ETH-TH-92-2
Fermilab Library
0 1160 0009317 3

ETH–TH/92–2

January 1992

Dup

PARTICLE PRODUCTION, INDUCED ENERGY–MOMENTUM TENSOR, AND THE TRACE ANOMALY IN (1+1)–DIMENSIONAL CURVED SPACE–TIMES

Christophe M. MASSACAND[1] and Christoph SCHMID[2]

Institut für Theoretische Physik, ETH–Hönggerberg, CH–8093 Zürich, Switzerland.

ABSTRACT

We present a gauge invariant particle concept and Fock space construction for scalar and spinor fields in (1+1)–dimensional curved space–times. It is based on gauge independent normal ordering with respect to the energy measured on a geodesic hypersurface. We derive the normal ordered expression for the induced energy–momentum tensor of massless scalar and spinor fields. Our computation is finite step by step and gives the energy–momentum tensor, which is covariantly conserved and has the standard trace anomaly $R/24\pi$. We apply our methods to four applications: 1) We show that a Rindler observer (accelerated observer in Minkowski space) interprets the Minkowski vacuum as his own vacuum, *i.e* he computes $< T^{\mu\nu} >= 0$ and $< n_k >= 0$. 2) We consider an inhomogeneous patch $R \neq 0$ in an otherwise flat space–time and compute the production of particles, $< n(k,t) >$, and energy–momentum, $< T^{\mu\nu}(x,t) >$, by the inhomogeneity. We show that $< n(k,t) >$ and $< T^{\mu\nu}(x,t) >$ are nonzero and consistent with each other. Our results contradict Parker's result that there is no particle production if space–time is conformally flat and quantum fields are conformally coupled. 3) We compute the production of energy density and the evolution of pressure of a quantum field in external FRW and inflationary space–times in 1+1 dimensions. 4) We consider the gravitational field of collapsing shells of classical matter in 1+3 dimensions, and we compute the production of Hawking radiation everywhere inside a linear wave guide in the radial direction. The results show the origin of the radiation (trace anomaly) and resolve the paradox of 't Hooft.

---

[1] partially supported by the Swiss National Science Foundation

[2] Lecture presented by C.S. at the 10th Seminar on "Relativistic Astrophysics and Gravitation", October 21–25 1991, in Potsdam. To be published by World Scientific

1. Topic and question

The topic is *particle production* (spin 0, 1/2) by gravitational tidal forces, *i.e.* curvature. This is relevant in the *very early universe* (*e.g* during inflation) and around *black holes* (Hawking radiation). The work for 1+1 dimensions is finished; the work in 1+3 dimensions has been started, but will not be reported here.

The question: for a given, arbitrary external curved space–time (CST), and for a given arbitrary initial state of the quantum field (most often for the initial vacuum state $|0, in\rangle$), one has the task of predicting the expectation value of the *induced energy–momentum–stress tensor*,

$$\langle 0, in | T^{\hat{a}\hat{b}}(P) | 0, in \rangle_H = \quad ?$$

The question must specify the point P in space–time and the local orthonormal frame (LONF with indices $\hat{a}, \hat{b}$) at P. This is a gauge invariant question (gauge invariant $\equiv$ independent of choice of coordinate system).

The LONF chosen (at a given point P in space–time) is a basis in the tangent vector space at P. The basis vector $e_{\hat{0}}(P)$ could be taken to be the instantaneous 4–velocity $u(P)$ of some observer, $e_{\hat{0}}(P) = u(P)$. It makes no difference whether the observer is free–falling or accelerated, since we only consider the instantaneous 4–velocity $u(P)$.

Our new method is *gauge invariant normal ordering*. This is connected to a gauge invariant notion of particle and of vacuum.

The *trace anomaly*, $\langle T^{\hat{a}}{}_{\hat{a}} \rangle = R/24\pi$ for massless particles, turns out to be the key concept. The trace anomaly governs the production of Hawking radiation near a black hole and the production of energy in the early universe.

2. Critique of frequency normal ordering

Previous definitions of particle, vacuum, Fock space, normal ordering were (in the words of Birrell and Davies [1]): "slippery, ambiguous, and not directly connected to $T_{\mu\nu}$".

Our key observation: previous work was based on a fundamental distinction between *positive and negative frequency modes*. The frequency ω is the eigenvalue of the Hamiltonian H, which is the time evolution operator. But time, frequency, and $\langle H \rangle$ are *gauge dependent*. A separation between positive and negative frequency modes is gauge dependent and therefore cannot have a fundamental physical meaning.

3. Our method: gauge invariant normal ordering

3.1 External electromagnetic fields

Our new method, gauge invariant normal ordering, will first be demonstrated in the case of quantum field theory in *external electromagnetic fields* in Minkowski space [2].

