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CLASSICAL ORBITS IN POWER LAW POTENTIALS

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ABSTRACT

The motion of bodies in power-law potentials of the form $V(r) = \lambda r^\alpha$ has been of interest ever since the time of Newton and Hooke. Aspects of the relation between powers α and $\bar{\alpha}$, where $(\alpha+2)(\bar{\alpha}+2) = 4$, are derived for classical motion and the relation to the quantum-mechanical problem is given. An improvement on a previous expression for the WKB quantization condition for nonzero orbital angular momenta is obtained. Relations with previous treatments, such as those of Newton, Bertrand, Bohlin, Fauré, and Arnol'd, are noted, and a brief survey of the literature on the problem over more than three centuries is given.

I. INTRODUCTION

A student of introductory quantum mechanics often learns that the Schrödinger equation is exactly soluble (for all angular momenta) for two central potentials: the Coulomb problem, with $V(r) \sim r^{-1}$, and the harmonic oscillator, with $V(r) \sim r^2$. Less frequently, she or he is made aware of the relation between these two problems, which are linked by a simple change of independent variable $r_{\rm Coul} = r^2_{\rm osc}$. Under this substitution, energies and coupling constants trade places, and orbital angular momenta are rescaled. Thus, there is really only one quantum mechanical problem, not two, which can be exactly solved for all orbital angular momenta. Still more rarely, it is noticed that the classical or quantum-mechanical problem with a power-law potential proportional to r^α is related to another one with a potential proportional to $r^{\bar{\alpha}}$, where $(\alpha+2)(\bar{\alpha}+2) = 4$.

The relation between the Coulomb (or Kepler) problem and the harmonic oscillator has roots in classical physics going back to the time of Newton [1] and Hooke [2]. Other observations of historical interest include ones by Bertrand [3], who studied conditions leading to closed orbits; Bohlin [4], who first used a conformal mapping technique to relate the Kepler and oscillator problems; Jauch and Hill [5], who discussed the connection between these two problems in quantum mechanics; and Fauré [6], who generalized this connection to arbitrary pairs of power-law potentials.

Recent discussions of Bohlin's results and their generalization to pairs of power-law potentials with other powers are given for classical motion by Arnol'd and Vasil'ev [7] and by Arnol'd [8]. The corresponding quantum-mechanical relations have been noted, for example, in Refs. [9] and [10] for the connection between the oscillator and the Coulomb problem, and in Refs. [11] and [12] for more general pairs of powers.

Within the past year, these relations have been treated in the present journal [13,14]. Our purpose in writing this article is three-fold.

First, we derive the classical relation between motions in pairs of power-law potentials using a change of variables in the energy integral. This treatment, which is in the spirit of Ref. [10], makes clear the role of orbital angular momenta in the classical problem, and the corresponding existence of a *family* of solutions of the equations of motion.

Second, we show how quantum mechanics leads to a unique choice of a single member of this family of solutions for any given set of quantum numbers, and apply Newton's original discussion of perturbations about circular orbits for arbitrary power-law potentials to derive an improvement of the semiclassical quantization condition for arbitrary orbital angular momenta.

Third, we add some remarks of a historical nature. In this respect we have greatly benefitted from conversations with Prof. S. Chandrasekhar regarding his forthcoming study of the *Principia* [15].

We discuss classical results in Section II, passing to the quantum-mechanical problem in section III. Relations with historical approaches are sketched in Section IV, while Section V concludes.

II. CLASSICAL RESULTS

The starting point for our discussion is a familiar expression giving the angular coordinate θ in terms of a radial integral. We begin from the Lagrangian for a particle of mass $m = 1$ moving under the influence of a central potential $V(r)$:

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{1}$$

The Euler-Lagrange equation for θ yields the conservation of angular momentum, which reads $r^2\dot{\theta} = \ell$. Using this in the radial equation of motion yields

$$\ddot{r} = \frac{\ell^2}{r^3} - \frac{dV}{dr} \tag{2}$$

Multiplying both sides by $\dot{r}$ and integrating in the usual way gives an expression for the energy:

$$\frac{1}{2}\dot{r}^2 + \frac{\ell^2}{2r^2} + V(r) = E. \tag{3}$$

From the conservation of angular momentum and Eq. (3) we have

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell/r^2}{\sqrt{2[E - V(r)] - \ell^2/r^2}}, \tag{4}$$

so that, up to an additive constant,

$$\theta(r) = \int^r \frac{\ell/r^2}{\sqrt{2[E - V(r)] - \ell^2/r^2}}\, dr. \tag{5}$$

Inverting this relation to find r in terms of θ gives the form of the orbit in a spherical oscillator potential: we find

$$\frac{1}{r^2} = \frac{E}{\ell^2} + \sqrt{\frac{E^2}{\ell^4} - \frac{2\lambda}{\ell^2}} \sin 2\theta, \tag{14}$$

which we recognize as an ellipse with its center at the origin, as expected.

