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Mathematical Feasibility of the Linearized Harmonic Gauge

and

the Question of Validity on Hilbert's Proof

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Abstract

In general relativity the notion of gauge for weak gravity, independent of its validity in physics, is also questionable in mathematics. In 1917, Hilbert claimed that he had proved the general applicability of the linearized harmonic gauge for weak gravity and this is currently used as a theoretical basis. However, Eddington rejected this claim. Einstein accepted it with an explicit reservation that the resulting metric must remain bounded and weak. It is shown that Hilbert's proof is actually incomplete and a counter example is provided to show Hilbert's claim is not generally valid such that Einstein's requirement is satisfied.

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I. Introduction

In this analysis, attention is called to a questionable, though more than 80 years old, belief that the linearized Harmonic gauge (see eq. [7]) is always applicable to weak gravity. In 1917 Hilbert [1,2] provided a "proof" for this belief after he pointed out that, for Einstein's equation, a solution is not unique due to the freedom of choosing a coordinate system. In this paper, it will be shown that Hilbert's proof is actually incomplete and a counter example to his claim will be provided (in section 5). Thus, the choice of a gauge is actually restricted by physical requirements.

In 1921 Einstein [3] already recognized the need of caution in the usage of such a gauge (see also section 4). Wald [4] pointed out that the linearized harmonic gauge and the linear field equation (eq. [8]) for weak gravity implies no radiation for a massive source. This inconsistency suggested that this gauge may not be generally applicable. Also, Eddington [5] may have discovered this already in 1923. In justifying the linear equation for weak gravity, he rejected the notion of gauge. Instead, he pointed out that the linear field equation would imply its differences from the Einstein equation is of second order terms. Thus, one may argue that the linearized harmonic gauge may still be approximately valid. It seems that the linear equation would provide a first order approximation for the Einstein equation.

However, Hilbert's claim of unconditional applicability of the linearized harmonic gauge should still be questioned. The compatibility between Einstein's equation and the harmonic gauge has not been really resolved since Fock [6] has discovered that, for massive sources in a dynamic situation, to obtain a solution for Einstein equation by extending the solution of the linear equation would inevitably lead to logarithmic divergence. On the other hand, it is known that the linear equation (eq. [8]) for weak gravity is the basis of Einstein's radiation formula [7], which is supported by observations on the PSR 1913+16 binary pulsars [8]. Also, it has been shown that the linear field equation with a massive source term is derivable in terms of extending Newtonian gravity with the physical principles that led to general relativity [9]. Then, based on the principle of causality that causes are identifiable, it is proven that there is no bounded dynamic solution for an Einstein equation with the source term of massive energy-stress tensor [8,10]. In other words, applicability of the gauge to a dynamic solution for massive matter cannot be tested directly.

However, since the linear field equation with a non-massive source term has not been justified, it may still be possible that an Einstein equation with a source of non-massive matter has a bounded dynamic solution for gravity. This would allow us to show that, in terms of mathematics, Hilbert's claim is not generally valid. Although he has shown that there is an equation for the gauge vector, it remains to show that it has a bounded solution for this vector as re-

quired by Einstein's notion of weak gravity (section 4). In other word, the general applicability of the linearized gauge has not been established in mathematics.

From the viewpoint of physics, Hilbert's preposition on the linearized harmonic gauge should also be questionable since there is no physical condition other than the smallness of the metric. Thus, it is not surprising that Einstein [3] cautioned its usage. Wald [4] even discovered that this gauge could lead to inconsistency in physics (section 3). Thus, the "current" belief of the arbitrariness of a gauge is questionable in terms of both mathematics and physics [11]. A counter example will be provided in this paper to show that Hilbert's claim is not general valid in mathematics.

This paper is organized as follows: In section 2, the harmonic gauge is described in connection with the Einstein equation. The inadequacy of the current notion of gauge in general relativity is pointed out as due to ignoring physical requirements such as the equivalence principle[1]. This problem is illustrated with a simple example. In section 3, the notion of weak gravity, as a physical requirement, is discussed in connection with field equations. In particular, compatibility with the linear field equation for weak gravity is considered such that related theoretical problems for a valid solution can be addressed. In section 4, the mathematical "proof" of Hilbert for the general applicability of the linearized harmonic gauge for weak gravity is analyzed and its incompleteness is pointed out as failing to show the satisfaction of the weak gravity notion. His claim is also analyzed in terms of physics in addition to mathematics. Then, a counter example to his claim is given in section 5. In section 6, the importance of physical considerations in theoretical consistency and even in mathematical analysis is remarked.

