Collimated Propagation of Magnetohydrodynamic Waves near a Schwarzschild Black Hole

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Abstract

We study non-stationary and axisymmetric perturbations of a magnetohydrodynamic accretion onto a nonrotating (Schwarzschild) black hole. The unperturbed stationary inflow is assumed to be magnetically dominated, and our analysis is limited to the fast-magnetosonic perturbations produced near the black hole. The key point is that the plasma accretion can be highly disturbed as a result of a slight perturbation of the poloidal magnetic field. In fact we find an interesting perturbed structure of the plasma velocity with a large peak in some narrow region just beyond the fast-magnetosonic surface. If the critical surface has an oblate shape for radial poloidal field lines, this perturbation of plasma motion is induced by outgoing waves propagating from the equatorial super-fast-magnetosonic region to the polar sub-fast-magnetosonic one. Thus, the effective acceleration of particles due to the outgoing fast-magnetosonic waves will work mainly in the polar direction as a mechanism of trigger of jet production in active galactic nuclei.

Key words: Accretion – Galaxies: active – Relativity

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1. Introduction

A few percent of the known galaxies in the universe contain hyperactive central region. Compared with the size of a galaxy the seat of this activity is extraordinarily compact, as indicated by rapid variability of non-thermal X-ray flux on time scales as short as minutes (e.g., Matilsky, Shrader, and Tannenbaum 1982; Kunieda et al. 1992). Emerging from the central compact source there is often a sharply focused jet, or two jets pointing in opposite directions. It is generally accepted that the jets are powered by the associated active galactic nuclei (AGNs), where accreting mass is converted into energy with several percent efficiency. We must therefore explain how the energy goes into a small fraction of the gas in the gravitational field of the prime mover, probably a supermassive black hole (e.g., Rees 1984).

Possible flows of this type were suggested by Shakura and Sunyaev (1973) in the context of a thick supercritical accretion disk which exhibited inflow along the equator and outflow near the poles. If the radiation-supported rotating gas adopts a hydrostatic toroidal configuration, then a pair of funnels will be defined which could be responsible for the production of jets along the rotational axis (Lynden-Bell 1978). Furthermore, the magnetohydrodynamic (MHD) disturbances produced at a galactic nucleus with a compact nuclear disk will be strongly collimated polewards resulting in jets, if the Alfvén velocity in the disk is much higher than its surroundings (Sofue 1980). Such a collimation will lead to an increase of wave amplitude resulting in shock waves which are conjectured to develop into a strongly compressed region of magnetic field. In this region, high energy particles are likely to be accelerated in the perpendicular direction to the equatorial nuclear disk.

The purpose here is to explore a little further into the problem whether MHD waves can convey energy from the equatorial region to the poles in the close vicinity of the central black hole of an AGN. Let us first describe a probable model of a rotating black hole magnetosphere and discuss the importance of considering the propagation of MHD waves in it. We expect that inflows start from an equatorial disk surface and that they stream along the magnetic field lines connecting the disk and the event horizon (Fig. 1). Magnetic field lines, which are denoted by thick solid curves in Figure 1, are considerably bent because of the balance among inertial forces, electromagnetic forces, and gravitational forces (Wilson 1975; Nitta et al. 1991). Sufficient matter is expected to be supplied into the magnetosphere along these bent field lines from the dense disk, as sketched here, so that we may adopt the MHD approximation. Causality requires that the MHD inflows should pass through the fast-magnetosonic point, which is designated as F in Figure 1, and become super-fast-magnetosonic at the horizon. The investigation of the so-called critical condition that the inflow should pass through this point smoothly is, in fact, the key to understand MHD interactions in a black hole magnetosphere.

The MHD interactions are expected to work most effectively in the magnetically dominated limit that the rest-mass energy density of particles is negligible compared with the magnetic energy density. In this limit, the fast-magnetosonic point is located very close to the horizon (e.g., Phinney 1983); as a result, general relativistic treatment is required. Analyzing the critical condition in a stationary and axisymmetric magnetically dominated black hole magnetosphere, Hirotani et al. (1992, hereafter paper I) showed that roughly 10% of the rest-mass energy and significant fraction of the initial angular momentum are transported from the fluid to the magnetic field during the infall. Furthermore, if a small-amplitude perturbation is introduced into the magnetosphere, a lot of perturbation energy is transported from the magnetic field to the fluid near the fast-surface (Figure 1) in the short-wavelength limit; accordingly, the plasma accretion becomes highly disturbed there (Hirotani, Tomimatsu, and Takahashi 1993, hereafter paper II). If several portions of such a fluid disturbance, which is originally supplied from the equatorial disk, are transported polewards in the form of MHD waves, then the magnetic energy may be changed into bulk kinetic energy or fast-particle energy of outflows observed as jets in AGNs. This paper is therefore intended as an investigation of the structure and propagation of MHD waves in a magnetically dominated black hole magnetosphere.

