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Instability of MSW equation for Neutrino Mixing in Periodic Matter Distribution

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Abstract

We show that, in periodic matter distribution, the MSW equation for neutrino mixing does not have stable solution, in general. However, there may exist some special situations where stable solutions exist.

The MSW[1] (Mikheyev–Smirnov–Wolfenstein)equation for neutrino mixing gives an important clue to investigate neutrino phenomena in matter. Analytical solutions of this eguation were obtained for linear,[2] exponential[3] and hyperbolic tangent[4] matter distribution. However more complicated matter distribution would exist in nature.
In this note we consider a periodic matter distribution.[5]

We express neutrino flavors with α and β,their mass eigenstates with 1 and 2 . We assume ν_α interact with matter. Matter distribution is expressed by f(t), t is time from ν_α creation. We use natural unit ($\hbar$=c=1)and assume neutrino velocity neally egual to c. Then MSW equation for $\nu_\alpha(t)$ is given as

$$\left(i\frac{\partial}{\partial t}-m_1\right)\left(i\frac{\partial}{\partial t}-m_2\right)\nu_\alpha(t) = i\frac{\partial f(t)}{\partial t}\nu_\alpha(t)+f(t)\left(i\frac{\partial}{\partial t}-m_\beta\right)\nu_\alpha(t) \tag{1}$$

where m_1,m_2 are diagonalized masses of two neutrions and m_β is mass of flavor β neutrino. Relation between m_β and m_1,m_2 are well known. We used two kind of masses in order to make equation simple form.

We put $\nu_\alpha(t)=e^{-im_1t}U(t)$,and furthermore put $U(t)=\exp\left\{\int^t p(s)\,ds\right\}V(t)$ and take $p(s)=-i\{f(s)-m_1+m_2\}/2$.

Then resulting equation for V(t) is

$$\frac{\partial^2}{\partial t^2}V(t)+\left\{\frac{1}{4}\left[f(t)-m_1+m_2\right]^2 + \frac{i}{2}\frac{\partial}{\partial t}f(t)+f(t)(m_1-m_\beta)\right\}V(t)=0 \tag{2}$$

Here we take periodic matter distribution

$$f(t)=A\cos Bt, \tag{3}$$

then Eq. (2) is

$$\frac{\partial^2}{\partial t^2}V(t)+\left\{\frac{1}{4}(m_1-m_2)^2+\frac{1}{4}A^2\cos^2 Bt + \left[\frac{1}{2}(m_1+m_2)-m_\beta\right]A\cos Bt-\frac{i}{2}AB\sin Bt\right\}V(t)=0 \tag{4}$$

The form of this equation is just egual to the Newton's classical equation of motion with complex periodical coefficients.
Period is $2\pi/B$. Such equations of motion with real periodic coefficient were already extensively studied.[6] So we may apply those arguments for solutions of Eq.(4).

We rewrite Eq.(4) as

$$\frac{\partial^2}{\partial t^2}V(t) + \{\lambda - q_1(t) - i\,q_2(t)\}V(t) = 0 \qquad (5)$$

where

$$\lambda = \frac{1}{2}(m_1 - m_2)^2$$

$$q_1(t) = -\frac{1}{4}A^2\cos^2 Bt - \left\{\frac{1}{2}(m_1+m_2) - m_\beta\right\} A\cos Bt$$

$$q_2(t) = \frac{1}{2}AB\sin Bt$$

We express two independent soiutions of Eq(5) as $V_1(t)$ and $V_2(t)$. Then $V_1(t+C)$ and $V_2(t+C)$, $C=2\pi/B$,are also solutions,that is,$V_1(t+C)$and $V_2(t+C)$ can be expressed as linear combinations of $V_1(t)$ and $V_2(t)$.

$$V_1(t+C)=a_{11}V_1(t)+a_{12}V_2(t)$$
$$V_2(t+C)=a_{21}V_1(t)+a_{22}V_2(t) \qquad (6)$$

Furthermore,we can construct another combination W(t)

$$W(t)=\gamma V_1(t)+\delta V_2(t)$$

which satisfies

$$W(t+C)=\rho\, W(t),$$

where ρ is some constant. If $|\rho|>1$, then W(t) is unstable.

