_2)$-module with crystal lattice L, and let $\phi^t : V_m \otimes V_l \to V$ be a linear map satisfying (2.4.1)–(2.4.3). If $\phi^t(v_0^m \otimes v_0^l) \in L$, then $\phi^t(L_m \otimes L_l) \subset L$.*

Proof. Without loss of generality we may assume that V is irreducible. Let $\phi : V_l \to V_m \otimes V$ be the intertwiner corresponding to ϕ^t. It can be shown directly or by using (2.4.1)-(2.4.3) that $\phi^t(v_0^m \otimes v_0^l) \in L$ implies $\phi(v_0^l) \in L_m \otimes L$. Since ϕ commutes with e_1, f_1 it also commutes with $\tilde{e}_1$ and $\tilde{f}_1$, hence $\phi(L_l) \subset L_m \otimes L$. □

PROPOSITION 2.3. *The following are equivalent.*
(i) $\Phi_{jn}(L(\lambda)) \subset L(\mu)$ *for all* j, n,
(ii) $\Phi^t(L(\mu) \otimes L(\lambda)) \subset L$,
(iii) $\Phi^t(u_\mu \otimes u_\lambda) \in L$.

Proof. The equivalence of (i) and (ii) follows from the fact that $\{u \in V(\mu) \mid (L(\mu), u) \subset A\} = L(\mu)$ ([8] Proposition 5.1.1).

Clearly (ii) implies (iii). Assuming (iii), let us show that $u \in L(\mu)_{\mu-\xi}$ and $v \in L(\lambda)_{\lambda-\eta}$ $(\xi, \eta \in Q_+)$ imply $\Phi^t(u \otimes v) \in L$ by induction on $|\xi| + |\eta|$. The case $|\xi| + |\eta| = 0$ is valid by the assumption. Suppose $|\xi| > 0$. Then there exists an i such that $u = \tilde{f}_i u'$ with some $u' \in L(\mu)_{\mu-\xi+\alpha_i}$. Consider the decomposition $u = \sum_{0 \le j} f_i^{(j)} u_j$, $v = \sum_{0 \le k} f_i^{(k)} v_k$, $e_i u_j = e_i v_k = 0$. Then $u \in L(\mu)$ (resp. $v \in L(\lambda)$) implies $u_j \in L(\mu)$ (resp. $v_k \in L(\lambda)$) ([8], Proposition 2.3.2). Moreover we have $u_0 = 0$. Therefore the induction hypothesis applies, and by Lemma 2.2 we get $\Phi^t\left(f_i^{(j)} u_j \otimes f_i^{(k)} v_k\right) \in L$ for all j, k. The case $|\eta| > 0$ is similar. □

2.5 *Admissible triples*

DEFINITION. *Let $\lambda, \mu \in (P_+^0)_k$ and $b \in B$. We say that the triple (μ, b, λ) is admissible if $u_\mu \otimes b \in B(\mu) \otimes B$ is a highest weight vector of weight $cl(\lambda)$; or equivalently if*

$$wt\, b = cl(\lambda - \mu), \qquad \tilde{e}_i^{\langle h_i, \mu \rangle + 1} b = 0 \quad \text{for any } i \in I.$$

The following result has been communicated by Kashiwara. The assumption (2.2.3) is used only at this place.

PROPOSITION 2.4. *The following are equivalent.*

(i) (μ, b, λ) is admissible,

(ii) There exists a $v \in L$ such that $wt\, v = cl(\lambda - \mu)$, $v \bmod qL \in B$ and $\tilde{e}_i^{\langle h_i, \mu \rangle + 1} v = 0$ for any $i \in I$,

(iii) There exists a $\Phi^t : V(\mu) \otimes V(\lambda) \longrightarrow V$ satisfying (2.4.1)-(2.4.3) such that $\Phi^t(u_\mu \otimes u_\lambda) \in L$ and $\Phi^t(u_\mu \otimes u_\lambda) \bmod qL \in B$.

Proof. Implication (iii)⇒(ii)⇒(i) is obvious. Conversely (i)⇒(ii) holds because the *upper* global base $v = G^{up}(b)$ corresponding to b has the desired property ([8], Lemma 5.1.1).

To prove (ii)⇒(iii) let $U'(\mathfrak{b}_+)$ be the subalgebra of U' generated by e_i $(i \in I)$ and q^h $(h \in (P_{cl})^*)$, and let $\mathbf{C}_\lambda = \mathbf{C}1_\lambda$ be the one dimensional $U'(\mathfrak{b}_+)$-module $e_i 1_\lambda = 0$, $q^h 1_\lambda = q^{\langle h, \lambda \rangle} 1_\lambda$. We note first that

$$\mathrm{Hom}_{U'}(V(\lambda), \widehat{V}(\mu) \otimes V) \xrightarrow{\sim} \mathrm{Hom}_{U'(\mathfrak{b}_+)}(\mathbf{C}_\lambda, \widehat{V}(\mu) \otimes V). \tag{2.5.1}$$

Clearly the canonical map (2.5.1) is well defined and injective. To see that it is surjective, pick a $v \in \widehat{V}(\mu) \otimes V$ such that $\mathrm{wt}\, v = cl(\lambda)$ and $e_i v = 0$ for all $i \in I$. Then from the proof of Proposition 2.1 there exists an $N \in \mathbf{Z}_{>0}$ such that $f_i^N v = 0$ for $i \in I$, hence v generates an integrable U'-module isomorphic to $V(\lambda)$.

The right hand side of (2.5.1) equals

$$\mathrm{Hom}_{U'(\mathfrak{b}_+)}(V^*(\mu) \otimes \mathbf{C}_\lambda, V) \simeq \{v \in V \mid \mathrm{wt}\, v = cl(\lambda - \mu),\ e_i^{\langle h_i, \mu \rangle + 1} v = 0\ \forall i \in I\}.$$

In the last equality we used the fact that as $U'(\mathfrak{b}_+)$-module $V^*(\mu)$ is presented as $U'(\mathfrak{b}_+) / \left(\sum_i U'(\mathfrak{b}_+) e_i^{\langle h_i, \mu \rangle + 1} + \sum_i U'(\mathfrak{b}_+)\left(t_i - q_i^{-\langle h_i, \mu \rangle}\right)\right)$. This shows that (ii) implies (iii). □

Proposition 2.4 shows that given $\lambda, \mu \in (P_+^0)_k$ the space of vertex operators has a basis indexed by admissible triples

$$\{\Phi_\lambda^{\mu\, b}(z) \mid (\mu, b, \lambda) : \text{admissible}\}$$

The basis elements are uniquely fixed by requiring

$$\widetilde{\Phi}_\lambda^{\mu\, b}(z) v_\lambda = v_\mu \otimes G^{up}(b) + (\text{terms of the form } v \otimes w, \operatorname{wt} v \neq \lambda). \tag{2.5.2}$$

2.6. *Composition of vertex operators* In the subsequent sections we need to extend the base field from $\mathbf{Q}(q)$ to $K = \mathbf{Q}((q))$. We set $V^K(\lambda) = V(\lambda) \otimes K$ and so forth. From Theorem 4.12 b) in [14] it follows that the integrable highest weight U-module $V(\lambda)$ is absolutely irreducible, hence in particular $V^K(\lambda)$ is an irreducible $U^K = U \otimes K$ module. The extension of vertex operator will be denoted by the same letter $\Phi : V^K(\lambda) \to \widehat{V}^K(\mu) \otimes V^K$.