The outline of this paper is as follows. In section 2.1 we briefly review an analysis of the stationary MHD accretion near the fast-magnetosonic point in the magnetically dominated limit. We next consider
non-stationary axisymmetric perturbations of the MHD accretion in section 2.2. In the short-wavelength limit, it is known that the fluid becomes highly variable near the fast-magnetosonic point owing to the MHD interactions (paper II). In this paper, however, we do not adopt the short-wavelength limit; instead, we assume that characteristic scale of the radial variations of perturbed quantities is comparable with that of unperturbed quantities. Under this assumption, we can derive the wave equations of perturbed quantities on the poloidal plane. Solving the wave equations, we show in section 3 that the fluid becomes most variable at slight inside of the fast-surface; this result partly supports what was obtained in paper II in the short-wavelength limit. We further demonstrate in section 4 that the disturbance can escape outwards as fast-magnetosonic waves by propagating polewards. Some implications of the results in relation to jets or outgoing winds observed from AGNs are also briefly discussed in this final section.

2. Magnetically Dominated Accretion

We will begin by considering basic equations describing a magnetosphere around a non-rotating black hole. Since the self-gravity of the electromagnetic field and plasma around the black hole is very weak, the background geometry of the magnetosphere is described by the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,$$

where $M$ is the mass of the hole. Throughout this paper we use geometrized units such that $c = G = 1$.

Under ideal MHD conditions that the electric field vanishes in the fluid rest frame, we have $F_{\mu\nu}U^\mu = 0$, where $F_{\mu\nu}$ is the electromagnetic field tensor satisfying Maxwell equations, $F_{[\mu\nu,\lambda]} = 0$ and $U^\mu$ is the fluid four velocity. The motion of the fluid in the cold limit is governed by the following equations of motion:

$$T^{\mu\nu} = \left[m_p\pi^{\mu\nu}U^\nu + \frac{1}{4\pi}F^{\mu\rho}F_\rho^\nu + \frac{1}{4}\delta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]_{\pi} = 0,$$

where the semicolon denotes a covariant derivative and $m_p$ the rest-mass of a particle. The proper number density $(n)$ obeys the continuity equation, $(nU^\mu)_{\mu} = 0$. We adopt these basic equations for a description of a stationary and axisymmetric black hole magnetosphere in section 2.1 and for an analysis of perturbed states afterwards.

2.1 Unperturbed Magnetosphere

Before turning to a closer examination of a perturbed magnetosphere, a few remarks should be made concerning the unperturbed stationary and axisymmetric state. We first define the unperturbed electromagnetic field at a point $(r, \theta)$ as those measured by a distant observer at rest. They can be written as

$$E_r = F_{rt}^{(0)}; \quad E_\theta = F_{t\theta}^{(0)},$$
$$B_r = \frac{F_{\phi}^{(0)}}{\sqrt{-g}}; \quad B_\theta = \left(1 - \frac{2M}{r}\right)\frac{F_{\phi r}^{(0)}}{\sin\theta}, \quad B_\phi = \sqrt{-g}F^{r\phi}^{(0)},$$

where $\sqrt{-g} \equiv r^2\sin\theta$; the subscript and the superscript (0) indicate unperturbed quantities.

In order to look closely at the problem how the existence of the horizon affects the MHD interactions, we would like to focus attention on the analysis near the horizon $(1 - 2M/r \ll 1)$. Since the poloidal magnetic field near the horizon tends to be bent into radial shape owing to fluid inertia even in the
magnetically dominated limit (paper I), it may safely be assumed that the unperturbed field is radial ($F_{\phi r}^{(0)} = 0$). Because particles are frozen to the radial magnetic field lines, flow lines also become radial ($f^{(0)} = 0$). Moreover, from the Maxwell equations and the frozen-in conditions, we can see that the unperturbed electric field becomes meridional:

$$E_r = 0, \quad E_\theta = -\sqrt{-g} \Omega_F B_r = -\Omega_F \Psi_\theta,$$

where $\Psi_\theta (\equiv F_{\theta r}^{(0)})$ is the derivative of the $\phi$ component $\Psi(\theta)$ of the unperturbed electromagnetic vector potential with respect to $\theta$, and the angular frequency $\Omega_F$ of a magnetic field line is conserved along a poloidal flow line $\Psi = \text{const}$. (Bekenstein and Oron 1978; Camenzind 1986a,b). In addition to $\Omega_F$, particle number flux per unit magnetic flux tube ($\eta$) is also conserved. Using $\eta$, we can express $B_r$ in terms of $U^r$ as

$$B_r = -\frac{\eta}{\eta} U^r. \quad (5)$$

Furthermore, in a stationary and axisymmetric magnetosphere, total energy ($E$) and angular momentum ($L$) of MHD flows are also conserved along a poloidal flow line. They are composed of fluid and electromagnetic parts as follows:

$$E = m_p U_r^{(0)} - \frac{\sqrt{-g} \Omega_F}{4\pi \eta} F^{\phi\theta (0)}, \quad (6)$$

and

$$L = -m_p U_\theta^{(0)} - \frac{\sqrt{-g}}{4\pi \eta} F^{r\phi (0)}. \quad (7)$$

Note that in the magnetically dominated limit we have

$$\frac{m_p}{E} \ll 1, \quad \text{and} \quad \frac{m_p}{\Omega_F \eta} \ll 1, \quad (8)$$

so that the electromagnetic parts in equations (6) and (7) dominate the fluid parts. For a Schwarzschild black hole, we always have $E > 0$. These four conserved quantities ($\Omega_F$, $\eta$, $E$, and $L$) are functions of $\Psi$ only.