2. The Einstein Equation of 1915 and the Notion of Gauge

The non-linear Einstein's field equation of 1915 [7] is

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = - KT(m)_{ab} , \tag{1a}$$

where R_{ab} is the Ricci curvature tensor, g_{ab} is the space-time metric, K is the coupling constant, and $T(m)_{ab}$ is an energy-stress tensor of massive matter, which generally depends on g_{ab}.

The Einstein tensor G_{ab} has the identity [7],

$$\nabla^c G_{cb} \equiv 0. \tag{2}$$

Owing to eq. (1), identity (2) implies the conservation law,

$$\nabla^{c}T(m)_{cb} = 0. \tag{3}$$

Due to identity (2), four more conditions are needed to solve eq. (1) uniquely. These additional conditions are called gauge conditions and represent a choice of space-time coordinates [2]. However, such a choice is actually not entirely arbitrary due to the fact that a space-time metric must satisfy physical requirements (for example, Einstein's equivalence principle [3,12]) that may not be compatible with a given gauge. However, the current notion of gauge, which is based on mathematical diffeomorphism [4], requires only the proper metric signature. Note that Eddington [5] who understood the equivalence principle does not accept this gauge notion.

An often-used gauge condition is the harmonic gauge [2,7]

$$\frac{\partial}{\partial x^{\mu}}(|g|^{\frac{1}{2}} g^{\mu\nu}) = 0, \tag{4}$$

where g is the determinant of the metric. Recently, it has been found, in confirmation with Eddington's caution, that this gauge can be incompatible with the equivalence principle [3] and the principle of causality (that causes are identifiable). For example, a solution for gravity of an electromagnetic plane wave would satisfy this gauge condition, but violate the physical principles [11,13]. Moreover, there are unphysical solutions with the proper metric signature, but none of them can be diffeomorphic to a physically realizable space-time [10,11].

A major problem in current theory of general relativity is its inability to distinguish a physical space from merely a mathematical manifold, which has the proper metric signature [14]. Consider the simple metric,

$$ds^2 = \alpha^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{5a}$$

where $\alpha(\geq 2c)$ is a constant. Metric (5a) is a solution of the Einstein equation $G_{ab} = 0$. Then, $ds^2 = 0$ implies that the velocity of light is α. One might argue that metric (5a) can be transformed to

$$ds^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2, \tag{5b}$$

by the following diffeomorphism,

$$dx' = dx, \qquad dy' = dy, \qquad dz' = dz, \qquad \text{and} \qquad dt' = \alpha dt/c. \tag{5c}$$

Eq. (5c) implies, however, that the units of t and t' are distinct and the light speed remains α but not c. Thus, eq. (5c) is a rescaling, and only the physical units, but not the physics, are changed. For example, the light speed can be expressed as 186,000 miles per second. If $\alpha = 2c$, metric (5a) implies that the light speed would be 2c, i.e., 372,000 miles/sec; and metric (5b) implies that the light speed is 186,000 miles/half-sec.

On the other hand, if metric (5b) were considered as a local Minkowski space, then the local coordinate transformation (5c) is invalid in physics since Einstein's equivalence principle implies that (5c) must be obtained through a suitable acceleration [12]. Since (5a) implies all the Christoffel symbols are zero, there is no gravitational acceleration. Thus, the equivalence principle is not satisfied. Also, in such a non-rotating free falling, the velocity of an observer is a constant. According to special relativity, this observer carries with himself a new coordinate system, which must be obtained by a Lorentz transformation. But, a Lorentz transformation cannot transform metric (5a) to a local Minkowski space. In conclusion, the cause of an incorrect light speed is due to the failure of satisfying the equivalence principle. And metric (5a), just like the Galilean transformation, has the problem of being not physically realizable.

3. The Notion of Weak Gravity and the Linear Field Equation

From observation, Einstein [3] required that a weak source would produce a weak gravity. Theoretically, his notion of weak gravity is based on the principle of correspondence and the principle of causality and as such is a physical requirement. Thus, whether this requirement can be satisfied by a field equation, must be proven. It turns out that the weak gravity requirement is satisfied by eq. (1) for static problems, but not dynamic problems [8] (see also § 4).