Suppose V, W satisfy (2.2.1), (2.2.2), and let $\Phi{:}V(\mu) \to \widehat{V}(\nu) \otimes V$, $\Psi{:}V(\lambda) \to \widehat{V}(\mu) \otimes W$ be vertex operators satisfying the conditions of Proposition 2.3. For an invertible element $x \in \mathbf{Q}(q)$ we set $\Phi^x = (\tau_x \otimes \mathrm{id})^{-1} \circ \Phi \circ \tau_x$ where $\tau_x \in \operatorname{End}_{\mathbf{Q}(q)}(V(\lambda))$ is defined by

$$\tau_x v = x^n v \quad \text{for } v \in V(\lambda)_{\mu + n\delta}, \quad \langle \mu, d \rangle = 0. \tag{2.6.1}$$

It is easy to check that Φ^x is a U'-linear map $V(\mu) \to \widehat{V}(\nu) \otimes V_x$. In the component form $(\Phi^x)_{jn} = x^{-n}\Phi_{jn}$; in the form (2.3.2) we have simply $\widetilde{\Phi}^x(z) = \widetilde{\Phi}(xz)$.

The composition of the vertex operators in the sense of (2.3.2) is defined as a formal series in z_1, z_2:

$$(\widetilde{\Phi}(z_1) \otimes \mathrm{id})\widetilde{\Psi}(z_2) = \sum \Phi_{jm} \circ \Psi_{kn} \otimes v_j z_1^{-m} \otimes w_k z_2^{-n}.$$

In general the substitution $z_1 = xz_2$ ($x \in \mathbf{Q}(q)^\times$) is meaningless.

PROPOSITION 2.5. *If $x = q^{-s}$ with s a positive integer, then the composition*

$$(\widetilde{\Phi}^x \otimes id) \circ \widetilde{\Psi} \ : \ V^K(\lambda) \to \widehat{V}^K(\nu) \otimes V^K \otimes W^K \tag{2.6.2}$$

is well defined over K.

Proof. The statement means that if $u \in L(\lambda)$ then for each $l \in \mathbf{Z}$ the sum $\sum_{m+n=l} q^{ms} \Phi_{jm} \circ \Psi_{kn} u$ is convergent in the q-adic topology. From Proposition 2.3 it follows that each term $\Phi_{jm} \circ \Psi_{kn} u$ belongs to $L(\nu)$. Since $\Psi_{kn} u = 0$ for $n \gg 0$, the assertion is clear. □

We give below a slightly stronger statement which says that for each $v \in V(\lambda)$ we have $\lim_{n \to \infty} \Phi_{jn} v = 0$.

PROPOSITION 2.6. *Suppose the conditions in Proposition 2.3 are valid. Then for any $v \in V(\lambda)$ and $p \in \mathbf{Z}_{>0}$ there exists an n_0 such that $\Phi_{jn}v \in q^p L(\mu)$ for all $n \geq n_0$. In particular (2.6.2) is well defined also for $s = 0$.*

Proof. We show first that for weight vectors $u \in L(\mu)$, $v \in L(\lambda)$ we have

$$e_i u = e_i v = 0,\ \Phi^t(u \otimes v) \in L \Longrightarrow \Phi^t(f_i^{(k)} u \otimes v) \in q_i^k L. \tag{2.6.3}$$

Let $l = \langle h_i, \mathrm{wt}\, u\rangle$, $m = \langle h_i, \mathrm{wt}\, v\rangle$, and write $w = \Phi^t(u \otimes v) = \sum_{0 \leq n} f_i^{(n)} w_n$, $e_i w_n = 0$. From (2.4.1) and $e_i v = 0$ we have

$$\begin{aligned}(-1)^k \Phi^t(f_i^{(k)} u \otimes v) &= q_i^{k(m+1)} \sum_{k \leq n} e_i^{(k)} f_i^{(n)} w_n \\ &\in \sum_{k \leq n} q_i^{k(1+m-n-\langle h_i, wt\, w\rangle)} A f_i^{(n-k)} w_n.\end{aligned}$$

Because $f_i^{l+1} u = 0$ the n in the sum can be at most l. Hence $\mathrm{wt}\, w = cl(\mathrm{wt}\, v - \mathrm{wt}\, u)$ implies $m - n - \langle h_i, \mathrm{wt}\, w\rangle = l - n \geq 0$, and we find (2.6.3).

Now let $v \in L(\lambda)_{\lambda - \xi}$ ($\xi \in Q_+$), $\nu \in P$ and $p \in \mathbf{Z}_{>0}$. By induction on $|\xi|$ we shall prove the following statement:

$$\begin{gathered}\text{For each } v, \nu, p \text{ there exists an } n_0 \text{ such that} \\ u \in L(\mu)_{\nu - n\delta}, n \geq n_0 \Longrightarrow \Phi^t(u \otimes v) \in q^p L.\end{gathered} \tag{2.6.4}$$

Suppose $|\xi| = 0$, $v = u_\lambda$. Using (2.6.3) and arguing similarly as in the proof of Proposition 2.3, it can be shown that $u \in L(\mu)_{\mu - \eta}$, $\eta = \sum_i n_i \alpha_i$ implies $\Phi^t(u \otimes u_\lambda) \in \prod_{i \in I} q_i^{n_i} L$.

Next consider the general case. Let $v = f_i^{(k)} v_0$, $e_i v_0 = 0$, $k \geq 0$. Assuming (2.6.4) for v we prove it for $f_i v$ using (2.4.2).