Before we leave the discussion over the unperturbed magnetosphere, we should notice that the fast-magnetosonic point ($r = r_F$) is located very close to the horizon ($r = r_H \equiv 2M$) in the magnetically dominated limit. This can be clearly seen if we write down its position as (paper I)

$$\frac{r_F - r_H}{r_H} \approx (-K_H) \sqrt{K_I - K_H} \frac{m_p}{E}, \quad (9)$$

where the nondimensional factor $\sqrt{K_I - K_H}$ is of order of unity; $K_I$ and $K_H$ are defined by $K_I \equiv (E - \Omega_F L)^2/m_p^2$ and $K_H \equiv -4M^2 \Omega_F^2 \sin^2 \theta$, respectively.

### 2.2 Perturbed Magnetosphere

We next consider a non-stationary and axisymmetric perturbation superposed on the unperturbed state discussed in the last subsection. Let the actual poloidal component of the electric and magnetic fields, as a result of the disturbance, be $E_A + e_A$ and $B_A + b_A$ ($A = r, \theta$), respectively; the small letters ($e_r, e_\theta, b_r, b_\theta$) are the Eulerian perturbations of the corresponding quantities. Let $b_T$ denote a perturbation of $F_{\phi r}$. Since $F_{\phi r}$ vanishes for the unperturbed state, we use $F_{\phi r}$ as a perturbed toroidal electric field. Furthermore, we introduce $\nu^r$, $u^r$ such that the actual component of the poloidal fluid velocity field, as a result of the disturbance, becomes $U^A + u^A$. Let $u_i$ and $-u_\phi$ be the Eulerian
perturbations of fluid energy and angular momentum per unit mass, respectively. In what follows, we omit the subscript and the superscript (0), because no confusion arises.

Making use of (4), we can simplify $t$, $\phi$, and $\theta$ components of the frozen-in conditions as follows:

$$-U^t e_r - U^\phi F_{\phi t} + \Omega_F \Psi_\theta u^\theta = 0,$$

$$-(1 - \frac{2M}{r})^{-1} U^t F_{\phi t} + \sin \theta \left(1 - \frac{2M}{r}\right)^{-1} U^r b_\theta - \Psi_\theta u^\theta = 0,$$

$$
(1 - \frac{2M}{r})^{-1} U^r (\epsilon_\theta - b_T) - \sqrt{-g} U^\phi b_r
+ \left(1 - \frac{2M}{r}\right)^{-1} (\Omega_F \Psi_\theta u_t + r^2 F^{\phi \theta} u^r) + \frac{\Psi_\theta}{r^2 \sin^2 \theta} u_\phi = 0.
$$

Our purpose here is to reduce these coupled perturbation equations to a single partial differential equation for one unknown function. If we assume that the characteristic scales of meridional variations in the perturbed state are much shorter than those in the unperturbed state, the perturbation equations reduce significantly. For axisymmetric perturbations, all perturbed quantities are therefore assumed to be proportional to $\exp(i\omega t - ik_\theta \theta)$, where $k_\theta \gg 1$. Under this approximation, the Maxwell equations reduce to

$$\sin \theta b_\theta = -\frac{i}{\omega} \frac{dF_{\phi t}}{dr_*},$$

$$b_r = \frac{1}{\omega \sqrt{-g}} k_\theta F_{\phi t},$$

$$e_r = \frac{i}{k_\theta} \left(1 - \frac{2M}{r}\right)^{-1} \left(-i\omega b_T + \frac{d\epsilon_\theta}{dr_*}\right),$$

where $r_*$ is the tortoise coordinate defined by

$$\frac{dr_*}{dr} \equiv \left(1 - \frac{2M}{r}\right)^{-1}. $$

It is convenient to introduce this coordinate when we describe waves near the horizon, because the interval $(r_H, \infty)$ in the $r$-coordinate is stretched to $(-\infty, \infty)$ in $r_*$. We assume that the characteristic scale of the radial variations is comparable with that of the gravitational field, that is, $|df/dr_*| \approx |f/M|$, where $f$ denotes some perturbed quantity. In this paper we do not take the short-wavelength limit ($|df/dr_*| \gg |f/M|$) which was adopted in paper II.