It is crucial to distinguish between the canonical and the kinetic energy–momentum,

$$\left.\begin{array}{l}\vec{k} = \vec{p} - e\vec{A} \\ \epsilon = H - eA^0\end{array}\right\} \rightarrow \begin{array}{l} k_\mu = p_\mu - eA_\mu \\ iD_\mu = i\partial_\mu - eA_\mu \end{array} .$$

$\langle p_\mu \rangle = \langle H, \vec{p} \rangle$ is the *canonical* energy–momentum. It is *gauge dependent*. It is needed for Hamiltonian *dynamics* (*time evolution*), for conservation laws (Noether theorem), and for canonical quantization.

$\langle k_\mu \rangle = \left\langle \epsilon, \vec{k} \right\rangle$ is the *kinetic* energy–momentum. It is *gauge invariant*. It is the *measured* energy–momentum. It is given by the *state* (at a given time).

The new thing is that our gauge invariant normal ordering separates between positive and negative eigenvalues of the kinetic energy operator ϵ (= measured energy).

3.2 External curved space–times

In external (1+1)–dimensional *curved space–time* our gauge invariant normal ordering goes as follows:

1) What is the *hypersurface* on which to do the mode expansion and the normal ordering? In Minkowski space it was a hypersurface of fixed time. In curved space–time, for any given point P and for any given LONF $e_{\hat{a}}(P)$ at P, it is the geodesic extension of $e_{\hat{1}}(P)$, and this geodesic hypersurface will be denoted by Σ_g; this is a gauge invariant concept (in previous work people used the gauge dependent notion of fixed time hypersurface, Σ_t). We do *not* assume asymptotic flatness, neither for $t \rightarrow \pm\infty$ nor for $x \rightarrow \pm\infty$. But we have assumed that space is open, not closed.

2)What is the *operator* to be used for normal ordering? For external electromagnetic fields in Minkowski space it was the measured energy. In curved space–time it is the locally *measured energy* (measured with cesium clocks) integrated over the geodesic hypersurface Σ_g (with the LONF's parallel to Σ_g at each point of Σ_g):

$$\epsilon = \int_{\Sigma_g} d\xi \, T^{\hat{0}}_{\hat{0}} = \int_{\Sigma_g} d\Sigma_\mu \, T^\mu_{\hat{0}} = \text{ gauge invariant.}$$

Here $d\xi$ is the proper length along Σ_g. Previously people used the Hamiltonian

$$H = \int_{\Sigma_t} d\Sigma_\mu \, T^\mu_0 = \text{ gauge dependent.}$$

Both the measured energy ϵ and the Hamiltonian H are formed from the energy–momentum tensor. The difference between ϵ and H is seen to arise in two places:

a) geodesic hypersurface Σ_g (gauge invariant) instead of a fixed–time hypersurface Σ_t (gauge dependent),

b) in the operator: the free index (*i.e.* the lower index) is a gauge invariant LONF–index $T^\mu_{\hat{0}}$ instead of a gauge dependent coordinate index T^μ_0.

The necessity of this normal ordering prescription in curved space–time has been demonstrated in ref.[3].

4. Main results

4.1 Densities of energy–momentum

For the *densities* of energy–momentum $T^{\hat{0}\hat{b}}$ on Σ_g we obtain the following result (derived for the case of no particle–particle interaction and no mass):

$$\left\langle N_{g.i}\, T^{\hat{0}\hat{b}}(P)\right\rangle = \left\langle T^{\hat{0}\hat{b}}(P)\right\rangle^{renorm.} \quad .$$

The left–hand side is computed with our new gauge invariant normal ordering prescription, $N_{g.i.}$; the computation is finite at each step. One might call this method the *"particle method"*. The right–hand side is computed with the standard *"field method"*: first point–splitting, second subtracting the divergent terms known from the Minkowski space calculation, third enforcing $\nabla_\mu T^{\mu\nu} = 0$ by hand (finite subtraction). Both methods, the old "field method" and our new "particle method" give the same result. This result agrees with the Wald tensor for 1+1 dimensions [4], which follows from the Wald axioms.

Note that although the notion of particle is related to the notion of normal ordering, there is a big difference between the two. The notion of particle needs the entire *geodesic hypersurface* Σ_g. On the other hand normal ordering of $T^{\hat{0}\hat{b}}(P)$ only needs the first derivative (along Σ_g) of the LONF at P; it only needs the condition that the spin connection $\omega_{\hat{1}}$ at P is zero, which is a *local condition*. This local condition is easily extended to 1+3 dimensions: $\omega_{\hat{a}} = 0$, $\hat{a} = 1, 2, 3$.