Note that $\langle h_i, \mathrm{wt}\, u\rangle = \langle h_i, \nu\rangle$ is independent of n. Since V is finite dimensional, there exists an $M_1 \in \mathbf{Z}_{>0}$ such that $f_i L \subset q_i^{-M_1} L$. Therefore if $\Phi^t(u \otimes v) \in q_i^s L$ then the first term of the right hand side of (2.4.2) belongs to $q_i^{s + \langle h_i, \nu\rangle - M_1} L$. As for the second term, we can take M_2 such that $e_i^{M_2} \Phi_{jn} v = 0$ for all j, n by Proposition 2.1. Write $u = \sum_{j \geq 0} f_i^{(j)} u_j$, $e_i u_j = 0$. Then $(f^{(j)} u_j, \Phi_{jn} v) = 0$ for $j > M_2$, and

$$q_i^{1 + \langle h_i, wt\, u\rangle} \Phi^t(e_i u \otimes v) \in \sum_{1 \leq j \leq M_2} q_i^{-j+1} A \Phi^t(f_i^{(j-1)} u_j \otimes v).$$

Hence if n_0 is large enough, the right hand side belongs to $q^p L$ for $n \geq n_0$. This completes the proof. □

3. Connection matrices

3.1. *Connection matrix* Fix $\lambda, \nu \in (P_+^0)_k$. Take finite dimensional U' modules $V, W \in \mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ satisfying (2.2.1)-(2.2.3) with the crystal lattice (L_V, B_V) and (L_W, B_W) respectively. Suppose that triples (ν, b_1, μ), (μ, b_2, λ) are admissible for some $\mu \in P_{cl}(k)_+$, $b_1 \in B_V$ and $b_2 \in B_W$. Then by Proposition 2.4 there exist vertex operators $\Phi_{\mu V}^{\nu b_1}(z_1)$ and $\Phi_{\lambda W}^{\mu b_2}(z_2)$.

The connection matrix is defined as follows. By the assumption $V[z_1, z_1^{-1}] \otimes W[z_2, z_2^{-1}]$ has an R matrix $\check{R}_{VW}(z_1/z_2)$ normalized as in (2.2.5). A result of [10] says that there exists a matrix $C_{VW}(z)$ such that

$$\begin{aligned}&(\mathrm{id} \otimes \check{R}_{VW}(z_1/z_2)) \left(\Phi_{\mu V}^{\nu b_1}(z_1) \otimes \mathrm{id}\right) \Phi_{\lambda W}^{\mu b_2}(z_2)\\ &\qquad = \sum_{b_1', b_2', \mu'} \left(\Phi_{\mu' W}^{\nu b_1'}(z_2) \otimes \mathrm{id}\right) \Phi_{\lambda V}^{\mu' b_2'}(z_1) C_{VW}\begin{pmatrix} \lambda & b_2 & \mu \\ b_2' & & b_1 \\ \mu' & b_1' & \nu \end{pmatrix}(z_1/z_2)\end{aligned} \tag{3.1.1}$$

holds. Here the matrix elements

$$C_{VW}\begin{pmatrix} \lambda & b_2 & \mu \\ b_2' & & b_1 \\ \mu' & b_1' & \nu \end{pmatrix}(z)$$

are understood to be zero unless (ν, b_1, μ), (μ, b_2, λ), (ν, b_1', μ') and (μ', b_2', λ) are all admissible triples. Since the composition of vertex operators (in the sense of matrix elements)

$$\left(\Phi_{\mu V}^{\nu b_1}(z_1) \otimes \mathrm{id}\right) \Phi_{\lambda W}^{\mu b_2}(z_2)$$

has meaning a priori only when $|z_1| \gg |z_2|$, the RHS of (3.1.1) should be understood as a result of analytic continuation to the region $|z_2| \gg |z_1|$. Furthermore the matrix C satisfies the Yang–Baxter equation ([10] Theorem 6.3.)

$$\begin{aligned}&\sum_{\substack{b_7, b_8, b_9 \\ \nu}} C_{V_1V_2}\begin{pmatrix} \lambda_6 & b_9 & \nu \\ b_5 & & b_8 \\ \lambda_5 & b_4 & \lambda_4 \end{pmatrix}(x)\, C_{V_1V_3}\begin{pmatrix} \lambda_1 & b_1 & \lambda_2 \\ b_6 & & b_7 \\ \lambda_6 & b_9 & \nu \end{pmatrix}(xy)\, C_{V_2V_3}\begin{pmatrix} \lambda_2 & b_2 & \lambda_3 \\ b_7 & & b_3 \\ \nu & b_8 & \lambda_4 \end{pmatrix}(y)\\ &= \sum_{\substack{b_7, b_8, b_9 \\ \nu}} C_{V_2V_3}\begin{pmatrix} \lambda_1 & b_7 & \nu \\ b_6 & & b_9 \\ \lambda_6 & b_5 & \lambda_5 \end{pmatrix}(y)\, C_{V_1V_3}\begin{pmatrix} \nu & b_8 & \lambda_3 \\ b_9 & & b_3 \\ \lambda_5 & b_4 & \lambda_4 \end{pmatrix}(xy)\, C_{V_1V_2}\begin{pmatrix} \lambda_1 & b_1 & \lambda_2 \\ b_7 & & b_2 \\ \nu & b_8 & \lambda_3 \end{pmatrix}(x)\,.\end{aligned} \tag{3.1.2}$$

As a direct consequence of (2.2.6), the matrix C satisfies the first inversion relation:

$$\sum_{b_1', b_2', \mu'} C_{VW}\begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix}(z)\, C_{WV}\begin{pmatrix} \lambda & b_1' & \mu' \\ \tilde{b}_1 & & b_2' \\ \tilde{\mu} & \tilde{b}_2 & \nu \end{pmatrix}(z^{-1}) = \delta_{b_1\tilde{b}_1}\delta_{b_2,\tilde{b}_2}\delta_{\mu,\tilde{\mu}}. \tag{3.1.3}$$

PROPOSITION 3.1.

$$C_{VV}\begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix}(1) = \delta_{b_1 b_1'}\delta_{b_2,b_2'}\delta_{\mu,\mu'}.$$

Proof. Thanks to Proposition 2.6, the composition $\left(\Phi_{\mu V}^{\nu b_1}(z)\otimes \mathrm{id}\right)\Phi_{\lambda V}^{\mu b_2}(z)$ gives rise to a well defined intertwiner $V(\lambda) \longrightarrow \widehat{V}(\nu)\otimes V\otimes V$ over $\mathbf{Q}((q))$. Moreover they are linearly independent since

$$\left(\Phi_{\mu V}^{\nu b_1}\otimes \mathrm{id}\right)\Phi_{\lambda V}^{\mu b_2}u_\lambda \equiv u_\nu\otimes b_1\otimes b_2 \mod q\widehat{L}(\nu)\otimes L\otimes L\otimes \mathbf{Q}[[q]]$$

where $\widehat{L}(\lambda) = \{v\in \widehat{V}(\lambda) \mid v = \sum v_\nu, v_\nu \in L(\lambda)_\nu\}$. Setting $z_1 = z_2$ in (3.1.1) and noting that $\check{R}_{VV}(1) = \mathrm{id}$ we get the Proposition. □

3.2. *Energy function* Now let us consider the limit $q \longrightarrow 0$. By the construction we have

$$\Phi_{\lambda V}^{\mu b}(z)u_\lambda \mod qL(\mu)\otimes L = z^{\Delta_\mu-\Delta_\lambda}u_\mu\otimes b.$$

By the assumption (2.2.2) and Proposition 4.3.2 of [6], we have

$$\check{R}_{VV}(z)\mid_{q=0}(b_1\otimes b_2) = z^{-H(b_1\otimes b_2)}b_1\otimes b_2.$$

Here H denotes the energy function of $\check{R}_{VV}$ (see [6], Sect. 4).