Based on weak gravity, Einstein "derived" the Newtonian approximation with the linearized harmonic gauge [3]. The notion of weak gravity requires that the deviations $\gamma_{ab} = (g_{ab} - \eta_{ab})$ from the flat metric η_{ab} is small ($|\gamma_{ab}| \ll 1$). If all the terms of explicit second order of deviations are neglected, then eq. (1) and eq. (4) are reduced to

$$G_{ab}^{(1)} \approx -KT(m)_{ab}, \text{ where } G_{ab}^{(1)} \equiv \frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} + H^{(1)}_{ab}, \tag{6a}$$

where

$$H^{(1)}{}_{ab} \equiv -\frac{1}{2}\partial^c[\partial_b \bar{\gamma}_{ac} + \partial_a \bar{\gamma}_{bc}] + \frac{1}{2}\eta_{ab}\partial^c\partial^d \bar{\gamma}_{cd}, \tag{6b}$$

$$\bar{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma, \qquad \text{and} \qquad \gamma \equiv \eta^{cd}\gamma_{cd} \tag{6c}$$

and

$$\partial^c \bar{\gamma}_{cb} = 0 . \tag{7}$$

Eq. (7) is the linearized harmonic gauge (or the Hilbert condition).

Gauge (7), "provided these conditions do not conflict with $|\gamma_{ab}| << 1$ [3] " reduces eq. (6) to

$$\frac{1}{2}\partial^c\partial_c \bar{\gamma}_{ab} = -K\, T(m)_{ab}. \tag{8a}$$

However, eqs. (7) and (8) imply the linearized conservation law,

$$\partial^c T(m)_{cb} = 0 . \tag{9}$$

which, as pointed out by Wald [4] and Yu [15], implies no radiation for a massive source although, for a non-massive source tensor T_{ab}, radiation may still be possible.

To see whether eq. (8a) is compatible with the notion of weak gravity, consider the solution of eq. (8a) [4,7],

$$\bar{\gamma}_{ab}(x^i, t) = -\frac{K}{2\pi}\int \frac{1}{R} T(m)_{ab}[y^i, (t - R)]\, d^3y, \quad \text{where} \quad R^2 = \sum_{i=1}^{3} (x^i - y^i)^2 . \tag{8b}$$

Solution (8b) would represent a wave if $T(m)_{ab}$ has a dynamical dependency on time t' (= t - R) [2,7]. An implicit gauge condition is that the flat metric η_{ab} is the asymptotic limit at infinity. Note that the integral in (8b) is finite and bounded since $T(m)_{ab}$ is non-zero only in a finite region. Moreover, eq. (8a) can be directly justified by the physical principles [9]. Thus, independent of eq. (1), the requirement of the existence of gravitational waves is assured.

Some characteristics of an exact solution for the weak gravity due to massive matter can be obtained. They are:

i) Solution (8b) manifests that the first order approximation of the space-time metric includes a propagating wave, and is an almost periodic function of time for a source in an almost periodic motion. This is consistent with the principle of causality, which requires that the exact solution is also an almost periodic function.

ii) Moreover, as shown be Eddington [5], from eq. (8) one obtains

$$\partial^a \bar{\gamma}_{ab}(x^i, t) = -\frac{K}{2\pi}\int \frac{1}{R}\,\partial^a T(m)_{ab}[y^i, (t - R)]\, d^3y, \tag{8c}$$

If eq. (8a) gives a first order approximation, $\partial^a \bar{\gamma}_{ab}$ should be of second order [5,8] due to the conservation law, $\nabla^a T(m)_{ab} = 0$ (which is independent of the notion of gauge).

iii) By definition, an exact space-time metric element (in a Cartesian coordinate system) for weak gravity is a small deviation from the flat metric [3] and therefore must be bounded (i.e. $|g_{ab}| <$ constant). On the other hand, according solution (8b), the first order approximation of γ_{ab} is also bounded and is small for a small source (with a finite extend). Thus, eq. (8) is consistent with the notion of weak gravity. Moreover, for the case of including singular mass distributions, in the region too close to the singular source, eq. (8a) is not valid for this problem. Thus, one must remodel $T(m)_{ab}$ such that weak gravity can be applied.

Note that i) and iii) are also satisfied by the electromagnetic wave and condition ii) is similar to the Lorentz gauge. Thus, a gravitational wave should have a fruitful analogy with electromagnetism [7].