The map from $V(r) \sim r^\alpha$ to $V(r) \sim r^{\bar{\alpha}}$ can also be done in the quantum mechanical case, with similar results [11]. In the classical case we found that the mapping interchanges the roles of E and λ via the transformation $E/\ell^2 \to -\lambda/\ell^2$, $\lambda/\ell^2 \to -E/\ell^2$. In the quantum case, we find that the variables E, λ and ℓ referring to the potential $V(r) \sim r^\alpha$ are related to $\bar{E}$, $\bar{\lambda}$, and $\bar{\ell}$ referring to $V(r) \sim r^{\bar{\alpha}}$ by the equations

$$\bar{E} = -\lambda(\bar{\alpha}/\alpha)^2, \quad \bar{\lambda} = -E(\bar{\alpha}/\alpha)^2, \text{ and } \quad (\bar{\ell} + 1/2) = -(\bar{\alpha}/\alpha)(\ell + 1/2). \tag{15}$$

Here we find that E and λ undergo a transformation similar to that found in the classical case: we have

$$\frac{E}{(\ell + 1/2)^2} = -\frac{\bar{\lambda}}{(\bar{\ell} + 1/2)^2}, \quad \frac{\lambda}{(\ell + 1/2)^2} = -\frac{\bar{E}}{(\bar{\ell} + 1/2)^2}. \tag{16}$$

In the limit of large ℓ, these reduce to the relations found in the classical case. The quantum case is distinguished, however, by the fact that $\bar{\ell}$ is given uniquely in terms of ℓ. In the classical case, it is always possible to rescale the angular momentum when carrying out the transformation, so long as compensating changes are made in E and λ; in the quantum case this is no longer possible.

The existence of pairs of "dual" power-law potentials has been recognized for quite some time. In fact, Newton himself knew that the oscillator problem could be mapped into the Kepler problem by means of a simple geometrical construction. In addition, he demonstrated that the "self-dual" potential $V(r) = -1/r^4$ has circular orbits which pass through the center of force. This potential is one of two (the other being the logarithmic potential, corresponding to $\alpha = 0$) which goes into itself under the mapping given above. Finally, he cites a relation equivalent to Eq. (8) in discussing the rate of rotation of precession of orbits which deviate slightly from circular ones in arbitrary power-law potentials.

III. RELATION TO QUANTUM MECHANICS

Using the method of Bohr-Sommerfeld quantization, it is possible to derive certain results regarding the energy levels of a particle bound by a power-law potential. A similar treatment is given in Ref. [10] for the oscillator and Kepler problems.

The quantization procedure begins from the so-called action variables, which are defined by

$$J_i = \oint p_i \, dq_i, \tag{17}$$

where q_i is a co-ordinate of the system, p_i is its conjugate momentum, and the integration runs over a full cycle of the motion. To obtain the energy levels of the system,

We can use Eq. (5) to derive a one-to-one correspondence between different pairs of power-law potentials. Setting $V(r) = \lambda r^\alpha$ in Eq. (5), we obtain

$$\theta(r) = \int^r \frac{1}{r^2\sqrt{2(E - \lambda r^\alpha)/\ell^2 - 1/r^2}}\, dr. \tag{6}$$

We can transform this into the corresponding integral for a different power-law potential by making the change of variable $u = r^{-\alpha/\bar{\alpha}}$, with $\bar{\alpha}$ to be determined later. Eq. (6) then becomes

$$\theta(r) = -\frac{\bar{\alpha}}{\alpha}\int^{u=r^{-\alpha/\bar{\alpha}}} \frac{1}{u^2\sqrt{2\{Eu^{-[2(\bar{\alpha}/\alpha)+2]} - \lambda u^{-[2(\bar{\alpha}/\alpha)+2+\bar{\alpha}]}\}/\ell^2 - 1/u^2}}\, du. \tag{7}$$