Note also that our method of normal ordering, discussed here and carried out in ref. [5], works for *arbitrary (1+1)–dimensional space–times.*

As long as we work *on one geodesic* hypersurface Σ_g (a line), *there exists no notion of curvature.* The *observables* on one geodesic hypersurface are the densities of energy and momentum, but not the flows. In *Hamiltonian formulation* of single–particle classical mechanics this would be $(\vec{x}, \vec{p})$, which are used as initial conditions on the initial hypersurface $\Sigma_g^{(i)}$. When discussing the *initial state* on the initial hypersurface (or the *state* on any fixed hypersurface), the notion of *time evolution* does not yet enter; there is no *time coordinate* yet (except that one may choose $t_i = 0$ on $\Sigma_g^{(i)}$), no lapse and shift function yet, *no Hamiltonian* yet, *no velocities* $\dot{\vec{x}}$ yet. Since quantum field theory on the geodesic hypersurface Σ_g contains no information about the curvature R, the energy–momentum density in the vacuum state must be zero:

$$\langle 0, \Sigma_g | T^{\hat{0}\hat{b}}(P) | 0, \Sigma_g \rangle = 0$$

for $P \in \Sigma_g$ and $e_{\hat{0}}(P)$ orthogonal to Σ_g. In this precise sense vacuum polarization effects do not give an energy–momentum density in the vacuum [3].

4.2 Fluxes of energy–momentum and the trace anomaly

The computation for the *fluxes* $T^{\hat{1}\hat{b}}$ of energy–momentum [5] through the timelike geodesic hypersurface extended from $e_{\hat{0}}(P)$ can be done using three different methods; the quickest method is to compute $\left\langle T^{\hat{0}\hat{b}}(P)\right\rangle$ in two different LONF's at P and

afterwards use frame covariance. Our result is

$$\left\langle T^{\hat{1}\hat{1}}(P)\right\rangle = \left\langle N_{g.i}\, T^{\hat{1}\hat{1}}(P)\right\rangle - \frac{R(P)}{24\pi} \quad ,$$

and $\left\langle T^{\hat{1}\hat{0}}\right\rangle = \left\langle T^{\hat{0}\hat{1}}\right\rangle$. In this expression the gauge invariant normal ordering is done on the geodesic hypersurface extending from $e_{\hat{1}}(P)$, although the fluxes $T^{\hat{1}\hat{b}}$ "live" (can be integrated) only on a hypersurface extending from $e_{\hat{0}}(P)$. This is the origin of the extra term on the right–hand side. This, together with $\left\langle T^{\hat{0}\hat{0}}\right\rangle$, gives the *trace anomaly*

$$\langle T^{\hat{a}}{}_{\hat{a}}(P)\rangle = \frac{R(P)}{24\pi} \quad .$$

After our gauge invariant normal ordering the computations are *finite at each step* (there are no divergences to be subtracted) and they give $\nabla_\mu T^{\mu\nu} = 0$; no need to enforce this equation "by hand". In the language of pertubation theory (our calculations are exact): after having subtracted the one–point functions, *e.g.* $\langle 0|\, T^{\hat{0}\hat{0}}\, |0\rangle$, there do not arise new divergences to be subtracted in two–point functions (in 1–loop calculations), *e.g.* in $\langle 0|\, T^{\hat{0}\hat{0}}(x) T^{\hat{0}\hat{0}}(y)\, |0\rangle$.

If we give as the initial state (on a spacelike geodesic hypersurface) the vacuum $|0, \Sigma_g\rangle$, and if we compute at a point P on Σ_g the fluxes (through a timelike geodesic hypersurface), the quantum field theoretic problem involves time evolution and tidal forces (*i.e.* the curvature R). Therefore it is no surprise that our computation gives a nonzero *pressure* (momentum flow) in the vacuum state:

$$\langle 0, \Sigma_g|\, T^{\hat{1}\hat{1}}(P)\, |0, \Sigma_g\rangle = -\frac{R(P)}{24\pi} \, .$$

for $P \in \Sigma_g$ and $e_{\hat{0}}(P)$ orthogonal to Σ_g. In this precise sense *vacuum polarization effects* produce *pressure* (*momentum flow*) but not energy–momentum density nor energy flow.

4.3 Improved point–splitting

A second method, an *improved point–splitting method*, is directly motivated by our gauge invariant normal ordering. As in the usual point–splitting, our point–splitting is done along a geodesic. The new thing is that for a given point P and a given LONF, the *densities* $T^{\hat{0}\hat{b}}$ are computed with the geodesic tangential to $e_{\hat{1}}(P)$, while the *fluxes* $T^{\hat{1}\hat{b}}$ are computed with the geodesic tangential to $e_{\hat{0}}(P)$. The motivation is that the densities $T^{\hat{0}\hat{b}}$ "live" (can be integrated) only on a hypersurface orthogonal to $e_{\hat{0}}$, and the fluxes $T^{\hat{1}\hat{b}}$ live only on a hypersurface tangential to $e_{\hat{0}}$. After the standard subtraction of the divergent terms arising in Minkowski space one obtains the correct $\langle T^{\mu\nu}\rangle$ directly without the need to enforce $\nabla_\mu T^{\mu\nu} = 0$ by hand (as in the old point–splitting methods).