Also, there is a crucial difference between the argument based on the arbitrary applicability of gauge (7) for weak gravity and Eddington's argument. From linear field eq. (8), if the source is non-zero in a finite region, the first order approximation of a space-time metric is bounded. It follows the conservation law, $\nabla^a T(m)_{ab} = 0$ that $\partial^a \bar{\gamma}_{ab}$ is of the second order, and this is independent of the gauge freedom.

On the other hand, Einstein equation (1) allows the term $\partial^a \bar{\gamma}_{ab}$ to be of the first order, and thus additional conditions are required to restrict them to be the second order. Also, since eq. (1) includes four constraint equations [7], it may not have a bounded solution for a dynamic problem. In fact, eq. (1) and eq. (8) are not compatible for a dynamic problem although they are for static problems [8,9]. The argument of Eddington assumes directly that eq. (8) and eq. (1) to be compatible; and thus the possibility of being incompatible is still open; besides $\partial^a \bar{\gamma}_{ab}$ being of the second

order is different from of $\partial^a \bar{\gamma}_{ab} = 0$ (see section 4). Apparently, Einstein's notion of weak gravity also implies that the freedom of space-time coordinates is restricted [9].

Moreover, that Einstein equation (1) does not have dynamic solutions (which include gravitational waves) for weak gravity [8] is supported by the fact that nobody has been able to show the existence of a dynamic solution. Although Damour and Schmidt [16] claimed the existence of dynamic solutions for eq. (1), they have not shown that such solutions are compatible with Einstein's notion of weak gravity as physics requires [9]. Also, although Christodoulou and Klainerman [17] claimed the existence of source free global solutions describing weak gravitational waves, they have not been able to justify these waves with dynamic sources. Moreover, the assumed dynamic nature of their initial condition has not been proven as valid. Thus, their claim is actually groundless. On the other hand, their initial conditions are incompatible with Einstein's radiation formula [18]. Thus, their claim is also invalid.

In fact, as early as 1936, Einstein [19] himself discovered that his eq. (1) does not admit a propagating wave solution. The subsequent "plane-waves" proposed by Bondi, Pirani and Robinson [20] have no weak limit. Hogarth [21] conjectured in 1953 that there is no physical dynamic solution unless the gravitational energy tensor is added to eq. (1), and Lo [8] proved that eq. (8) is actually an approximation of the modified Einstein equation,

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = - K[T(m)_{ab} - t(g)_{ab}] , \qquad (10)$$

where $t(g)_{ab}$ is the gravitational energy-stress tensor.

It has been proven that there is no bounded dynamic solution for eq. (1) with a massive source [8,18,23]. Thus, it is impossible to illustrate the incompatibility between eq. (1) and gauge (7) through a dynamic solution. Perhaps, this is a main reason that the conditional validity of the gauge was not discovered. Nevertheless, we may analyze the static solutions that can be made satisfying the linearized gauge, and find out the difficult in extending such a feature to a dynamic situation. If an exact solution is insisted, a non-massive source term must be used in eq. (1). Note also for a non-massive source tensor T_{ab}, eq. (8) could be invalid since the physical principles can justify eq. (8) only the case of a massive source [9]. In other words, eq. (8) may not be generally applicable (see metric [23] in section 5).

4. The Questionable Claim of Hilbert and His Incomplete Proof

Since a gauge must be compatible with physical requirements such as Einstein's equivalence principle, and etc., in contrast to the belief of Fock [22], the validity of a gauge can only be conditional. Since Einstein's notion of weak gravity is a physical requirement, the application of gauge (7) may lead to a violation of weak gravity for some situations as Einstein worried. Here, it will be shown directly that Hilbert's claim is invalid in mathematics.

Surprisingly, in Hilbert's proof [1] for the freedom of gauge (7), there is no requirement for γ_{ab}, except their smallness. Hilbert considered an infinitesimal coordinate transformation [1,4,7]

$$x_a = x_a' + \varepsilon\zeta_a . \tag{11}$$

It follows eq. (11) one has

$$g'_{ab} = g_{ab} + \varepsilon g_{ac}\frac{\partial\varsigma_b}{\partial x_c} + \varepsilon g_{bc}\frac{\partial\varsigma_a}{\partial x_c} + \ldots \tag{12}$$

Then, for $g_{ab} = \eta_{ab} + \gamma_{ab}$ and $g'_{ab} = \eta_{ab} + \gamma'_{ab}$, one has

$$\gamma'_{ab} \approx \gamma_{ab} + \varepsilon\partial_a\zeta_b + \varepsilon\partial_b\zeta_a . \tag{13}$$

Eq. (13) is called a gauge transformation, and ζ_a is a gauge vector.