If we now choose $\bar{\alpha}$ to satisfy

$$(\alpha + 2)(\bar{\alpha} + 2) = 4, \tag{8}$$

we find that Eq. (7) reduces to

$$\theta(r) = -\frac{\bar{\alpha}}{\alpha}\int^{u=r^{-\alpha/\bar{\alpha}}} \frac{1}{u^2\sqrt{2(-\lambda + Eu^{\bar{\alpha}})/\ell^2 - 1/u^2}}\, du. \tag{9}$$

Apart from a few simple substitutions, this is identical to the original integral given in Eq. (6). Consequently we have obtained the desired correspondence between pairs of power-law potentials. For, consider the orbit of a particle with energy E and angular momentum ℓ in a potential $V(r) = \lambda r^{\bar{\alpha}}$. This orbit can be specified by the parameters E/ℓ^2 and λ/ℓ^2, and the dependence of θ on r can be written as

$$\theta = \theta_\alpha(E/\ell^2, \lambda/\ell^2; r). \tag{10}$$

From this, we can construct the equation of a corresponding orbit in the potential $V(r) \sim r^{\bar{\alpha}}$ by making use of Eq. (9): we find that $\theta_\alpha(r)$ and $\theta_{\bar{\alpha}}(r)$ are related by

$$\theta = \theta_\alpha(E/\ell^2, \lambda/\ell^2; r) = -\frac{\bar{\alpha}}{\alpha}\theta_{\bar{\alpha}}(-\lambda/\ell^2, -E/\ell^2; r^{-\alpha/\bar{\alpha}}) \tag{11}$$

As an example of this mapping, we take the case $\bar{\alpha} = -1$, corresponding to the Coulomb potential. From the relation (8), we find that $\alpha = 2$, corresponding to the spherical oscillator potential. In this case, direct evaluation of the integral in Eq. (6) leads to an equation for the orbits in a Coulomb potential:

$$\theta = \theta_{\bar{\alpha}=-1}(E/\ell^2, \lambda/\ell^2; r) = \arcsin\left[\frac{1/r + \lambda/\ell^2}{\sqrt{(\lambda/\ell^2)^2 + 2E/\ell^2}}\right] \tag{12}$$

To obtain the corresponding relation for the oscillator potential, we make use of Eq. (11) to write

$$\theta_{\alpha=2}(E/\ell^2, \lambda/\ell^2; r) = \frac{1}{2}\theta_{\bar{\alpha}=-1}(-\lambda/\ell^2, -E/\ell^2; r^2) = \frac{1}{2}\arcsin\left[\frac{1/r^2 - E/\ell^2}{\sqrt{(E/\ell^2)^2 - 2\lambda/\ell^2}}\right] \tag{13}$$

It is interesting to note that Newton derived this very result in his study of nearly circular orbits [16]. Using (25), Eq. (22) becomes

$$\frac{\partial E}{\partial J_r} = \frac{1}{\sqrt{\alpha+2}}\frac{\partial E}{\partial J_\theta}, \tag{26}$$

from which it follows that E depends on J_r and J_θ only in the combination $J_r + J_\theta/\sqrt{\alpha+2}$. If we now quantize J_r and J_θ by setting $J_r = n\hbar$, $J_\theta = \ell\hbar$, we find that the approximate semi-classical energies of the system have the form

$$E_{n\ell} = F(\alpha, n + \ell/\sqrt{\alpha+2}), \tag{27}$$

where the dependence on α has been made explicit.

The above form reduces to the correct values for the Coulomb ($\alpha = -1$) and harmonic oscillator ($\alpha = 2$) potentials. It represents the leading $\ell-$dependent correction to the dependence on n; additional terms of subleading order have been found in Ref. [12]. It is a considerable improvement with respect to the functional dependence quoted in Ref. [11], where it was argued that the combination $n + \ell/2 + 1/4$ appeared for all $\alpha > 0$.

IV. RELATIONS TO PREVIOUS WORK

The existence of certain pairs of "dual" power-law potentials has been recognized for quite some time; in fact, the earliest observations on this subject date back to Newton, who demonstrated that there is a certain correspondence between the harmonic oscillator problem and the Kepler problem.