Therefore at $q = 0$ the equation (3.1.1) gives

$$C_{VV}\begin{pmatrix} \lambda & b_2 & \mu \\ b_2' & & b_1 \\ \mu' & b_1' & \nu \end{pmatrix}(z)\Bigg|_{q=0} = \delta_{b_1,b_1'}\delta_{b_2,b_2'}\delta_{\mu\,\mu'}z^{\Delta_\lambda+\Delta_\nu-2\Delta_\mu-H(b_1\otimes b_2)}\mathrm{id}. \tag{3.2.1}$$

In this sense we find that the energy function for the connection matrix C coincides with that of vertex model in the sense of [6].

3.3 *Dual modules* Let ι denote the anti-automorphism of U given by

$$\iota(e_i) = -e_i,\quad \iota(f_i) = -f_i,\quad \iota(q^h) = q^{-h}.$$

In Sect.3.3–3.4 we shall consider another U-module structure $V^{*\iota}$ using ι in place of a (2.1.1). We recall below from Sect. 5.1 [6] some facts concerning the dual modules.

It is easy to see that $(V^{*\iota})^{*\iota} = V$, $(V_x)^{*\varphi} = (V^{*\varphi})_x$ for $\varphi = a, \iota$. One has an isomorphism of U'-modules

$$\begin{aligned} &F\ :\ V_x^{*\iota} \xrightarrow{\sim} V^{*a},\quad v^* \quad\longmapsto\quad q^{p(\lambda_0^*)-p(wtv^*)}v^*, \\ &(\ x = q^{-rh^\vee},\ p(\lambda) = (af(\lambda), af(\lambda)+2\rho)\ ), \end{aligned} \tag{3.3.1}$$

where v^* is a weight vector and λ_0^* is a fixed weight of V^* (e.g. as in (2.2.4)). Note that the power of q is integral. Composing the maps $V = (V_x^{*\iota})_x^{*\iota} \to (V_x^{*\iota})^{*a} \to (V^{*a})^{*a}$ and adjusting the power of q appropriately one obtains also an isomorphism

$$\begin{array}{rcccl} \tilde{F} : V_y & \xrightarrow{\sim} & (V^{*a})^{*a}, & & y = q^{-2rh^\vee} \\ v & \mapsto & q^{-4(\rho, af(wtv))}\nu(v) & & \end{array} \tag{3.3.2}$$

Here $\nu : V \xrightarrow{\sim} V^{**}$ signifies the canonical isomorphism of vector spaces.

Following [6] define the pairing $\langle\ ,\ \rangle_\iota$ of $V^{*\iota}$ and V by

$$\langle v^*, v\rangle_\iota = q^{(af(\lambda_0), af(\lambda_0)) - (af(\lambda), af(\lambda))}\langle v^*, v\rangle$$

with $\langle\ ,\ \rangle$ being the canonical pairing. If (L, B) is a crystal base of V then

$$L^{*\iota} = \{v^* \in V^* \mid \langle v^*, L\rangle_\iota \subset A\}$$

is a crystal lattice. Moreover the dual base $B^{*\iota} \subset L^{*\iota}/qL^{*\iota}$ of B with respect to $\langle\ ,\ \rangle_\iota$ gives rise to a crystal pseudo-base $B^{*\iota} \sqcup (-B^{*\iota})$. If B is perfect of level N then $B^{*\iota} \sqcup (-B^{*\iota})/\{\pm 1\}$ is also a perfect crystal of level N. Hereafter we shall also assume that

$$V^{*\iota} \text{ has a crystal base } (L^{*\iota}, B^*), \tag{2.2.2*}$$

$$V^{*\iota} \text{ has a global base.} \tag{2.2.3*}$$

Remark. Finite dimensional modules having crystal pseudo-base (L, B) with perfect B were studied in detail in [7]. We do not know a general criterion to find modules satisfying more stringent conditions (2.2.3), (2.2.2)* and (2.2.3)*. They are true for 'simple' representations such as the vector representation of $U_q'(\widehat{\mathfrak{sl}}_n)$ and the spin $l/2$ representation V_l of $U_q'(\widehat{\mathfrak{sl}}_2)$.'

LEMMA 3.2. *Let $\lambda, \mu \in (P_+^0)_k$, $b \in B$, $b^* \in B^*$ so that $\langle b, b^*\rangle_\iota = \pm 1$. Then*

$$(\mu, b, \lambda) \textit{ is admissible} \iff (\lambda, b^*, \mu) \textit{ is admissible}\ .$$

Proof. In view of the definition in Sect. 2.5 it suffices to check that the following are equivalent for each $i \in I$:

(i) $\tilde{e}_i^{\langle h_i, \mu\rangle + 1} b = 0$,

(ii) $\tilde{e}_i^{\langle h_i, \lambda\rangle + 1} b^* = 0$.

The condition (ii) is equivalent to

$$\langle \tilde{e}_i^{\langle h_i, \lambda\rangle + 1} b^*, b'\rangle_\iota = 0 \quad \text{for any } b' \in B. \tag{3.3.3}$$

Since the left hand side is $(-1)^{\langle h_i, \lambda\rangle + 1}\langle b^*, \tilde{e}_i^{\langle h_i, \lambda\rangle + 1} b'\rangle_\iota$, (3.3.3) is equivalent to the condition that there is no $b' \in B$ such that $b = \tilde{e}_i^{\langle h_i, \lambda\rangle + 1} b'$. Let $\epsilon_i(b)$ and $\varphi_i(b)$ have the same meaning as in [6] (2.2.5). We have $\varphi_i(b) \le \langle h_i, \lambda\rangle$. Recalling that $\varphi_i(b) - \epsilon_i(b) = \langle h_i, \lambda - \mu\rangle$ one gets $\epsilon_i(b) \le \langle h_i, \mu\rangle$, which is equivalent to (i). □

3.4. *Preparation* This subsection is devoted to some preliminary considerations necessary for deriving the second inversion relation in Sect. 3.5.