Let us finally consider the equations of motion. Eliminating perturbed fluid density by using the continuity equation and taking the leading orders in the expansions of $(1 - 2M/r) \ll 1$, we obtain the $r$, $\theta$ and $\phi$ components of the equations of motion after lengthy manipulations. They are written as

$$4\pi m_p n \left(1 - \frac{2M}{r}\right)^{-1} U^r \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \left(-i\omega + \frac{d\epsilon_\theta}{dr_*}\right) u^r - ik_\theta \partial U^\theta \frac{\partial \Psi_\theta}{\partial r_*} = 0$$

$$- \frac{\Omega_F \Psi_\theta \sin \theta}{\sqrt{-g}} \left(1 - \frac{2M}{r}\right)^{-1} \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \left(-i\omega \epsilon_\theta + \frac{d\epsilon_\theta}{dr_*}\right) = 0.$$
\begin{align*}
&[K_H \left(1 - \frac{2M}{r}\right)^{-2} \left(-\frac{1}{M} - i\omega + \frac{d}{dr_r}\right) \left(-i\omega + \frac{d}{dr_r}\right) - \left(\frac{k_\theta}{2M}\right)^2 \left(-i\omega + \frac{d}{dr_r}\right) F_{\phi t} \\
&- \omega k_\phi \Omega_F \sin^2 \theta \left(1 - \frac{2M}{r}\right)^{-1} \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_r}\right) (e_\theta - b_T) = 0 \quad (18)
\end{align*}

\begin{align*}
&4\pi m_p n \left[\left(1 - \frac{2M}{r}\right)^{-1} U^r \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_r}\right) \left(-i\omega + \frac{d}{dr_r}\right) u_\phi + ik_\phi \frac{\partial U_\phi}{\partial r} u^t\right] \\
&- \frac{\Psi_\phi \sin \theta}{\sqrt{-g}} \left(1 - \frac{2M}{r}\right)^{-1} \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_r}\right) \left(-i\omega \epsilon_\theta + \frac{d\epsilon_T}{dr_r}\right) = 0, \quad (19)
\end{align*}

where $U_t$ and $-U_\phi$ express unperturbed fluid energy and angular momentum respectively, $n$ denotes the unperturbed fluid density. Furthermore, the definition of the proper time gives

\begin{equation}
u_t + u^r = \frac{1}{r^2 \sin^2 \theta} \left(1 - \frac{2M}{r}\right) \frac{1}{U^r} \left[\frac{(U_\phi)^2 + r^2 \sin^2 \theta}{2U^r} u^r - U_\phi u^\phi\right], \quad (20)
\end{equation}

Instead of solving the $t$ component of the equations of motion, we use equation (20).

We analyze the system which is formed by 10 equations (3 frozen-in conditions, 3 Maxwell equations, and 4 equations of motion) in 10 unknown functions $(\nu_t, b_\theta, e_\phi, b_T, F_{\phi t}, n_1, u^r, u^\phi, \nu_r, \nu^\theta)$. Perturbed fluid density would be calculated if we solved the perturbed continuity equation. These 10 equations give some relations between the amplitude of arbitrary two perturbed quantities and are further combined into a single differential equation. As a relation between $\nu^r$ and $\epsilon_r$, we present here (Appendix)

\begin{equation}
\frac{k_\theta}{2M} \frac{u^r}{U^r} = i K_H \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{1}{2M} - i\omega + \frac{d}{dr_r}\right) \frac{\epsilon_\omega}{E_F r / 2M}, \quad (21)
\end{equation}

If we combine (21) with another independent relation between $\nu^r$ and $\epsilon_r$, we obtain the single differential equation which $\nu^r$ obeys. Since we are interested in the fast-magnetosonic mode, we obtain the following second-order single differential equation which is appropriate to the problem on hand (Appendix);

\begin{equation}
\Lambda \nu^r = 0, \quad (22)
\end{equation}

where the second-order differential operator $\Lambda$ is defined by

\begin{equation}
\Lambda \equiv \left(\frac{1}{2M} - i\omega + \frac{d}{dr_r}\right) \left[-U^r F_{FM}^2 - \left(U_\phi^2 + U_{FM}^2\right) i\omega + \left(U_\phi^2 - U_{FM}^2\right) \frac{d}{dr_r}\right] \\
+ U_{FM}^2 \left(1 - \frac{2M}{r}\right) \left(\frac{k_\theta}{2M}\right)^2. \quad (23)
\end{equation}

The fast-magnetosonic speed $U_{FM}$ near the horizon in the magnetically dominated limit can be written as (paper 1)

\begin{equation}
U_{FM} \equiv \frac{\sqrt{K_H - K_H}}{(2M)^2 \Omega_F^2 \sin^2 \theta} \frac{E_{max}}{m_p} \gg 1. \quad (24)
\end{equation}

To obtain the wave equation describing the oscillation of $\nu^r$, let us replace $k_\theta$ with $i \partial_\theta$ and $\omega$ with $-i \partial_\theta$. The resultant equation reads, introducing nondimensional variables $x \equiv (r - r_H)/r_H$ and $\tau \equiv tf(2M)$,
Considering only the spatial dependence, we can see that this equation is elliptic in the sub-fast region \((x > x_F)\) while it becomes hyperbolic in the super-fast region \((x < x_F)\). It must be noted that equation (25) is valid only for perturbations with \(k_\phi \gg 1\), which ensures local analysis in the meridional direction. Since fluid’s energy \((u_t)\) and angular momentum \((-u_\phi)\) obey the same differential equation as (25), it is entirely fair to say that this equation describes fluid’s disturbance. In section 3 and 4, we shall therefore mainly discuss (25) in detail.