5. Applications

5.1 The Rindler problem

In the first application we consider the "Rindler problem": given Minkowski space, given as the initial state of the quantum field the Minkowski vacuum, given the Lagrangian for freely propagating fields in external space–time without any detector present (in the Lagrangian). Task: an accelerated man (Rindler, using Rindler coordinates) and a free–falling man (using Minkowski coordinates) both must predict $\left\langle T^{\hat{a}\hat{b}}(P)\right\rangle$. The answer of the free–falling man is clearly zero, but since $\left\langle T^{\hat{a}\hat{b}}\right\rangle$ is a tensor, the Rindler man must also predict zero. Next question: predict the particle number $\langle n_k\rangle =?$ Our gauge invariant particle concept gives $\left\langle n_k^{g.i.}\right\rangle = 0$, whether one works in Minkowski gauge or in Rindler gauge. On the other hand the frequency normal ordering in Rindler gauge gives $\langle n_k^{\omega}\rangle_{Rindler} \neq 0$:

$$\left\langle T^{\hat{a}\hat{b}}(P)\right\rangle = 0 \quad , \quad \left\langle n_k^{g.i.}\right\rangle = 0 \quad , \quad \langle n_k^{\omega}\rangle_{Rindler} \neq 0 \quad .$$

The Rindler particles are fictitious, they do not correspond to nonvanishing $\left\langle T^{\hat{a}\hat{b}}(P)\right\rangle$; they are of no help whatsoever in predicting $\left\langle T^{\hat{a}\hat{b}}(P)\right\rangle$.

5.2 Particle production by an inhomogeneous patch

In the second application we consider an inhomogeneous patch with $R \neq 0$ in an otherwise flat space–time. Curvature effects (tidal forces) produce a burst of particles,

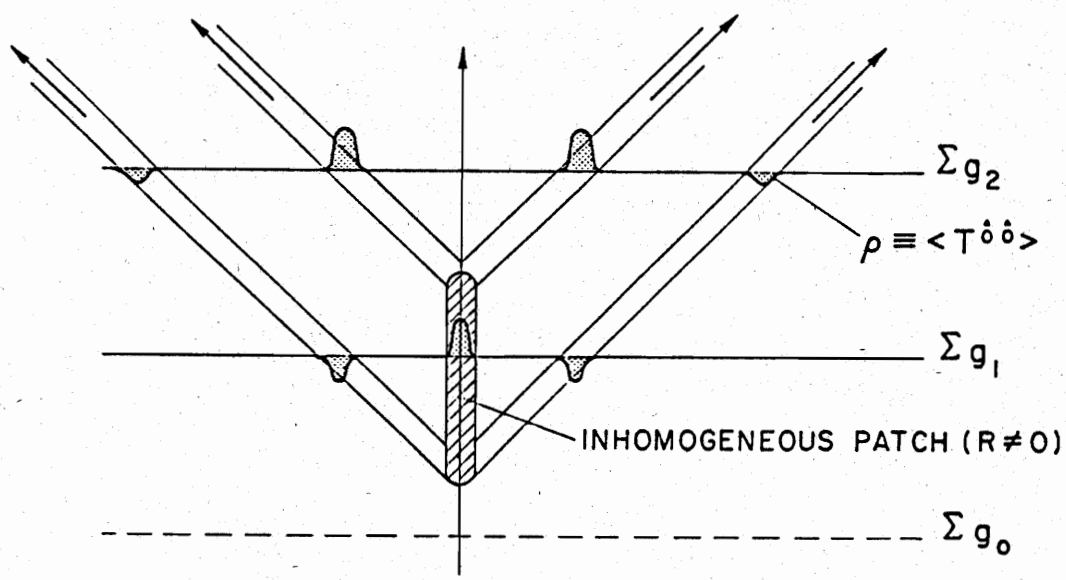

Figure 1a

$\langle n(k,t)\rangle \neq 0$, and of energy–momentum, $\left\langle T^{\hat{a}\hat{b}}(x,t)\right\rangle \neq 0$. To simplify we choose as the initial state of the quantum field the Minkowski vacuum. For the inhomogeneity in

fig. 1a we choose a thin elongated patch along a geodesic, with constant proper width $\Delta\xi$, and with $R = constant$ except for switching on at the beginning and switching off at the end (shaded area in fig. 1a). The produced energy density $\left\langle T^{\hat{0}\hat{0}}(\xi,t)\right\rangle$ is shown cross-hatched in fig. 1a; we see energy bursts caused by the switching on and the switching off. The produced particle number $\langle n(k,t)\rangle$ is shown in fig. 1b.