LEMMA 3.3. *Let $\lambda, \lambda' \in (P_+^0)_k$. Then*

$$\begin{aligned} Hom_{U'}(V^K(\lambda), \widehat{V}^K(\lambda')) &= K \quad \text{if } \lambda = \lambda', \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Proof. Let $\Psi : V^K(\lambda) \to \widehat{V}^K(\lambda')$ be a U'-linear map, and let Ψ_n be its weight components, so that Ψ_n maps $V^K(\lambda)_\nu$ to $V^K(\lambda')_{\nu+n\delta}$. Each Ψ_n is a U'-linear map from $V^K(\lambda)$ to $V^K(\lambda')$ sending u_λ to a highest weight vector of weight $\lambda + n\delta$ in $V^K(\lambda')$. Since the latter is irreducible we find that $\Psi_n = 0$ $(n \neq 0)$ and that Ψ_0 is a scalar which can be nonzero only for $\lambda = \lambda'$. □

For $\lambda, \mu \in (P_+^0)_k$ we set

$$\begin{aligned} B_\lambda^\mu &= \{b \in B \mid (\mu, b, \lambda) \ : \ \text{admissible}\}, \\ B^{*\lambda}_{\ \mu} &= \{b^* \in B^* \mid (\lambda, b^*, \mu) \ : \ \text{admissible}\}. \end{aligned}$$

Lemma 3.2 says that up to signs B_λ^μ and $B^{*\lambda}_{\ \mu}$ are dual bases to each other with respect to $\langle\ ,\ \rangle_\iota$. In this subsection we assume that they are non-empty.

Let $b \in B_\lambda^\mu$, $b^* \in B^{*\lambda}_{\ \mu}$, and consider the corresponding intertwiners $\Phi_{\lambda V}^{\mu b}$, $\Phi_{\mu V^{*\iota}}^{\lambda b^*}$ normalized as in (2.5.2). Using (3.3.1) we define $\Phi_{\mu V^{*a}}^{\lambda b^*} \in \mathrm{Hom}_{U'}(V(\mu), \widehat{V}(\lambda) \otimes V^{*a})$ by

$$\Phi_{\mu V^{*a}}^{\lambda b^*} = (\tau_x^{-1} \otimes F) \circ \Phi_{\mu V^{*\iota}}^{\lambda b^*} \circ \tau_x, \qquad x = q^{-rh^\vee}$$

where τ_x is defined in (2.6.1).

Define a linear map $T_{V(\lambda)} \in \mathrm{End}_{\mathbf{Q}(q)}(V(\lambda))$ by

$$T_{V(\lambda)} v = q^{(2\rho, \lambda - \nu)} v \quad \text{for } v \in V(\lambda)_\nu.$$

Similarly, for $V \in \mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ we define $T_V \in \mathrm{End}_{\mathbf{Q}(q)}(V)$ by

$$T_V v = q^{(2\rho, \lambda_0 - af(\nu))} v \quad \text{for } v \in V_\nu$$

where again λ_0 is the fixed weight of V (cf.(2.2.4)). From the definition one verifies the relations

$$\Phi_{\mu V^{*a}}^{\lambda b^*} = (T_{V(\lambda)}^{-1} \otimes F T_{V^*}^{-1}) \circ \Phi_{\mu V^{*\iota}}^{\lambda b^*} \circ T_{V(\mu)} \times q^{(2\rho, \lambda + \lambda_0^* - \mu)}, \tag{3.4.1}$$

$$\langle F v^*, v \rangle = \langle T_{V^*} v^*, v \rangle_\iota. \tag{3.4.2}$$

Now take a weight basis $\{v_j\}$ of V in such a way that $v_j \bmod qL \in B$, and let $\{v_j^*\} \subset V^*$ be the dual basis with respect to the canonical pairing. In the notation (2.3.1) we set

$$\begin{aligned} \Phi_{\lambda V}^{\mu b}(z) &= z^{\Delta_\mu - \Delta_\lambda} \widetilde{\Phi}_{\lambda V}^{\mu b}(z) = \sum_j \Phi_{\lambda V}^{\mu b}(z)_j \otimes v_j, \\ \Phi_{\mu V^*}^{\lambda b^*}(z) &= z^{\Delta_\lambda - \Delta_\mu} \widetilde{\Phi}_{\mu V^*}^{\lambda b^*}(z) = \sum_j \Phi_{\mu V^*}^{\lambda b^*}(z)_j \otimes v_j^*. \end{aligned} \tag{3.4.3}$$

PROPOSITION 3.4. *For $b \in B^{\mu}_{\lambda}$, $b^* \in B^{*\lambda}_{\ \mu}$ there exist scalars*

$$g^{\mu bb^*}_{\lambda} = q^{rh^\vee(\Delta_\mu - \Delta_\lambda)}\tilde{g}^{\mu bb^*}_{\lambda}, \quad g^{*\lambda b^* b}_{\ \mu} = q^{rh^\vee(\Delta_\lambda - \Delta_\mu)}\tilde{g}^{*\lambda b^* b}_{\mu},$$
$$\tilde{g}^{\mu bb^*}_{\lambda}, \tilde{g}^{*\lambda b^* b}_{\mu} \in \mathbf{Q}((q)),$$

such that

$$\sum_j \Phi^{\lambda' b^*}_{\mu V^*}(z)_j \circ \Phi^{\mu b}_{\lambda V}(z)_j = \delta_{\lambda\lambda'} g^{\mu bb^*}_{\lambda}\, id_{V(\lambda)}, \tag{3.4.4}$$

$$\sum_j q^{-4\rho_j}\Phi^{\mu' b}_{\lambda V}(q^{-2rh^\vee}z)_j \circ \Phi^{\lambda b^*}_{\mu V^*}(z)_j = \delta_{\mu\mu'} g^{*\lambda b^* b}_{\ \mu}\, id_{V(\mu)}. \tag{3.4.5}$$

Here $\rho_j = (\rho, af(wt(v_j)))$, and the components are given in (3.4.3).