### 3. Structure of Fluid Disturbances

A study on non-stationary perturbations of magnetically dominated black hole magnetosphere was made in paper II, and it revealed that the plasma accretion should become highly variable near the fast-surface in the short-wavelength limit under the assumption of the neglect of meridional propagation of MHD waves \((k_\phi \ll \sqrt{E/m_p k_r})\). Considering meridional propagation offers, however, the key to an understanding of the energy transport from the equatorial region to the poles in the form of MHD waves.

First of all, we therefore have to inquire into the question whether the same conclusions hold when the meridional propagation becomes important. It is shown in this section that though the relative amplitude among fluid quantities and electromagnetic field becomes slightly different from what was obtained in paper II, the conclusion that the fluid becomes highly variable near the fast-surface remains true. We next investigate the spatial structure of fluid’s amplitude, which could not be treated in the framework of the short-wavelength limit, by solving equation (25).

Let us briefly examine (21) and (25) in the short-wavelength limit \((x \mid du^r/dx \ll k^r \mid x \mid)\). In this limit, we cannot treat mode conversion, spatial confinement of perturbations, and so on. Nevertheless, this limit permits us to clarify how the MHD interaction affects relative amplitude among perturbed quantities at each point along a flow line. Replacing \(\partial_r, x\partial_x, \) and \(\partial_\phi\) with \(i2M\omega, -i2Mk_r,\) and \(-i2Mk_{\perp}\) respectively, and with the aid of the dispersion relation for the outgoing fast-magnetosonic mode obtained from (25), we find that (21) becomes

\[
\frac{u^r}{U^r} \approx \frac{K_H k_r + \sqrt{k_r^2 + (x + x_F) k_{\perp}^2}}{E_r/(2M)} \frac{e_r}{E_r/(2M)}
\]

Here, the outgoing waves propagate outwards in the fluid’s co-moving frame. If we were to set \(k_{\perp} \approx k_r\) in order of magnitude, which indicates that the \(\theta\)-derivative term in (25) and hence the meridional propagation is negligible near the horizon, we would have the same relation \(u^r \approx (E/m_r)[e_r/(E_r/(2M))]\) which was presented in paper II.

We would like to consider, however, the case when the meridional propagation in the dispersion relation obtained from (25) is essential. Accordingly, let us postulate \(k_{\perp} \approx \sqrt{E/m_p k_r}\). Then we have

\[
u^r, u_r, -u_\phi \approx \sqrt{\frac{E}{m_r E_r/(2M)}} \frac{e_r}{E_r/(2M)}
\]

in order of magnitude, in a somewhat extended region near the fast-surface \((x \approx x_F \approx m_p/E)\). Moreover, setting \(k_{\perp} \approx \sqrt{E/m_p k_r}\) in other relations among perturbed quantities, we can see that other well-behaved components of the electromagnetic field have the same or smaller amplitude compared with \(e_r\) in order of magnitude near the fast-surface as follows:

\[
\]
It follows from (27) and (28) that in the magnetically dominated limit \((E/m_p \gg 1)\) the fluid suffers large amplitude disturbance compared with the electromagnetic field. This result partly supports what was obtained in paper II, in which the meridional propagation was neglected \((k_\perp \ll \sqrt{E/m_p} \text{ in order of magnitude})\). Although the perturbation energy is supplied mainly in the form of electromagnetic disturbances \(\langle e_r \rangle \approx \langle E/m_p \rangle m_p n \langle u^r \rangle\) far from the horizon, the equipartition of energy \(\langle e_r \rangle \approx m_p n \langle u^r \rangle^2\) is achieved near the fast-surface. In other words, a lot of perturbation energy is transported from the electromagnetic field to the fluid during the infall as a result of the effective MHD interactions. In a realistic magnetically dominated black hole magnetosphere, the fluid will become highly variable \(\langle u^r \rangle \approx \langle u^r \rangle\) near the fast-surface for a very small input of perturbation energy \(\langle e_r \rangle \approx (m_p/E)(E e_p/2M)^2\) from the surroundings. This effect is due to the redshift effect near the horizon and disappears if the fast-magneto sonic point is located far from it.