Such W can exist when

$$\begin{vmatrix} a_{11}-\rho, & a_{12} \\ a_{21}, & a_{22}-\rho \end{vmatrix} = 0 \qquad (7)$$

This equation is called discriminant. This is a second order equation for ρ,

and we express two solutions as ρ_1, ρ_2 and then we can construct two W's

$$W_1(t+C) = \rho_1 W_1(t)$$

$$W_2(t+C) = \rho_2 W_2(t)$$

Above discriminant Eq. (7) is independent of the selection of V_1 and V_2. So we take as V_1 and V_2 which satisfy following initial conditions

$$V_1(0) = e^{i\theta}, \qquad \dot{V}_1(0) = 0$$

$$V_2(0) = 0, \qquad \dot{V}_2(0) = e^{i\theta}$$

where θ is a constant. Wronskian for these V_1, V_2 is $e^{2i\theta}$ and therefore they are linearly independent. Then, from Eq.(6), putting t=0, we obtain

$$a_{11} = e^{-i\theta} V_1(C), \quad a_{12} = e^{-i\theta} \dot{V}_1(C)$$

$$a_{21} = e^{-i\theta} V_2(C), \quad a_{22} = e^{-i\theta} \dot{V}_2(C)$$

and discrminant is

$$\rho^2 - e^{-i\theta}\{V_1(C) + \dot{V}_2(C)\}\rho + e^{-2i\theta}\{V_1(C)\dot{V}_2(C) - \dot{V}_1(C)V_2(C)\} \tag{8}$$

$$= \rho^2 - e^{-i\theta}\{V_1(C) + \dot{V}_2(C)\} + 1 = 0$$

Here we may suppose that coeficient of the second term of Eq.(8),

$$e^{-i\theta}\{V_1(C) + \dot{V}_2(C)\}$$

would be, in general,complex value,and therefore solutions of Eq.(8) ρ_1 and ρ_2 also complex value,because V_1 and V_2 are solutions of complex differential equation, Eq.(5). However this supposition seems to be rather abstrct at present stage. So we wish to present expressions which show relations between $V_{1,2}$ and $q_{1,2}$ more explicitely.

We can construct solution $V_1(t)$ and $V_2(t)$ as follows; define σ, $u_0(t)$ and $v_0(t)$ as

$$\lambda = \sigma^2,$$

$$u_0(t) = e^{i\theta}\cos\sigma t, \quad v_0(t) = e^{i\theta}\sin\sigma t/\sigma$$

and construct $u_n(t)$ and $v_n(t)$ successively

$$u_n(t) = \int_0^t \frac{\sin\sigma(t-s)}{\sigma}\{g_1(s) + i g_2(s)\}\, u_{n-1}(s)\, ds$$

$$v_n(t) = \int_0^t \frac{\sin\sigma(t-s)}{\sigma}\{g_1(s) + i g_2(s)\}\, v_{n-1}(s)\, ds$$

Then V_1, V_2 are obtained as

$$V_1(t) = \sum_{n=0}^{\infty} u_n(t), \qquad V_2(t) = \sum_{n=0}^{\infty} v_n(t) \tag{9}$$

In fact we can confirm these V_1 and V_2 satisfy Eq.(5).

Thus we may consider V_1 and V_2 are generally complex value and therefore ρ_1 and ρ_2 are also complex.

Thus complex number ρ_1 and ρ_2 must satisfy

$$\rho_1\rho_2 = 1$$

$$\rho_1 + \rho_2 = e^{-i\theta}\{V_1(c) + \dot{V}_2(c)\}$$

However, clerarly there do not exist such ρ_1 and ρ_2 with $|\rho_1| = |\rho_2| = 1$ and W_1, W_2 are unstable. $\nu_\alpha(t)$ can be represented as linear combination of these W's,so in general unstable.

Finally we wish to add some comments;

1) Last part of our arguments may be not rigorous but rather general and abstract. So we can not exclude special cases where stable solutions exist.

2) If stable solutions are found they would give much interesting relations between parameters λ,A,B etc, that is, neutrino mass spectrum and matter distribution parameters.

3) For the case of three neutrino mixing, the discriminant is dependent on the choice of linear independet solutions and therefore we can not develop analogous arguments in general form.

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