Proof. We take the base field to be $K = \mathbf{Q}((q))$ and omit writing K. Let $x = q^{-rh^\vee}$, and consider the composition of U'–linear maps

$$V(\lambda) \longrightarrow \widehat{V}(\lambda') \otimes V^{*\iota}_x \otimes V \xrightarrow{id \otimes F \otimes id} \widehat{V}(\lambda') \otimes V^{*a} \otimes V \xrightarrow{id \otimes \langle\ ,\ \rangle} \widehat{V}(\lambda'). \tag{3.4.6}$$

Here the first arrow is given by the composition of $\Phi^{\mu b}_{\lambda V}$ with $(\tau_x^{-1} \otimes \mathrm{id})\Phi^{\lambda' b^*}_{\mu V^{*\iota}}\tau_x$, which is well defined thanks to Proposition 2.5. By Lemma 3.3 (3.4.6) must be a scalar $\tilde{g}^{\mu bb^*}_{\lambda}$. It is easy to see that up to a power (3.4.4) coincides with (3.4.6). The formula (3.4.5) can be derived in a similar manner by taking $V^{*\iota}$ in place of V and using the isomorphism (3.3.2). □

Remark. Strictly speaking we must extend the base field further to include the fractional powers of q appearing in (3.4.4) and (3.4.5). These are the artifact of the overall power $z^{\Delta_\mu - \Delta_\lambda}$ in the normalization of the vertex operator (2.3.1), and are inessential for the subsequent discussions.

LEMMA 3.5. *Notations being as above, we have*

$$\tilde{g}^{\mu bb^*}_{\lambda} \in q^{\kappa}(\langle b, b^*\rangle_\iota + q\mathbf{Q}[[q]]), \quad \kappa = (2\rho, \lambda + \lambda_0^* - \mu). \tag{3.4.7}$$

In particular the matrices $g^{\mu}_{\lambda} = (g^{\mu bb^*}_{\lambda})_{bb^*}$, $g^{*\lambda}_{\ \mu} = (g^{*\lambda b^* b}_{\ \mu})_{b^* b}$ are invertible.

Proof. The latter assertions are clear from (3.4.7) and the definition of $g^{*\lambda}_{\ \mu}$. To see (3.4.7) note that (3.4.6) can be written as

$$q^{\kappa}(T^{-1}_{V(\lambda)} \otimes \langle\ ,\ \rangle_\iota) \circ \Psi, \quad \text{where } \Psi = (\Phi^{\lambda b^*}_{\mu V^{*\iota}} \circ T_{V(\mu)} \otimes \mathrm{id}) \circ \Phi^{\mu b}_{\lambda V}.$$

Here we have used (3.4.1), (3.4.2).

On the other hand, since $(2\rho, \alpha) \in \mathbf{Z}_{>0}$ for $\alpha \in Q$, $T_{V(\mu)}$ preserves the crystal lattice $L(\mu)$ and $T_{V(\mu)}u_\mu = u_\mu$. Using the normalization of the vertex operators we have then

$$\Psi u_\lambda \equiv u_\lambda \otimes b^* \otimes b \quad \mathrm{mod}\ q\widehat{L}(\lambda) \otimes L^{*\iota} \otimes L \otimes \mathbf{Q}[[q]].$$

Noting that $\langle L^{*\iota}, L\rangle_\iota \subset A$ we find (3.4.7). □

Now set

$$G_\lambda = q^{2rh^\vee\Delta_\lambda}\chi_\lambda, \quad \chi_\lambda = \mathrm{tr}_{V(\lambda)}T^2_{V(\lambda)} \in \mathbf{Q}((q)). \tag{3.4.8}$$

Note that χ_λ is the principally specialized character of the irreducible $\mathfrak{g}^\vee$-module with highest weight λ, where $\mathfrak{g}^\vee$ is the dual Kac-Moody Lie algebra.

LEMMA 3.6.

$$G_\lambda g_\lambda^{\mu b b^*} = G_\mu g^{*\lambda b^* b}_{\ \ \mu}.$$

Proof. We are to show that

$$q^{2rh^\vee \Delta_\lambda} \mathrm{tr}_{V(\lambda)} \Big(T^2_{V(\lambda)} \circ \sum_j \Phi^{\lambda' b^*}_{\mu V^*}(z)_j \circ \Phi^{\mu b}_{\lambda V}(z)_j \Big)$$
$$= q^{2rh^\vee \Delta_\mu} \mathrm{tr}_{V(\mu)} \Big(T^2_{V(\mu)} \circ \sum_j q^{-4\rho_j} \Phi^{\mu' b}_{\lambda V}(q^{-2rh^\vee} z)_j \circ \Phi^{\lambda b^*}_{\mu V^*}(z)_j \Big).$$

On the other hand we have

$$T^2_{V(\mu)} \circ q^{-4\rho_j - 2rh^\vee(\Delta_\lambda - \Delta_\mu)} \Phi^{\mu b}_{\lambda V}(q^{-2rh^\vee} z)_j = \Phi^{\mu b}_{\lambda V}(z)_j \circ T^2_{V(\lambda)}.$$

By the cyclic property of the trace and the fact that each hand side is convergent in q-adic topology, the assertion is clear. □

3.5. *Second inversion relation* The second inversion relation for the R matrix can be obtained by applying (2.2.7) twice together with (3.3.2):

$$\alpha_{VW}(z)\big(\big((R_{VW}(z)^{-1})^{t_1}\big)^{-1}\big)^{t_1} = (\tilde{F} \otimes \mathrm{id}) R_{VW}(zq^{-2rh^\vee})(\tilde{F}^{-1} \otimes \mathrm{id}), \quad (3.5.1)$$
$$\alpha_{VW}(z) = \beta_{VW}(z)/\beta_{V^*W}(z).$$

We shall give its counterpart for the connection matrices.

PROPOSITION 3.7.

$$\sum_{b_1, b_1', \lambda} \frac{G_\lambda G_\nu}{G_\mu G_{\mu'}} C_{VV} \begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix} (z^{-1})\ C_{VV} \begin{pmatrix} \lambda & b_1' & \mu' \\ b_1 & & \tilde{b}_2' \\ \mu & \tilde{b}_2 & \nu' \end{pmatrix} (q^{-2rh^\vee} z)$$
$$= \alpha_{VV}(z) \delta_{b_2 \tilde{b}_2} \delta_{b_2' \tilde{b}_2'} \delta_{\nu\nu'},$$

where G_λ and α_{VV} are given in (3.4.8) and (3.5.1), respectively.