Newton's treatment [17] consisted of three steps. First, he considered the following question: Suppose that a particle with a given angular momentum travels in the same orbit whether acted upon by a force F_1 acting towards point C_1, or a force F_2 acting towards point C_2: what is the ratio of the forces F_1 and F_2? Having obtained the desired result, Newton went on to prove that the orbits in a field of force obeying Hooke's law are ellipses with their centers at the center of force. Using these results, Newton demonstrated that the gravitational force, varying as $1/r^2$, has elliptical orbits whose foci lie at the center of force.

A more modern treatment [15] similar in substance to Newton's approach begins from a purely geometrical expression for the centripetal force acting on a particle. We consider a particle moving with velocity v in an orbit with local radius of curvature ρ. The acceleration of the particle towards the center of curvature is given by $a_\rho = v^2/\rho$. Consequently, the acceleration of the particle towards the center of force C is given by

$$a = v^2/\rho \sin\epsilon, \tag{28}$$

where ϵ is defined in Fig. 1. Next we note that the angular momentum about C may be written as $\ell = mvr\sin\epsilon$, so that the velocity obeys

$$v \propto \frac{1}{r\sin\epsilon}. \tag{29}$$

we solve for the total energy of the system in terms of the action variables (17), and impose the quantum conditions $J_i = n_i\hbar$, where the n_i are integers. In the special case of a particle bound by a central potential, the action variables corresponding to r and θ are given by

$$J_\theta = 2\pi\ell, \tag{18}$$

where ℓ is the angular momentum, and

$$J_r = 2\int_{r_1}^{r_2} \sqrt{2[E - V(r)] - J_\theta^2/4\pi^2 r^2}\, dr, \tag{19}$$

where r_1, r_2 are the radial turning points. In general, it is not possible to analytically evaluate the integral appearing in (19). However it is possible to make certain quantitative statements about the dependence of E on J_r and J_θ. Treating J_r and J_θ as the independent variables, and differentiating Eq. (19) implicitly with respect to these variables gives

$$\frac{\partial E}{\partial J_r} = \left[\oint \frac{dr}{\sqrt{2[E - V(r)] - \ell^2/r^2}}\right]^{-1}, \tag{20}$$

and

$$\frac{\partial E}{\partial J_\theta} = \frac{1}{2\pi}\left[\oint \frac{\ell/r^2}{\sqrt{2[E - V(r)] - \ell^2/r^2}}\, dr\right]\frac{\partial E}{\partial J_r}. \tag{21}$$

In deriving Eq. (21), we have used Eqs. (20) and (18). Eq. (21) can be simplified by observing that the integral multiplying $\partial E/\partial J_r$ is the same integral which appears in Eq. (5). Consequently, we may re-write (21) in the form

$$\frac{\partial E}{\partial J_\theta} = \frac{\Delta\theta}{2\pi}\frac{\partial E}{\partial J_r}, \tag{22}$$

where $\Delta\theta$ is the angle through which the particle moves during one full cycle of the radial motion. We can give an approximate expression for $\Delta\theta$ in the case of nearly circular orbits. Setting $V(r) = \lambda r^\alpha$ in Eq. (2), we find that the radius of a circular orbit with angular momentum ℓ is $r_0 = (\ell^2/\alpha\lambda)^{1/(\alpha+2)}$. Expanding the right hand side of Eq. (2) to lowest order about this equilibrium value, we find that the displacement $x \equiv r - r_0$ satisfies

$$\ddot{x} = -\frac{(\alpha+2)\ell^2}{r_0^4}x. \tag{23}$$

Consequently, for nearly circular orbits we may write $r(t) \simeq r_0 + x_0\cos(\omega_r t + \delta)$, where $x_0 \ll r_0$, and the frequency ω_r of the oscillation is given by

$$\omega_r = \sqrt{\alpha+2}\frac{\ell}{r_0^2} \tag{24}$$

We can use this approximate form of $r(t)$ to obtain an expression for $\Delta\theta$. From the conservation of angular momentum, we have

$$\Delta\theta = \int_0^{2\pi/\omega_r} \frac{\ell}{r(t)^2}\, dt \simeq \int_0^{2\pi/\omega_r} \frac{\ell}{r_0^2}\left[1 - \frac{2x_0}{r_0}\cos(\omega_r t + \delta)\right] dt = \frac{2\pi}{\sqrt{\alpha+2}}. \tag{25}$$

Consequently, the repulsive oscillator potential is dual to itself. Finally, Newton observes that an attractive potential behaving as $1/r^4$ possesses circular orbits that pass through the center of force. From the relation (8), we see that this potential is also self dual.