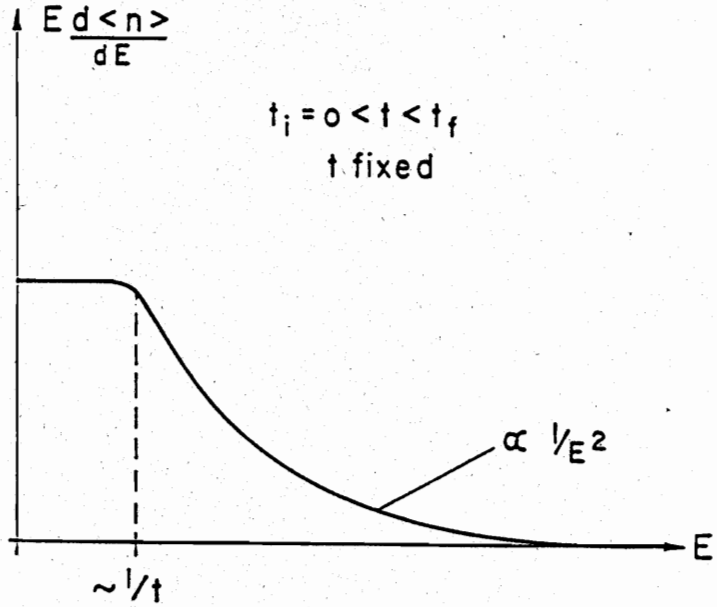

Figure 1b

Our particle density is consistent with the energy–momentum density, and the latter agrees with the Wald tensor [4]. All this is true not only in the Minkowski in/out regions, but also inside the patch of inhomogeneity.

Let us compare our results with *Parker's theorem* [6,7], which states: if space–time is conformally flat and the particles are conformally coupled, then there is no particle creation. Since in 1+1 dimensions space–time is always conformally flat and in 1+1 dimensions the minimal coupling of massless particles is conformal, there seems to be a *contradiction* between our result that particles are produced and Parker's theorem that no particles are produced. The *resolution* comes from the fact that Parker normal orders with respect to the conformal frequency, and this is inequivalent with our gauge invariant normal ordering. In the Minkowski in–region Parker's method can use Minkowski coordinates, and the initial state has $n_k = 0$ for both particle definitions. But if the conformal coordinates agree with Minkowski coordinates in the flat in–geometry, they must necessarily disagree with Minkowski coordinates in the flat out–geometry; the conformal normal ordering must disagree with the standard Minkowski normal ordering in the out–Minkowski space. The conformal particle number $n_k^{conf.}$ is conserved and remains zero for all times,

$$\left\langle n_k^{conf.}\right\rangle = 0 \quad ,$$

while our gauge invariant normal ordering agrees with standard Minkowski normal or-

dering (whenever one has Minkowski geometry) and gives $\left\langle n_k^{g.i.} \right\rangle \neq 0$ in the final state. Our particle density matrix is in quantitative agreement with Wald's energy–momentum density at all times (also in the intermediate geometry with $R \neq 0$), and this makes our particle definition relevant for physics:

$$\left.\begin{array}{l} \left\langle T^{\hat{0}\hat{b}}(x,t) \right\rangle \neq 0 \\ \left\langle n^{g.i.}(k,t) \right\rangle \neq 0 \end{array}\right\} \quad \text{consistent at all times.}$$

Parker's quanta clearly disagree with the standard particle notion in the out–Minkowski geometry.

We have computed the exact evolution starting from a pure initial state. Therefore the entropy $S = -tr(\rho \ln \rho)$ is zero at all times. If one would introduce small random phases into the density matrix ρ in the momentum basis, the entropy would increase and particle production would become accompanied by entropy production. The time scale needed for this decoherence process remains undetermined.

5.3 Energy production in Friedman–Robertson–Walker spaces

As our initial condition we assume that the energy density $\left\langle T^{\hat{0}\hat{0}} \right\rangle$ of the quantum field and its momentum density $\left\langle T^{\hat{0}\hat{1}} \right\rangle$ are zero on a hypersurface of constant Friedman time t (this hypersurface Σ_t is not a geodesic hypersurface). The trace anomaly $\langle T^{\mu}{}_{\mu} \rangle = R/24\pi$ and the covariant conservation of energy–momentum $\nabla_\mu T^{\mu\nu} = 0$ are sufficient to predict the creation of energy.

It is important to note that for external FRW geometries the following two equations are equivalent:

$$\nabla_\mu T^{\mu\nu} = 0 \overset{\text{FRW}}{\longleftrightarrow} dU = -p\, dV \quad ,$$

where V denotes a comoving volume, $p \equiv \left\langle T^{\hat{1}\hat{1}} \right\rangle$, $U \equiv \rho V$ with $\rho \equiv \left\langle T^{\hat{0}\hat{0}} \right\rangle$ in the FRW local orthonormal frame.