Proof. Let $R_{VV}(z)^{kl}_{ij}$ be the matrix elements of R_{VV}, that is,

$$R_{VV}(z) v_i \otimes v_j = \sum_{k,l} R_{VV}(z)^{kl}_{ij} v_k \otimes v_l.$$

Similar convention is used for $R_{VV^*}(z)$ and $R_{V^*V}(z)$ by taking the dual base $\{v_j^*\}$ of $\{v_j\}$. From (3.1.1) we have

$$\sum_{i,j} \Phi^{\nu b_2}_{\mu V}(z_1)_i \circ \Phi^{\mu b_1}_{\lambda V}(z_2)_j R_{VV}(z)^{lk}_{ij}$$
$$= \sum_{b_1', b_2', \mu'} \Phi^{\nu b_2'}_{\mu' V}(z_2)_k \circ \Phi^{\mu' b_1'}_{\lambda V}(z_1)_l C_{VV} \begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix} (z) \quad (3.5.2)$$

Here we put $z_1/z_2 = z$. Operate $\Phi^{\mu' b_2^*}_{\nu V^*}(z_2)_k$ from the left to the both hand sides of (3.5.2) and sum over k. Setting $W = V^*$ in (3.1.1) and using (2.2.6) we have

$$\begin{aligned}
&\Phi^{\mu' b_2^*}_{\nu V^*}(z_2)_k \circ \Phi^{\nu b_2}_{\mu V}(z_1)_i \\
&= \sum_{\substack{b_1', b_1^*, \lambda' \\ j', l'}} \Phi^{\mu' b_1'}_{\lambda' V}(z_1)_{l'} \circ \Phi^{\lambda' b_1^*}_{\mu V^*}(z_2)_{j'} C_{V^* V} \begin{pmatrix} \mu & b_2 & \nu \\ b_1^* & & b_2^* \\ \lambda' & b_1' & \mu' \end{pmatrix} (z^{-1})\ R_{VV^*}(z)^{ik}_{l'j'}.
\end{aligned}$$

From the relation

$$R_{VV}(z)^{lk}_{ij} = (R_{VV}(z)^{t_2})^{lj}_{ik} = \beta(z^{-1})(R_{VV^*}(z)^{-1})^{lj}_{ik}$$

together with Proposition 3.4, we get

$$\begin{aligned}
&\beta_{VV}(z^{-1}) \sum_{b_1', b_1^*, \lambda} g_\lambda^{\mu b_1 b_1^*} \Phi^{\mu' b_1'}_{\lambda V}(z_1)_l C_{V^* V} \begin{pmatrix} \mu & b_2 & \nu \\ b_1^* & & b_2^* \\ \lambda & b_1' & \mu' \end{pmatrix} (z^{-1}) \\
&\qquad = \sum_{b_1', b_2', \mu'} g_{\mu'}^{\nu b_2' b_2^*} \Phi^{\mu' b_1'}_{\lambda V}(z_1)_l C_{VV} \begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix} (z)\ .
\end{aligned}$$

From Lemma 3.5 there exist inverse matrices $(\gamma_\lambda^{\mu b^* b})_{b^* b}$ and $(\gamma^{*\lambda}_{\ \mu}{}^{b b^*})_{b b^*}$ of g_λ^μ and $g^{*\lambda}_{\ \mu}$. Consequently we get

$$\beta_{VV}(z^{-1}) C_{V^* V} \begin{pmatrix} \mu & b_2 & \nu \\ b_1^* & & b_2^* \\ \lambda & b_1 & \mu' \end{pmatrix} (z^{-1}) = \sum_{b_1', b_2'} \gamma_\lambda^{\mu b_1^* b_1'} g_{\mu'}^{\nu b_2' b_2^*} C_{VV} \begin{pmatrix} \lambda & b_1' & \mu \\ b_1 & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix} (z)\ .$$

Similarly we can derive

$$\begin{aligned}
&\beta_{V^* V}(z)^{-1} C_{VV^*} \begin{pmatrix} \mu & b_1^* & \lambda \\ \tilde{b}_2 & & b_1 \\ \nu' & \tilde{b}_2^* & \mu' \end{pmatrix} (z^{-1}) \\
&\qquad = \sum_{\tilde{b}_1, \tilde{b}_2'} g^{*\lambda b_1^* \tilde{b}_1}_{\ \mu} \gamma^{*\mu' \tilde{b}_2' \tilde{b}_2^*}_{\ \nu'} C_{VV} \begin{pmatrix} \lambda & b_1 & \mu' \\ \tilde{b}_1 & & \tilde{b}_2' \\ \mu & \tilde{b}_2 & \nu' \end{pmatrix} (q^{-2rh^\vee} z)\ .
\end{aligned}$$

Using the first inversion relation (3.1.3) with $W = V^*$ and Lemma 3.6, we obtain the desired result. □

4. Restricted paths and one point functions

4.1 *Restricted paths* Recall that we assume B is perfect of level N (2.2.2). Hence for any $\eta \in (P_+)_N$ there exist sequences of weights $\eta_0 = \eta, \eta_1, \eta_2, \cdots \in (P_+)_N$ and a path $p_{gr} = (p_{gr}(n))_{n\geq 1}$, $p_{gr}(n) \in B$, such that for any n the following isomorphism of crystals holds:

$$B(\eta) \xrightarrow{\sim} \mathcal{P}(\eta_n; B) \otimes B^{\otimes n}$$

Here u_η is sent to $u_{\eta_n} \otimes p_{gr}(n) \otimes \cdots \otimes p_{gr}(1)$, and $\mathcal{P}(\eta; B)$ denotes the set of η paths (see [6] section 4). It is known that if $b \in B(\eta)$ corresponds to an η path $p = (p(n))_{n\geq 1}$ then

$$\begin{aligned} \operatorname{wt} b = \eta &+ \sum_{k=1}^{\infty} \Big(af(\operatorname{wt} p(k)) - af(\operatorname{wt} p_{gr}(k)) \Big) \\ &- \left(\sum_{k=1}^{\infty} k \big(H(p(k+1) \otimes p(k)) - H(p_{gr}(k+1) \otimes p_{gr}(k)) \big) \right) \delta, \end{aligned}$$

where H is the energy function of the corresponding $\check{R}_{VV}$. We shall identify $B(\eta)$ with $\mathcal{P}(\eta; B)$.

Let k be a positive integer with $k > N$. Fix $\xi \in (P_+)_{k-N}$ and $\eta \in (P_+)_N$. We set

$$High(\xi, \eta) = \{ u_\xi \otimes b \in B(\xi) \otimes B(\eta) \mid e_i(u_\xi \otimes b) = 0 \ \forall i \in I \}.$$

DEFINITION. *We say that $\tilde{p} = (a, p)$ is a restricted (ξ, η)-path in B if the following hold.*

(1) $a = (a(n))_{n\geq 0}$, $a(n) \in (P_+)_k$,
(2) $p = (p(n))_{n\geq 1} \in \mathcal{P}(\eta; B)$,
(3) the triple $(a(n), p(n), a(n-1))$ *is admissible for all* $n \geq 1$,
(4) $a(0) = \xi + af(wt\, p)$.