It is well known that $G_{ab}^{(1)}$ is invariant under transformation (13). Thus, for weak gravity, the order of G_{ab} is invariant under a gauge (13) if it maintains γ_{ab} being smallness of the first order; and therefore G_{ab} is either of the first order or the second order of deviations.

Then, to satisfy the linearized harmonic gauge $\partial^c \bar{\gamma}'_{cb} \approx 0$, one should obtain a gauge vector ζ_a such that

$$\varepsilon\partial^c\partial_c\zeta_b = -\partial^c\bar{\gamma}_{cb} , \tag{14a}$$

and

$$\varepsilon\zeta_b(x^i, t) = -\frac{1}{4\pi}\int\frac{1}{R}\partial^c\bar{\gamma}_{cb}[y^i, (t - R)]\, d^3y, \qquad \text{where} \quad R^2 = \sum_{i=1}^{3}(x^i - y^i)^2 . \tag{14b}$$

is a solution. Then, $\varepsilon\zeta_b$ would be at least of the same order as $\partial^c \bar{\gamma}_{cb}$. Moreover, if eq. (8) is valid, $\varepsilon\partial_a\zeta_b$ would have a higher order, and the first order of $\bar{\gamma}_{ab}$ is essentially the same since

$$\varepsilon\partial_a\zeta_b(x^i, t) = -\frac{1}{4\pi}\int\frac{1}{R}\,\partial_a\partial^c\bar{\gamma}_{cb}[y^i, (t - R)]\, d^3y, \tag{15}$$

and $\varepsilon\partial_a\zeta_b$ would be of the same order as γ_{ab}. An implicit assumption is that the integrals are finite and bounded.

However, since γ_{ab} can be non-zero almost everywhere, there is no guarantee that the integration in (14b) or (15) to be finite. Even if they are finite, they may not be bounded. For an unbounded $\varepsilon\partial_a\zeta_b$, it is meaningless to consider its order in terms of any parameter. (In classical electrodynamics, an unbounded gauge function is, nevertheless, acceptable since such a gauge function has no physical meaning.) Thus, Hilbert's proof is at least incomplete.

Thus, the above analysis supports Einstein's [3] worry of inconsistency that an arbitrary gauge condition may not be compatible with the requirement of weak gravity. Nevertheless, (7) is applicable to the static case. To see this, let us consider a static vacuum solution of eq. (1) [2],

$$ds^2 = \left(\frac{1-C/2r}{1+C/2r}\right)^2 dt^2 - \left(1 + \frac{C}{2r}\right)^4 (dx^2 + dy^2 + dz^2) \tag{16}$$

where $r^2 = x^2 + y^2 + z^2$, $C = KM/4$ for a spherically distributed mass M. The first order approximation of metric (16) gives exactly the same result as eq. (8) [3]. From metric (16), we have, at large r,

$$\bar{\gamma}_{tt} \approx O(K/r), \quad \text{but} \quad \partial^a\bar{\gamma}_{at} = 0; \tag{17a}$$

and

$$\partial^a\bar{\gamma}_{ax} = -\partial_x\left[\frac{c^2}{4r^2}\left(1 + \frac{c}{2r}\right)^2\right], \text{ and } \quad \partial_y\partial^c\bar{\gamma}_{cx} \approx O(C^2/r^4). \tag{17b}$$

Hence, integral (15) converges and $\varepsilon\partial_a\zeta_b$ is also bounded. A solution satisfying the harmonic coordinates [7] is

$$ds^2 = \left(\frac{1-c/r}{1+c/r}\right)^2 dt^2 - \left(1 + \frac{c}{r}\right)^2 (dx^2 + dy^2 + dz^2)\, ds^2 - \left(\frac{1-c/r}{1+c/r}\right)\frac{c^2}{r^4}(xdx + ydy + zdz) \tag{18}$$

There is a difference from merely satisfying eq. (4) approximately.

Now, consider the well-known Schwarzschild solution

$$ds^2 = (1 - \frac{2c}{r})dt^2 - (1 - \frac{2c}{r})^{-1}dr^2 - r^2 d\Omega^2 \tag{19}$$

From metric (19), we have

$$\gamma_{tt} = -\frac{2c}{r}, \qquad \gamma_{xx} = -\frac{2cx^2}{r^3}(1 - \frac{2c}{r})^{-1}, \quad \text{and} \quad \gamma_{xy} = -\frac{2cxy}{r^3}(1 - \frac{2c}{r})^{-1}. \tag{20}$$

The first order of eq. (20) was obtained by Einstein in 1916 [12].