A modern treatment, based on complex variable methods, has been described by Arnol'd [8] and by Mittag and Stephen [13]. This method is related to a theorem proved by Bohlin [4], and enables one to directly transform the equation of motion for one power into that for another.

The following results can be obtained elegantly using energy integrals (as in Sec. II) [13]. For a somewhat different approach, we may begin from the equation of motion for a particle moving under the influence of a central force $F \sim r^a$. Combining the co-ordinates x and y into the single complex variable $w = x + iy$, we find that the equation of motion may be written as

$$\ddot{w}(t) = -w(t) \mid w(t) \mid^{a-1} . \tag{35}$$

To transform this into the equation of motion for a different power-law, we use the mapping $z(\tau) = w(t)^\beta$, with β to be determined later, and τ a rescaled time variable chosen so that conservation of angular momentum is preserved under the mapping. Since the angular momentum is given by

$$\ell = \mid w(t) \mid^2 \frac{d}{dt} \arg w(t), \tag{36}$$

and since $\arg z(\tau) \propto \arg w(t)$, we can preserve conservation of angular momentum by choosing

$$\frac{d\tau}{dt} = \frac{\mid z(t) \mid^2}{\mid w(\tau) \mid^2} = \mid w(t) \mid^{2(\beta-1)}, \tag{37}$$

for then we have

$$\ell = \mid w(t) \mid^2 \frac{d}{dt} \arg w(t) \propto \mid z(t) \mid^2 \frac{d}{d\tau} \arg z(\tau). \tag{38}$$

To find the new equation of motion, we evaluate $d^2z(\tau)/d\tau^2$. Using Eq. (35) and making repeated use of Eq. (37), we find

$$\frac{d^2z(\tau)}{d\tau^2} = -2\beta(\beta-1)w(t)^\beta \left[\frac{1}{2} \mid \dot{w}(t) \mid^2 + \frac{\mid w(t) \mid^{a+1}}{2(\beta-1)}\right] \mid w(t) \mid^{2-4\beta} . \tag{39}$$

If we now choose β to satisfy $2(\beta-1) = a+1$, we see that the quantity in square brackets reduces to the energy E, given by $E = |\dot{w}|^2/2 + |w|^{a+1}/(a+1)$. Eq. (39) then takes the form

$$\frac{d^2z(\tau)}{d\tau^2} = -2\beta(\beta-1)Ez(\tau) \mid z(t) \mid^{\bar{a}-1} . \tag{40}$$

This is precisely the equation of motion for a particle moving under the influence of a force $F \sim r^{\bar{a}}$, where the new power $\bar{a}$ is related to the old power a by $(\bar{a}+3)(a+3) = 4$,

Combining Eqs. (28), (29), we have

$$F \propto a = \frac{v^2}{\rho \sin \epsilon} \propto \frac{1}{r^2 \rho \sin^3 \epsilon}. \tag{30}$$

Now consider a particle moving under the influence of a force F directed towards point C. Suppose that the same particle, with the same orbital angular momentum, moves in the *same* orbit when acted upon by a force F' directed towards point C'. It follows from Eq. (30) that the ratio of the forces F and F' is given by

$$\frac{F}{F'} = \frac{r'^2 \sin^3 \epsilon'}{r^2 \sin^3 \epsilon}, \tag{31}$$

where r' and ϵ' are defined in Fig. 1. Using (31), and the properties of orbits in a spherical oscillator potential, we can show that the force law whose orbits are ellipses with their foci at the center of force is precisely the inverse square law. We know that the orbits in a spherical oscillator potential are ellipses whose centers lie at the center of force C, as illustrated in Fig. 1. Now we ask what force F' must tend toward the focus $F_1 \equiv C'$ in order that the particle move in the same ellipse when acted on by F'. Using $F \propto r$ and Eq. (31), we find

$$F' \propto \frac{1}{r'^2} \left(\frac{r \sin \epsilon}{\sin \epsilon'} \right)^3. \tag{32}$$

Next, we observe [18] that the distance r to the center of the ellipse is related to the semi-major axis A of the ellipse by

$$A \sin \epsilon' = r \sin \epsilon. \tag{33}$$

Using this, we find that F' is given by

$$F' \propto \frac{A^3}{r'^2} \propto \frac{1}{r'^2}. \tag{34}$$

So we have shown that a force obeying the inverse square law supports elliptical orbits with one focus at the center of force. Newton employed a similar argument to derive the hyperbolic Kepler orbits from the corresponding orbits in a repulsive oscillator potential. In this way, Newton exploited the duality of the inverse square and Hooke force laws to derive the form of the Kepler orbits.