Having got the special case of the short-wavelength limit out of the way, we may now turn to the real subject. To examine the spatial structure of \(u^r(x, \theta)\), let us return to (25) and write it as

\[
\left\{ \left(1 - i \sigma + x \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial x} + i \sigma (x + x_F) \right) + x^2 \frac{\partial^2}{\partial \theta^2} \right\} u^r = 0,
\]

where we have replaced \(\theta_r\) with \(i \sigma\). Fluid's energy \(\langle u_t \rangle\) and angular momentum \(\langle -u_\phi \rangle\) also satisfy this equation. In general, solutions of this equation must be sought which satisfy appropriate boundary conditions. The fuller study of this subject by numerical method lies, however, outside the scope of this paper. Let us consider instead the subject from an analytic point of view and study the radial dependence of \(u^r\) by neglecting the \(\theta\)-derivative term as a first step. This approximation is valid only when meridional variations are not too large to violate the assumption \(1/\sqrt{x} \gg \ell_0 (\gg 1)\). Under this approximation, equation (29) gives the solution corresponding to the outgoing fast-magneto sonic mode, which is represented by the differential operator in the square bracket, as follows:

\[
u^r = C_1 \frac{x^i \sigma}{[x - x_F(\theta)]^{1 + i \sigma}}.
\]

where \(C_1\) is an integration constant. In order that the right-hand-side may not diverge at \(x = x_F(\theta)\), the real part of \(1 + 2i \sigma\) should be non-positive; this indicates that outgoing radial waves must decay, because no steady supply of perturbation energy across the fast-surface \(x = x_F(\theta)\) is possible. In this paper, we postulate a steady excitation of perturbation induced by plasma injection from the equatorial disk. It requires that \(\sigma\) should be real. Then, to avoid the divergence at \(x = x_F\), we must consider the \(\theta\)-derivative term in (28) at least near the fast-surface.

In fact, unless the fast-surface is exactly spherical \(x_F = \text{const.}\), the derivative \(\partial u^r / \partial \theta\) for the solution (30) becomes important in the limit \(x \to x_F\). In the narrow region \(| (x - x_F)/x_F | \ll 1\), we must therefore modify the solution into the form

\[
u^r = C_1 \frac{x^{i \sigma}}{[x - x_F(\theta) + \epsilon(\theta)]^{1 + i \sigma}}.
\]
where $\epsilon$ should be a complex function of $\theta$, so that we may obtain a regular solution $u^*$ for real $\sigma$. Inserting (31) into (29) and evaluating the equation in the limit $\left| (x - x_F + \epsilon)/x_F \right| \ll 1$, we obtain a non-linear first-order differential equation for $\epsilon$,

$$
\epsilon = \left( \frac{dx_F}{d\theta} - \frac{d\epsilon}{d\theta} \right)^2.
$$

(32)

From equation (9) the variable $x_F \equiv (r_F - r_H)/r_H$ turns out to be of order of $m_p/E \ll 1$ in the magnetically dominated limit. The we obtain the approximated solution

$$
\epsilon = \left( \frac{dx_F}{d\theta} \right)^2 + i\epsilon_I,
$$

(33)

where the imaginary part

$$
\epsilon_I = C \exp \left( - \int_0^\theta \left( \frac{dx_F}{d\theta} \right)^{-2} d\theta \right)
$$

(34)

is assumed to be much smaller than the real part. Note that for $dx_F/d\theta > 0$ the imaginary part $\epsilon_I$ rapidly decreases as $\theta$ increases, and the validity of equation (33) is assured for the high latitudes if the constant $C$ is chosen to be of order of $(m_p/E)^2$. Interestingly the structure of the outgoing fast-magnetosonic perturbation crucially depends on the shape of the fast-surface (i.e., $dx_F/d\theta$): The sharp peak of the amplitude appears at the surface $x = x_F - (dx_F/d\theta)^2$ just beyond the fast-surface. Further, for the oblate shape $(dx_F/d\theta > 0)$, the peak for the high latitudes is much smaller than that for the low latitudes. Various shapes of the critical surface will be possible according to the stationary models of the MHD accretion. For example, if the ingoing winds are injected from the disk surface (Fig.1), the particle flux should be mainly supplied into the low latitudes. Then the Alfvén Mach number $M_H(\theta) \equiv \sqrt{4\pi m_p \eta^2}$ at the horizon becomes an increasing function of $\theta$. Because $x_F$ is equal to $M_H^2(\theta)$ (paper I), the magnetosphere has an oblate critical surface. In fact, the ratio $K_H/E$ in equation (9) may remain nearly constant, by virtue of the boundary condition at the horizon which gives $\eta E \propto \sin^2 \Psi_0 \propto \sin^2 \theta$ for split monopole field (Blandford and Znajek 1977). Hence $x_F$ depends on $\theta$ through the factor $\eta \sqrt{K_I - K_H}$, in which the particle flux $\eta$ is an increasing function of $\theta$, and the conserved quantity $K_I$ equal to $(U_1^{(0)} + \Omega_F U^\prime_\phi^{(0)})^2$ is weakly dependent on $\theta$ if $U_1^{(0)} \approx 1$ and $\Omega_F U^\prime_\phi^{(0)} \ll 1$ at the injection points (the disk surface). Anyway, in this paper, we proceed with the study under the assumption $dx_F/d\theta > 0$ in relation to jet formation from the magnetosphere. We have found a large amplitude given by (31) at $x = x_F - (dx_F/d\theta)^2$ for the velocity perturbation. The important point is that this perturbation can extend to the sub-fast region ($x > x_F$) across the critical surface, where the solution (31) is regular. The infalling motion of plasma in the sub-fast region is mainly disturbed at slight outside this surface, because the outgoing waves disperse and the amplitude decays like $|u^*| \sim x^{-1}$ for $x \gg x_F$. The steady supply of the outgoing waves is due to the meridional propagation from the super-fast equatorial region to the sub-fast polar one. A large part of the outgoing waves can escape from the narrow super-fast region in the range $x_F - (dx_F/d\theta)^2 < x < x_F$, by virtue of the oblate shape of the critical surface ($dx_F/d\theta > 0$), before reaching the polar region. Hence the peak amplitude of $u^*$ at $x = x_F - (dx_F/d\theta)^2$ exponentially decreases as $\theta$ decreases. To confirm the structure of the fast-magnetosonic perturbation found in the solution (31), in the next section, we will discuss the problem of the meridional propagation in detail.
4. Collimated Propagation of MHD Waves