It follows eq. (20) that we have

$$\bar{\gamma}_{tt} = -\{\frac{2c}{r}(1-\frac{c}{r})(1-\frac{2c}{r})^{-1}\}, \qquad \text{but} \qquad \partial^a \bar{\gamma}_{at} = 0; \tag{21a}$$

and

$$\partial^a \bar{\gamma}_{ax} = -\partial_x\{\frac{2c}{r})(\frac{c}{r} - \frac{x^2}{r^2})(1-\frac{2c}{r})^{-1}\} + \partial_y\{\frac{2cxy}{r^3}(1-\frac{2c}{r})^{-1}\} + \partial_z\{\frac{2cxz}{r^3}(1-\frac{2c}{r})^{-1}\}$$

$$\approx O(C/r^2) + O(C^2/r^3) \tag{21b}$$

The terms of $O(C/r^2)$, do not make integral (14) to have a logarithmic divergence because there are cancellations in the integration. Since $\partial^a \bar{\gamma}_{ax}$ is of the first order, eq. (8) does not provide an approximation for metric (19).

However, for a dynamic problem, Hilbert's proof would not be applicable because eq. (1) may not have a bounded solution. For example, the unbounded "plane-wave" solution of Bondi et al. [20] illustrates that it may not always be possible to have a bounded time-dependent solution for eq. (1) [9]. In other words, for a dynamic situation, it may not be possible to reduce $\partial^a \bar{\gamma}_{ab}$ to second orders no matter how weak the source could be [11].

One may argue that for weak gravity, we may expect that the solution at large r, is essentially similar to (16) except that C becomes time-dependent. This would mean

$$\bar{\gamma}_{tt} \approx O(C/r), \quad \text{but} \quad \partial^a \bar{\gamma}_{at} \neq 0. \tag{22a}$$

Thus,

$$\partial_t \partial^a \bar{\gamma}_{at} \approx O(1/r) \qquad \text{if} \qquad \partial^x \bar{\gamma}_{xt} + \partial^y \bar{\gamma}_{yt} + \partial^z \bar{\gamma}_{zt} = 0. \tag{22b}$$

Then, it is not clear that $\partial^a \bar{\gamma}_{at}$ remains of the second order and the integrals (14) and (15) are finite.

5. A Counter Example for the Harmonic Gauge

For simplicity, let us consider a weak gravitational wave satisfying $\partial_c \partial^c \bar{\gamma}_{ab} \approx 0$ everywhere in a region. If gauge (7) is valid, $G^{(1)}{}_{ab}$ must be of second order. Since the order of G_{ab} is gauge invariant, gauge (7) can be applicable only if G_{ab} is of second order. Therefore, a metric as a counter example would have its G_{ab} to be of the first order, but $\partial_c \partial^c \bar{\gamma}_{ab} \approx 0$ everywhere. Thus, such a counter example would not satisfy the vacuum equation $G_{ab} = 0$ and has a dynamic tensor of the first order in the source. To illustrate this, let us consider a plane-wave as follows:

$$\gamma_{xx} = \gamma_{yy} = \gamma_{xy} = \frac{K}{4} A_0{}^2 [1 - \cos 2\omega\,(t - z)]\,, \tag{23a}$$

$$\gamma_{tt} = -\gamma_{zz} = [1 - 2\gamma_{xx}]^{-1/2} - 1\,, \tag{23b}$$

and otherwise $\gamma_{ab} = 0$. Obviously, metric (23) has the proper signature. The source term of the Einstein equation is

$$T_{ab} = \rho\,(t - z)\, P_a P_b \qquad \text{where} \quad P_t = -P_z = \omega\,, \qquad \text{and} \quad P_x = P_y = 0 \tag{23c}$$

and

$$\rho\,(t - z) = -\frac{A_0^2}{G} \cos 2\omega\,(t - z), \text{ where } G = g_{xx} g_{yy} - (g_{xy})2 = 1 - \frac{K}{2} A_0{}^2 [1 - \cos 2\omega\,(t - z)].$$

It is clear that metric (23) satisfies the weak gravity requirement and G_{ab} is of the first order in K. But eq. (8) is not valid because $\frac{1}{2}\partial^c \partial_c \bar{\gamma}_{ab} = 0$. *Thus, metric (23) is a counter example for the harmonic gauge. Since this is a demonstration of mathematics,* here we shall not consider the physics of this wave (which is discussed in reference [13]).