Zel'dovich [8] asked (in 1971) under what conditions one can have energy production (in a quantum field), if one starts with $\rho = 0$ at some initial time t_i. He made the ansatz for short times, $(t - t_i) \ll H^{-1}$,

$$\begin{aligned} \rho &\propto (t - t_i)^n \\ p &\propto (t - t_i)^m \quad . \end{aligned}$$

From $dU = -p\, dV$ it immediately follows that $n > m$, *i.e.* the pressure p must grow first, and afterwards the energy density ρ will follow. Specifically from $d\rho/dt \approx -p\,(dV/dt)V^{-1} \approx -(const.)p$, it follows that $n = m + 1$. But more importantly, from Zel'dovich's argument it follows that *energy production* (starting from zero energy) is possible if and only if there is a *trace anomaly*:

$$n = m + 1 \longrightarrow p \neq \rho \longleftrightarrow \langle T^{\mu}{}_{\mu} \rangle \neq 0 \quad .$$

Dimensional arguments tell us that the trace $\langle T^{\mu}{}_{\mu}\rangle$ can only be a pure number times R in 1+1 dimensions. Therefore the pressure exponent is $m = 0$ (pressure approximately constant), while the energy exponent is $n = 1$ (energy density growing linearly at the beginning).

The computation of the trace anomaly gives $\langle T^{\mu}{}_{\mu}\rangle = R/24\pi$, therefore the initial pressure is $p = -R/24\pi$. Our sign conventions are $\eta_{11} = -1$ and such that $R = -2(\dot{H}+H^2)$ for a FRW universe in 1+1 dimensions, where $H(t)$ is the Hubble parameter and $\dot{} \equiv \partial_t$.

For an external FRW geometry with a scale factor $a(t) \propto t^{\gamma}$, where $\gamma = 1/2$ or $2/3$, the curvature $R = 2\gamma(1-\gamma)/t^2$ is positive and the vacuum pressure of the quantum field is negative. Energy density and pressure will evolve from their initial vacuum values to an asymptotic regime for $(t - t_i) \gg H^{-1}(t_i)$, where *genuine energy production stops*; ρ and p are both positive and get merely *redshifted by adiabatic expansion*:

$$\left.\begin{aligned} &\frac{a^2(t)}{a^2(t_i)}\,\rho(t) \longrightarrow \frac{H^2(t_i)}{24\pi} \\ &\frac{p(t)}{\rho(t)} \longrightarrow 1 \end{aligned}\right. \qquad \text{independent of } t.$$

The asymptotic energy density and pressure of the quantum field are the same as for a *usual radiation field*, because the trace anomaly becomes negligible for $(t - t_i) \gg H^{-1}(t_i)$. The trace anomaly goes to zero as $(\rho - p) \sim t^{-2}$, while the energy density and the pressure get redshifted only as $\rho, p \sim t^{-2\gamma}$.

For an inflationary external curved space–time, $a(t) \propto e^{Ht}$, the curvature is negative and the vacuum pressure of the quantum field is positive. Energy density and pressure will evolve from their initial vacuum value to an asymptotic regime for $(t-t_i) \gg H^{-1}(t_i)$, where *energy and pressure reach a constant value*:

$$\left.\begin{aligned} &\rho(t) \longrightarrow -\frac{H^2}{24\pi} \\ &\frac{p(t)}{\rho(t)} \longrightarrow -1 \end{aligned}\right. \qquad \text{independent of } t.$$

Asymptotically we have $\langle T^{\mu\nu}\rangle \to \Lambda g^{\mu\nu}$, with the *induced cosmological constant* $\Lambda = -H^2/24\pi$. This induced cosmological constant is *opposite in sign* to the external effective cosmological constant (of the external vacuum energy/pressure).

5.4 Hawking radiation from collapsing shells

We consider a *collapsing thin spherical shell* (in 1+3 dimensions) of classical matter (*e.g.* many neutrons all at the same radius). For simplicity consider the special case $(m/E) \to 0$; therefore the shell follows the light–cone in the r–t–plane.

We now make a gedanken experiment. We consider a *thin linear wave guide* going radially across the shell to infinity. Instead of electromagnetic waves we consider a

massless scalar field inside the wave guide. The initial state (on a given spacelike hypersurface) is chosen to be the vacuum of the scalar field. The classical matter and the geometry are (1+3)–dimensional, the wave field is (1+1)–dimensional.

Question: what is the energy flow (*i.e.* the *Hawking radiation*) inside the tube *for any* r (not only far away!) measured with respect to some local orthonormal frame (LONF). In this talk we choose as an example the LONF adapted to Schwarzschild coordinates, *i.e.* the LONF of an observer instantaneously at rest in Schwarzschild coordinates.