The meridional propagation of outgoing fast-magnetosonic waves is essential to the perturbed structure near the fast-surface. Now we consider the problem from a different point of view, to explain the meridional propagation more clearly. Note that equation (29) becomes a hyperbolic type in the super-fast region. An useful approach is to calculate the characteristics of the wave equation (29), because in general they are identical with the ray paths along which disturbances propagate in the short-wavelength limit. If the short-wavelength limit breaks down, and as a result, if the wave can no longer be regarded as a plane wave, then the disturbance propagates within the "Mach cone" of which boundary is given by the characteristics.

The characteristics of equation (29) in the super-fast region are obtained by solving the non-linear differential equation,

\[ \frac{dx}{d\theta} = \mp \sqrt{x_F(\theta) - x}. \]  

Replacing \( x_F - x \) with \( \epsilon \), we find that this equation is identical with (32). Now, to show the propagation to the critical surface \( x = x_F \), we must obtain real solutions by exactly solving equation (35). For this purpose it is mathematically convenient to assume that \( \frac{dF}{d\theta}(\equiv \Delta) \) is a positive constant. This assumption does not lose any essential feature of the characteristics. Then equation (35) yields two characteristics as follows;

\[ \left(1 - \sqrt{x_F - x} \right) \exp \left[-(1 - \sqrt{x_F - x})^2 \right] = (\alpha + 1) \epsilon^{-\alpha} \exp \left\{ \frac{\theta_0 - \theta}{2\Delta} \right\}, \]

or

\[ \Delta^{-1} \sqrt{x_F - x - 1} \exp \left[\Delta^{-1} \sqrt{x_F - x} \right] = (\alpha - 1) \epsilon^\alpha \exp \left\{ \frac{\theta_0 - \theta}{2\Delta} \right\}, \]

where the integration constant is chosen so that the characteristic passes the point \( x = x_0 = x_F - \alpha^2(dx_F/d\theta)^2 = x_F - \alpha^2 \Delta^2 \) at \( \theta = \theta_0 \) (\( \alpha > 0 \)). The further a disturbance is caused inside the fast-surface, the larger we must give the value to \( \alpha \).

In the super-fast region any waves must propagate in the inward direction \( dx < 0 \). Hence the propagation to the lower latitudes (\( d\theta > 0 \)) and the higher ones (\( d\theta < 0 \)) corresponds to (36) and (37), respectively. Further we have \( d\theta < 0 \) only for the characteristic (37) started from \( x = x_0 > x_F - \Delta^2 \) (\( \alpha < 1 \)). On these ground we have come to the conclusion: The solution (36) corresponds to the characteristic along which a disturbance caused at \((x_0, \theta_0)\) propagates into the equatorial region and at last is swallowed by the hole, while the solution (37) with \( \alpha < 1 \) corresponds to that along which the disturbance propagates into the polar region and at last escapes to a sub-fast region. Moreover, the difference of \( dx/d\theta \) between (36) and (37) at \((x_0, \theta_0)\) is an increasing function of \( \alpha \). That is, the further a disturbance is caused inside the fast-surface, the smaller becomes the Mach angle. Figure 2 shows such a situation schematically. The angle \( \theta = \theta_X \) at which the characteristic (37) meets the fast-surface \( x = x_F \) is given by putting \( \sqrt{x_F - x} = 0 \). The result is

\[ \tan \theta_X = [(1 - \alpha) \epsilon^{\alpha} + \alpha \tan \theta_0] \Delta \tan \theta_0. \]

For the magnetically dominated limit the gradient \( \Delta \) of order of \( m_p/E \) should be very small. Equation (38) means that the difference \( \theta_0 - \theta_X \) also remains very small, unless \( \alpha \) is nearly equal to unity. The meridional propagation to escape to the sub-fast region \( \theta > \theta_X \) is, therefore, very effective. If a disturbance is caused inside the maximum-amplitude surface (i.e., \( \alpha > 1 \)), then the disturbance cannot propagate into the subsonic region; such a disturbance must be always swallowed by the hole.
words, fast-magnetosonic waves can propagate from any point of the narrow region \( x_F - (dx_F/d\theta)^2 < x < x_F \) into the sub-fast region by propagating polewards. The maximum-amplitude surface \( x = x_F - (dx_F/d\theta)^2 \) can be therefore referred to the fast-magnetosonic separatrix surface (Bogovalov 1992); one of the two characteristics which is oriented polewards as illustrated in Figure 2 leaves from this surface. This is the reason why \( |u^*| \) at this surface decreases with decreasing \( \theta \), as we noted in the last section. If MHD disturbances originate from the super-fast region, and thus the outgoing waves with \( k_\theta < 0 \) dominate in the sub-fast region, then the wave energy will collimate toward the polar axis.