Also, wave (23) satisfied $\partial^a T_{ab} = 0$, but this is not a guarantee for gauge (7) since

$$\partial_t \partial^a \bar{\gamma}_{at} = -K\,\omega^2 A_0{}^2 \cos 2\omega\,(t - z) \tag{24}$$

is of the first order, and is non-zero almost everywhere. Now, although the equation

$$\varepsilon \partial^c \partial_c (\partial_t \zeta_t) = -\partial_t \partial^c \bar{\gamma}_{ct}(t-z)\,, \tag{25}$$

has

$$\varepsilon\partial_t \zeta_t(x^i, t) = -\frac{1}{4\pi}\int \frac{1}{R}\,\partial_t\partial^c \bar{\gamma}_{ct}[y^i, (t - R)]\, d^3y, \qquad \text{where} \quad R^2 = \sum_{i=1}^{3} (x^i - y^i)^2 . \tag{26}$$

as a formal solution. But, integral (26) is not a bounded solution. If eq. (25) had a bounded solution, then the approximate linear equation would be

$$\partial_c\partial^c \bar{\gamma}_{ab} = -2\,K\,\rho\,(t - z)\,P_a P_b \tag{27}$$

But, eq. (27) does not provide an approximation, since it also does not have a bounded solution [13]. From the above analysis, it is not possible to transform weak wave (23) such that gauge (7) is satisfied while the transformed gravitational wave remains bounded and weak as Einstein [3] required (see also Appendix).

6. Remarks

In conclusion, the linearized harmonic "gauge condition" (7) is not always applicable. Mathematics is a valuable tool for physics because what is right in physics can be proven in mathematics and conversely one cannot prove a preposition, which is not correct in physics. It has been proven in terms of physics that the current notion of gauge based on diffeomorphism alone is inadequate [9,11]. But, Hilbert's claim suggested the contrary. Accordingly, it should be invalid in mathematics although this invalidity is not directly depending on the physical arguments.

Hilbert is not the only one who has mistaken on the question of choosing a gauge. Fock [22] had made a similar mistake of ignoring physical requirements and thus believed that, independent of the physical situation, only the harmonic gauge is valid. When he failed to show the effects of a uniform acceleration with a space-time metric, he claimed that Einstein's equivalence principle is invalid. Fock [22] and his followers [23] are of course wrong.

In fact, one does not have to go far to see such a mistake. For instance, the example of rotating disk in Einstein's 1916 paper has the metric [24],

$$ds^2 = (c^2 - \Omega^2 r^2)\, dt'^2 - dr'^2 - (1 - \Omega^2 r'^2/c^2)^{-1} r'^2\, d\phi'^2 - dz^2 \tag{28}$$

where Ω is the angular velocity of the disk. The rotating disk has a frame of reference (r', ϕ', z), and (28) is the only valid metric in physics. There are other mathematical coordinate systems, but (28) is the one valid in physics because

he is "obliged to define time in such a way that the rate of a clock depends upon where the clock may be [12]." Obviously, metric (28) does not satisfy the harmonic gauge. Nevertheless, (28) is diffeomorphic to the flat metric, $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, which satisfies the harmonic gauge. In other words, when a frame of reference is chosen, the gauge may have been determined. This example also serves to show how the theoretical bias can affect a theorist.

Experimental tests may not be considered as capable of verifying directly all aspects of a theory because it takes a theorist to identify implicit assumptions and theoretical inconsistency may not appear in a readily testable form. For instance, when covariance is tested, it is tested in a physical space. Since an unphysical space does not exist in nature, it is in principle futile to test unrestricted general covariance experimentally.

Some theorists believed that a physical justification for a solution should be obtained only from comparison with experiment [14]. Such an attitude and the above difficulty probably result in a tendency of not examining the physical validity of a solution. In other words, a physical problem is often treated from the viewpoint of only pure mathematics, and physical requirements are often not sufficiently considered in calculations. As pointed out by Kinnersley [14], most of the exact solutions describe situations, which are not physical.

More important, sometimes mathematical judgment can be impaired by prejudices due to insufficient physical considerations. If Hilbert [1] is not immune to such shortcomings, who is?