Newton also mentions several other classes of potentials which are "dual" to one another in a somewhat extended sense [15,19]. For instance, the attractive $1/r$ potential is dual to the repulsive $1/r$ potential in the sense that a particle moving in the attractive potential can describe a hyperbolic orbit with its focus at the center of force, while a similar particle in the repulsive potential will move in the conjugate hyperbola. In both cases, the orbits possess a similar shape, and we see that the two potentials are in some sense dual to one another. In much the same way, a particle moving in a repulsive oscillator potential can move on either branch of a hyperbola.

In the classical case, all motion is a function of the combinations E/ℓ^2 and λ/ℓ^2, where E, λ, and ℓ are the energy, potential strength, and orbital angular momentum, respectively. The classical motions in power-law potentials r^α and $r^{\bar{\alpha}}$, where α and $\bar{\alpha}$ are related as above, are described by variables in which energies and coupling strengths trade places: $E/\ell^2 = -\bar{\lambda}/\bar{\ell}^2$ and $\lambda/\ell^2 = -\bar{E}/\bar{\ell}^2$. Upon quantization, one recovers results obtained previously [11] for the Schrödinger equation.

We have shown that a consideration of classical motion in power-law potentials specifies how the radial quantum number n and orbital angular momentum ℓ enter into the WKB quantization condition. They appear in the form $n + \ell/\sqrt{\alpha+2}$, which reduces to the exact forms for the Coulomb and harmonic oscillator potentials.

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which is precisely what we found in Sec. II [recall that if $V \sim r^\alpha$, then $F \sim r^{\alpha-1}$; replacing a with $\alpha - 1$, we regain Eq. (8)]. Hence we have found the desired mapping from the equation of motion in a central field with $F \sim r^a$ to the equation of motion with $F \sim r^{\bar{a}}$. The special case of this result corresponding to $a = 1$, $\bar{a} = -2$ was first demonstrated by Bohlin [4] in 1911.

The work of Bertrand [3] on classical orbits in power law potentials is also of historical interest. Bertrand's theorem, first published in 1875, states that the only potentials which give rise to closed orbits (for arbitrary initial conditions) are the Coulomb and oscillator potentials. The proof of Bertrand's theorem is discussed in numerous textbooks [20,21], and so we merely sketch the details here. In the radial equation of motion, Eq. (2), we change the independent variable from t to θ, and the dependent variable from r to $u = 1/r$. We then find that, for a potential $V \sim r^\alpha$, u obeys

$$\frac{d^2u}{d\theta^2} = -u + \frac{\alpha\lambda}{\ell^2}u^{-\alpha-1}. \tag{41}$$

If we now consider nearly circular orbits, we can expand the right hand side of Eq. (41) about its root $u_0 = (\alpha\lambda/\ell^2)^{1/(\alpha+2)}$. Including linear terms gives rise to the equation of motion

$$\frac{d^2x}{d\theta^2} = -\omega^2 x + O(x^2), \tag{42}$$

where $x \equiv u - u_0$ is the displacement from equilibrium, and $\omega = \sqrt{\alpha+2}$. Hence we find that, in this approximation, any force law with ω rational will give closed, nearly circular orbits. For larger oscillations, we must include the quadratic and cubic terms in the expansion leading to Eq. (42). To find the resulting perturbed orbit, we expand $x(\theta)$ in a Fourier series in θ, including terms up to $\cos 3\omega\theta$. Substituting this Fourier series into the perturbed equation of motion, we find that the self-consistency of the Fourier series *Ansatz* requires

$$(2+\alpha)(1+\alpha)(2-\alpha) = 0. \tag{43}$$

Of the three roots, the root $\alpha = -2$ is excluded by the requirement $\omega > 0$. Consequently, the only potentials for which $x(\theta)$ can be expanded in a Fourier series are those with $\alpha = 2$ or $\alpha = -1$. But it is clear that $x(\theta)$ must be periodic in θ (and hence expandable in a Fourier series) in order for the orbits to be closed. Hence we find that the *only* potentials which give rise to closed orbits are the Coulomb and oscillator potentials.