Note that the $a(n)$ are uniquely fixed from ξ and p by (3), (4). We let $\mathcal{P}_{res}(\xi, \eta; B)$ denote the set of restricted (ξ, η)-path in B.

PROPOSITION 4.1. *The following is a bijection.*

$$\mathcal{P}_{res}(\xi, \eta; B) \longrightarrow High(\xi, \eta) \qquad (a, p) \mapsto u_\xi \otimes p$$

Proof. For $n \geq 0$ we define $v(n) \in B(\xi) \otimes B(\eta_n)$ by the following map induced from $B(\eta) \xrightarrow{\sim} B(\eta_n) \otimes B^{\otimes n}$:

$$\begin{array}{ccc} B(\xi) \otimes B(\eta) & \xrightarrow{\sim} & \big(B(\xi) \otimes B(\eta_n)\big) \otimes B^{\otimes n} \\ u_\xi \otimes p & \mapsto & v(n) \otimes p(n) \otimes \cdots p(1). \end{array}$$

First let $(a, p) \in \mathcal{P}_{res}(\xi, \eta; B)$. Then we have $\operatorname{wt} v(n) = \operatorname{wt} v(0) - \sum_{j=1}^{n} \operatorname{wt} p(j) = a(n) \in P$. Let us show $v(n) \in High(\xi, \eta_n)$ for all $n \geq 0$ by the induction on n.

For $n \gg 0$, we have $v(n) = u_\xi \otimes u_{\eta_n}$, so we get $v(n) \in High(\xi, \eta_n)$. Assume that $v(n) \in High(\xi, \eta_n)$. From the admissibility of the triple $(a(n), p(n), a(n-1))$, $v(n) \otimes p(n) = v(n-1)$ is a highest weight vector of $B(a(n)) \otimes B$. Setting $n = 0$, we find $v(0) = u_\xi \otimes p \in High(\xi, \eta)$.

Conversely if $u_\xi \otimes p \in High(\xi, \eta)$ then setting $a(n) = \mathrm{wt}\, v(n)$ we have $(a, p) \in \mathcal{P}_{res}(\xi, \eta; B)$. □

4.2 *IRF models and their one point functions*

Here we define IRF models whose Boltzmann weights are given by the connection matrices, and state results on their one point functions.

As before we fix $\mathfrak{g}, V \in \mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ and $k > N$. Take a two dimensional square lattice $\mathcal{L}$. Place variables λ, μ, ... (resp. b, b', ...) on vertices (resp. bonds) with values in $(P_+^0)_k$ (resp. B). For a configuration of variables around a face

$$\begin{matrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{matrix}$$

we associate the Boltzmann weight

$$C_{VV}\begin{pmatrix} \lambda & b_1 & \mu \\ b_1' & & b_2 \\ \mu' & b_2' & \nu \end{pmatrix}(z) .$$

Recall that it is zero unless the triples $(\mu, b_1, \lambda), (\nu, b_2, \mu)$, $(\mu', b_1', \lambda), (\nu, b_2', \mu')$ are admissble.

Although in our consideration we treated q to be an indeterminate, the matrix elements of C_{VV} have meaning as functions of q and z. Under such identification we restrict q to $0 < q < 1$ and z to $1 < z < q^{-rh^\vee}$.

Next we explain the ground states of our IRF model. Fix a particular site i. Consider the horizontal half infinite line l having i as the left end. The ground states are labeled by the pair (ξ, η) ($\xi \in (P_+^0)_{k-N}, \eta \in (P_+^0)_N$). The ground state corresponding to (ξ, η) is described as follows. Define the (ξ, η) -path a_{gr} such that $cl(a_{gr}(n-1) - a_{gr}(n)) = \mathrm{wt}\,(p_{gr}(n))$. Place $a_{gr}(0), a_{gr}(1), \cdots$ (resp. $p_{gr}(1), p_{gr}(2), \cdots$) on every site (resp. edge) on l starting from i. The ground state is uniquely determined by the condition that it is constant along the NE-SW direction.

Take a dominant integral weight $\lambda \in (P_+^0)_k$. We consider the probability of finding the variable on i being the value λ, and denote it by $P(\lambda|\xi, \eta)$. Here (ξ, η) signifies the choice of a boundary condition. Thanks to the Yang–Baxter equation (3.1.2), the initial condition (Proposition 3.1) and the second inversion relatin (Proposition 3.7), Baxter's corner transfer matrix method [5] applies. We have the following expression for the one point function.

$$P(\lambda|\xi, \eta) = \frac{\chi_\lambda F(\lambda|\xi, \eta; q^{2rh^\vee})}{Z},$$

where

$$F(\lambda|\xi,\eta;q) = \sum_{\tilde{p}\in\mathcal{P}(\xi,\eta;B)(\lambda)} q^{\omega(p)},$$
$$\mathcal{P}(\xi,\eta;B)(\lambda) = \{\tilde{p} = (a,p) \in \mathcal{P}(\xi,\eta;B) | a(0) = \lambda\},$$
$$\omega(p) = \sum_{k=1}^{\infty} k(H(p(k+1)\otimes p(k)) - H(p_{gr}(k+1)\otimes p_{gr}(k))),$$
$$Z = \sum_{\lambda'\in(P_+^0)_k} \chi_{\lambda'} F(\lambda'|\xi,\eta).$$

Here the energy function H is given in (3.2.1). From Proposition 4.1 we have the following.

PROPOSITION 4.2.

$$F(\lambda|\xi,\eta;q) = \sum_n \dim(V(\xi)\dot{\otimes} V(\eta))_{\lambda-n\delta} q^n.$$

Moreover if the generalized Cartan matrix of $\mathfrak{g}$ *is symmetric,*

$$P(\lambda|\xi,\eta) = \frac{\chi_\lambda F(\lambda|\xi,\eta;q^{2rh^\vee})}{\chi_\xi \chi_\eta}.$$

Remark 1. The quantity $q^{s_\xi+s_\eta-s_\lambda}F(\lambda|\xi,\eta;q)$ is called the branching coefficient [15], where $s_\lambda = \dfrac{(\lambda+\rho,\lambda+\rho)}{r(k+h^\vee)} - \dfrac{(\rho,\rho)}{rh^\vee}$ for $\lambda \in (P_+^0)_k$.

Remark 2. This type of results have been established by direct methods for higher spin representations of $U_q(\widehat{\mathfrak{sl}}_2)$ [16] and the vector representation of $U_q(\mathfrak{g})$ of classical types A, B, D [17]. (There are problems for the type C since the vector representation is not perfect.) Proposition 4.2 covers and generalizes these results, on the assumption (yet to be verified) that the connection matrices coincide with the Boltzmann weights constructed in [3][4].

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