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Appendix: Physical Solutions and Validity of a Field Equation

Although the linear equation (27) does not have any bounded (in magnitude) weak solution, as Einstein required, it is known that this equation have solutions. However, as shown in electrodynamics, a mathematical solution may not be valid in physics. A physical condition for weak gravity, as pointed out by Einstein [3], is that the resulting metric solution must remains bounded and weak. Hilbert [1] also claimed that such a condition of weak gravity was satisfied.

Let us examine the physics of some solutions of the linear equation,

$$\eta^{mn}\partial_m\partial_n F(x,y,z,t) = [4\partial_u\partial_v - \partial_x\partial_x - \partial_y\partial_y]F(u,v,z,t) = f(u) \qquad \text{(A1)}$$

where η^{mn} are the flat metric (+,-,-,-), u ≡ (t - z), and v ≡ (t + z).

If F is a function of only t and z, then the inhomogeneous solution depends not only on u but also v, and is

$$F(t, z) = \frac{v}{4}\int^{u} f(t)dt \tag{A2}$$

It should be noted that, due to the factor v, F(t, z) is not bounded even for a very small source term f(t). If F is also a function of x and y, then the inhomogeneous solution of eq. (A1) would have another form as follows:

$$F(x, y, z, t) = -\frac{1}{4}(x^2 + y^2)\, f(u) \tag{A3}$$

Note that solution (A3) of Penrose [25] is also not bounded. Moreover, the principle of causality implies that any parameter in a solution for physics must be related to some physical causes. If the cause for field F is independent of coordinates x and y, then solution (A3) violates the principle of causality. Solution (A3) depends on the parameters, the origins of the x-axis and y-axis, but they are unrelated to any physical cause.

The left-hand side of eq. (A1) can be considered as a Maxwell's equation or an equation of linearized gravity. Then, one may examine a field equation after the related physical cause has been identified.

For linearized gravity, the function F would relate to the deviations from a flat metric. If the physical cause is an electromagnetic plane-wave propagating in the z-direction, then the related source energy-stress tensor would be a function of u [13,14], and thus the source term in linearized gravity would have the form f(u). Then, according to solution (A2), the metric would have a factor v and is unbounded. On the other hand, the principle of causality implies that the metric is a function of u only [9.13]. This contradiction manifests that eq. (A1) is not an appropriate form since the electromagnetic plane-waves are supported as an idealizations [13].

In other words, the principle of causality implies that there are weak gravity exact solutions (e.g., of an electromagnetic wave), which cannot be approximated with linearized gravity.

For the case of Maxwell's equation, the principle of causality requires that the source term may not be in the form of a plane-wave f(u). Thus, a charged particle must be invariably massive, and therefore, according to special relativity, it is impossible to have a charged source, moving with the velocity of light. Some might argue that since Maxwell

equation in the Lorentz gauge has weak and bounded physical solutions for a weak source, (27) should have a bounded weak solution. It is now clear that such an argument based on similarity is irrelevant.

Although the question of causality is not clear for equation (27), the gauge terms would be added (see eq. [13]) to the initial solution and forms a new solution, which is subjected to causality. For instance, as Einstein pointed out, it must remain to be bounded and weak. According to (23), the cause of the gravitational wave is a function of only u. It follows that the principle of causality requires that a physical solution of (27) must also be a function of u. In conclusion, the analysis supports that there is no bounded solution for equation (27).

ENDNOTES

1) In many textbooks, the equivalence principle is actually Pauli's version [2] which "is now commonly but mistakenly regarded as Einstein's version of the principle [26]" in spite of Einstein strongly objected Pauli's version as a misinterpretation [26]. Pauli [2] regards the equivalence principle essentially as the mathematical existence of local Minkowski spaces. Different from Pauli, Einstein requires additionally: i) "the special theory of relativity applies to the case of the absence of a gravitational field [27]" and ii) a local Minkowski space is obtained by choosing the acceleration. Einstein [27] wrote, "... we must choose the acceleration of the infinitely small ("local") system of coordinates so that no gravitational field occurs; this is possible for an infinitely small region." The notion of acceleration with respect to a three-dimensional frame of reference is essential in Einstein's theory [3,22]. Moreover, Einstein's principle is proposed for a physical space [12,26], where physical requirements are sufficiently satisfied [10,18]. Fock [22] believed in Pauli, but regarded Einstein's equivalence principle as invalid, in part, because of misconceptions such as a frame of reference must be related to a Euclidean subspace [28].

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