V. CONCLUSIONS

The connection between classical motions in power-law potentials of the form r^α and $r^{\bar{\alpha}}$, where $(\alpha+2)(\bar{\alpha}+2) = 4$, is seen to be a very old one. For the Coulomb and harmonic oscillator potentials, it goes back to the days of Newton and Hooke. Newton appears to have known about at least other aspects of the general case, including behavior for nearly circular orbits in arbitrary powers.

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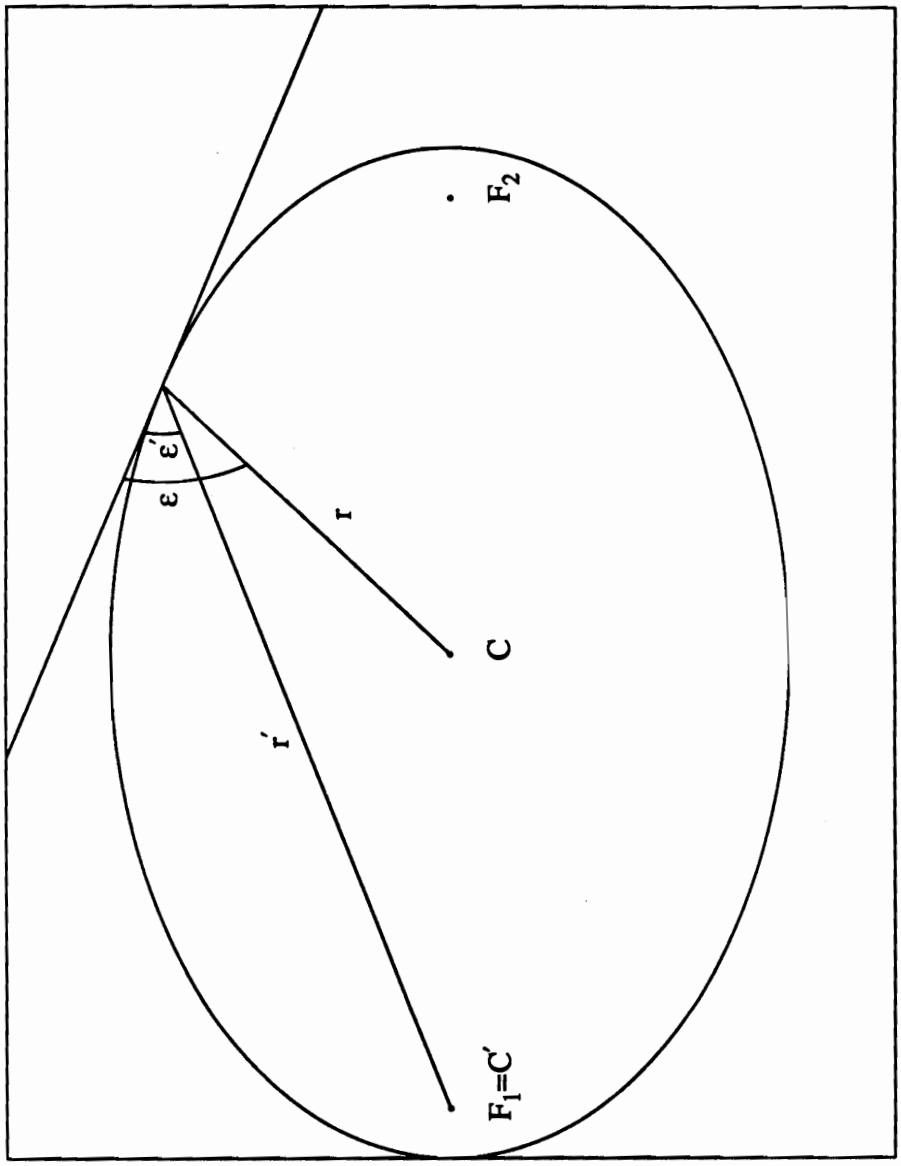
Figure 1: Orbital parameters for comparison of forces tending toward F_1, C.