Our *exact solution* is sketched in fig. 2. We see the collapsing shell following the light–cone, and the spacelike hypersurface Σ_i on which we choose our initial state (the vacuum of the quantum field). Since the geometry inside the shell is flat ($R = 0$) there is no trace anomaly and no energy–momentum production inside, $\left\langle T^{\hat{a}\hat{b}} \right\rangle \equiv 0$ inside.

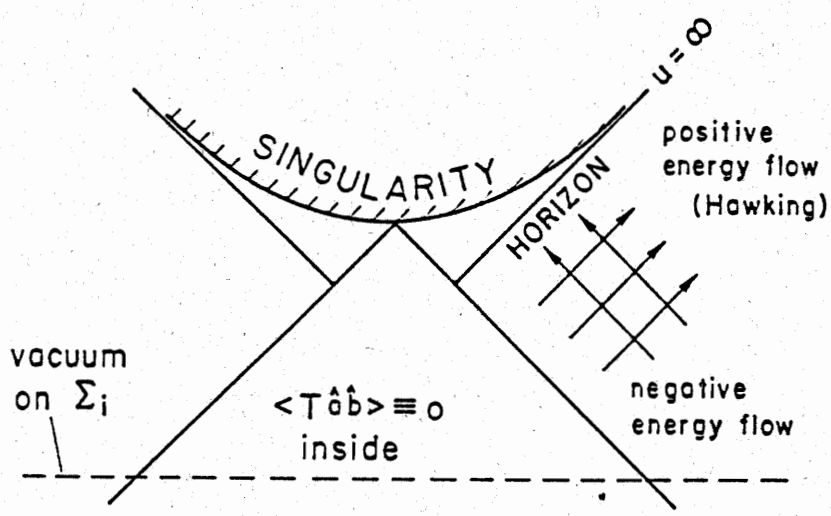

Figure 2

Outside the shell we have $R \neq 0$, and the trace anomaly causes the production of energy–momentum. We have shown that the outgoing and the ingoing wave fields decouple also at the level of quantum field theory. The outgoing waves have positive energy density, the ingoing waves have negative energy density. Our solution agrees with the solution of Davies, Fulling, and Unruh [9].

We now present the *problem of 't Hooft* [10]. He discussed an experiment where one photon is sent radially into a black hole from far away. He considered the collision of this ingoing photon with an outgoing Hawking photon. He computed the invariant mass (*i.e.* the total center–of–mass energy of the two photons) $[(p_1^\mu + p_2^\mu)^2]^{1/2}$. This invariant mass has a finite value far away, but it is blueshifted with the factor $[g_{00}^{(S)}(r)]^{-1/2}$ as one goes farther in. This blue shift is unbounded as we approach the Schwarzschild horizon, and the invariant mass of the two photons will exceed the Planck mass.

't Hooft concludes that fundamental modifications of our concepts are needed.

The *resolution* of this apparent difficulty will have the following bottom line: it is wrong to assume that one can trace back Hawking photons like classical point particles or a classical radiation field.

Our quantitative argument goes as follows: for a *classical point particle* moving radially in or out one has a conserved quantity,

$$\text{conserved: } p_{\mu=0} = e^{\hat{a}=\hat{0}}_{\mu=0}\, p_{\hat{a}=\hat{0}} \quad ,$$

where the index μ refers to Schwarzschild coordinates, $\hat{a}$ refers to a LONF aligned with Schwarzschild coordinates, and $e^{\hat{a}}_{\mu}$ is the 2–bein (LONF). The quantity $p_{\hat{a}=\hat{0}}$ is the locally measured energy, $p_{\mu=0}$ is the conserved energy (the Hamiltonian of section 3.2). For a *swarm of classical point particles* the *measured* flux of particle number (through 4π per sec.) increases farther in. We define light–cone coordinates u, v with $du = d(t - r^*)$, $dv = d(t + r^*)$, where t is the Schwarzschild time and dr^* is such that u (resp. v) is constant along outgoing (resp. ingoing) null geodesics. It follows that the flux of particle number for outgoing classical massless point particles *per fixed* du, dN/du, is conserved (as we follow the particles along their geodesics). From the two conservation laws (for a swarm of outgoing massless particles), $p_u = p_0 =$ conserved and $dN/du =$ conserved, it follows that

$$T^{(out)}_{uu} = \text{conserved} \longleftrightarrow \frac{\partial}{\partial v}\left[T^{(out)}_{uu}\right]_{class.} = 0 \quad .$$

The same *conservation law* can be shown to hold also for *classical field theory* of massless and conformally coupled fields. The argument of 't Hooft assumes that Hawking radiation can be traced back like classical radiation or like classical point particles. And this is equivalent to the above conservation law.

But *this conservation law is broken at the level of quantum field theory* because of the trace anomaly: $\nabla_\mu T^{\mu\nu} = 0$ together with the trace anomaly gives

$$\frac{\partial}{\partial v}\left\langle T^{(out)}_{uu}\right\rangle_{QFT} = -e^{2\varphi}\frac{\partial}{\partial u}\frac{R}{96\pi} \quad ,$$

where $e^{2\varphi}$ is the conformal scale factor in $ds^2 = e^{2\varphi}\, du\, dv$. *This equation determines the production of Hawking radiation locally.* The resulting $\left\langle T^{(out)}_{uu}\right\rangle$ is shown in fig. 3. $\left\langle T^{(out)}_{uu}\right\rangle$ starts at zero at the shell, it increases (as we go outwards) to an asymptotic value. In our figure we have chosen that value for u for which the shell crosses the horizon ($u = \infty$). For $u =$ fixed $= \infty$ the asymptotic value of $\left\langle T^{(out)}_{uu}\right\rangle$ (for $v \to \infty$, *i.e.* $r =$ Schwarzschild coordinate $\to \infty$) is the Hawking value. The horizontal dashed line is 't Hooft's naive "following back a Hawking photon" which disagrees with the exact result dramatically as we approach the shell.

If one only discusses Hawking radiation infinitely far away, there is no need to consider the trace anomaly. But if one wants to discuss the energy-momentum tensor close to the black hole, one cannot escape the fact of the trace anomaly. And if one wants to understand (derive) how the Hawking flux increases from zero (at the shell) to the Hawking value (at ∞), the trace anomaly is the key concept.

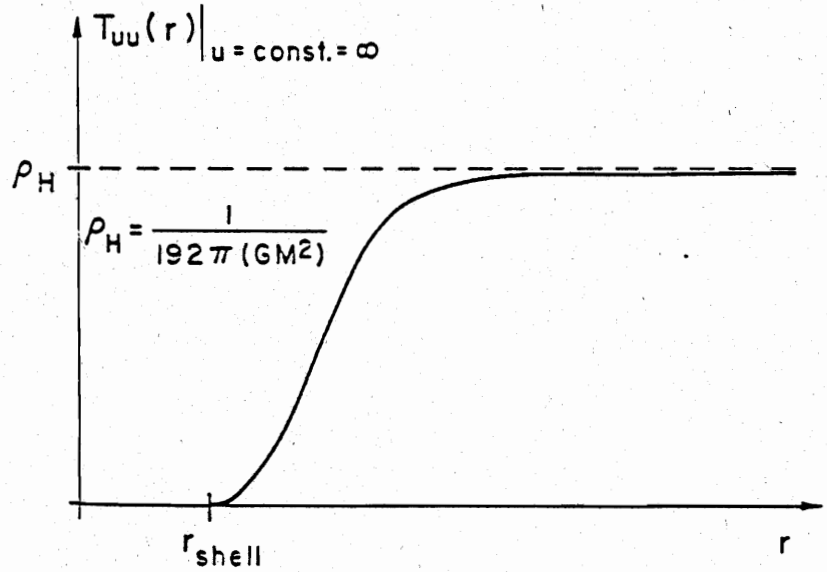

Figure 3

In the *external gravitational field approximation* (*i.e.* in the only treatment known) the collapsing shell will have *radiated away the equivalent of its total mass* by a certain time (the evaporation time), more precisely *before a certain null geodesic* $u = const. < \infty$ is reached. This happens *before the shell reaches the Schwarzschild radius and before a horizon is formed.* Near this null geodesic and beyond it, *back reaction* effects must become crucial and *change the picture totally.* Discussing what happens near the horizon is impossible without including back reaction.

References

[1] N. Birrell and P. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982); this book explains the standard concepts not presented in this seminar.

[2] K. Isler, C. Schmid, and C. Trugenberger, *Nucl. Phys.* **B301** (1988), 327

[3] K. Isler, C. Schmid, and C. Trugenberger, *Nucl. Phys.* **B339** (1990), 577

[4] R. Wald, *Ann. Phys.* **110** (1978), 472

[5] C. Massacand and C. Schmid, "Particle production, induced energy-momentum tensor, and the trace anomaly for bosons in (1+1)-dimensional curved space-time", preprint ETH-TH/92-3

[6] L. Parker, *Phys. Rev.* **183** (1969), 1057

[7] For a recent discussion see J. Frieman, *Phys. Rev.* **D39** (1989), 389

[8] Y. Zel'dovich, "The Creation of Particles and Antiparticles in Electric and Gravitational Fields", in *Magic without Magic: J.A. Wheeler*, edited by J. Klauder (Freeman and Company, 1972)

[9] P. Davies, S. Fulling, and W. Unruh, *Phys. Rev.* **D13** (1976), 2720

[10] G. 't Hooft, "On the Quantization of Space and Time", in *Quantum Gravity*, edited by M. Markov, V. Berezin, and V. Frolov (World Scientific, 1988)