The question why only the waves propagating to the polar regions can escape (and the waves propagating to the equatorial region are swallowed by the hole) can be explained quite naturally in terms of the characteristics as a result of the oblate shape of the fast-surface. If the fast-surface were prolate, on the contrary, only the waves propagating into the equatorial region could escape. In this paper we have assumed a radial flow line for the unperturbed system. We emphasize here that the similar wave propagation can occur if the fast-surface is not perpendicular to the flow lines. only if the fast-surface is perpendicular to the flow lines, any fast-magnetosonic waves can no longer escape.

Let us summarize the main points that have been made in this paper. We have studied non-stationary and axisymmetric perturbations of magnetically dominated accretion onto a non-rotating black hole. A slight fluctuation of the electromagnetic field is accompanied with the fluid motion which is highly disturbed in the super-fast region just beyond the fast-surface. Some portion of the energy of this fluid disturbance can escape to a sub-fast region by virtue of the meridional propagation and concentrate toward the polar sub-fast region in the form of outgoing fast-magnetosonic waves.

Viewed in this point, the following scenario can be established: Perturbation energy is supplied from the equatorial disk into the middle latitudes or low latitudes in the form of ingoing fast-magnetosonic waves. In a magnetically dominated magnetosphere, the perturbation energy is mostly supplied electromagnetically from the surroundings. During the propagation the ingoing waves will be scattered to produce the outgoing waves. Especially in the vicinity of the fast-surface, the active MHD interactions will establish some equipartition of energy between the fluid and the electromagnetic field \( (m_p u [u^*]^2 \approx |e^2|) \). In other words, a lot of perturbation energy is transported from the magnetic field to the fluid there. Therefore, some portion of the perturbation energy supplied originally from the equatorial disk can be finally transported to the polar regions in the form of outgoing fast-magnetosonic waves. It would be possible to argue (though the evidence is not compelling) that the poleward collimation of waves may bring about the acceleration of particles in jets. To understand how the wave energy is converted into particle’s kinetic energy is an important problem for developing the MHD scenario of jet formation in the very central region of AGNs. For example, in addition to the heating of the surrounding plasma by the dissipation of wave energy into thermal energy, the wave stress exerted on the expanding atmosphere may result in a direct transfer of momentum and energy from the waves to the outgoing winds (Jacques 1977). This will be the subject of future investigations.

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Appendix  Reduction of Basic Equations

In this appendix, we derive equations (21)-(23) from equations (10)-(15) and (17)-(20). Eliminating $u^\theta$ from (10) and (11), and making use of the relations (13) and (15), we find that $e_\theta$ can be expressed in terms of the toroidal fields $b_T$ and $F_{\phi T}$ as

$$\frac{dx_\theta}{dr} = i\omega b_T - \frac{k_\theta}{\omega} \Omega_F (-i\omega + \frac{d}{dr_\ast}) F_{\phi T}. \quad (A1)$$

With the help of (A1), we obtain a relation between $e_r$ and $F_{\phi T}$ from (15) as follows:

$$e_r = -\frac{i}{\omega} \Omega_F (1 - \frac{2M}{r})^{-1} (-i\omega + \frac{d}{dr_\ast}) F_{\phi T}. \quad (A2)$$

Combining (10) and (A2), we have

$$u^\theta = -\frac{i}{\omega} \frac{U^r}{\Psi_\theta} \left( 1 - \frac{2M}{r} \right)^{-1} (-i\omega + \frac{d}{dr_\ast}) F_{\phi T}. \quad (A3)$$

We can further calculate the relation which can be used to eliminate fluid angular momentum ($-u_\phi = r^2 \sin^2 \theta u^\phi$). Inserting (14) in (12), and making use of (20), we have

$$u^\phi = \frac{U^\phi - \Omega F/\sqrt{K_F} u^r}{U^r} - \frac{1}{\sqrt{K_F} \Psi_\theta} U^r \left[ \left( 1 - \frac{2M}{r} \right)^{-1} U^r (e_\theta - b_T) - \frac{k_\theta}{\omega} U^\phi F_{\phi T} \right]. \quad (A4)$$

We use equations (13), (14), (17)-(20), and (A1)-(A4) as independent equations. Operating $(-i\omega + d/dr_\ast)$ on (A4) and making use of (18), we obtain a relation among $u^\theta$, $u^r$, and $F_{\phi T}$. Substituting this relation and (A3) in (19), with the help of (17) and $\partial_r (U_r + \Omega_F U_\theta) = \partial_r [(E - \Omega_F L)/m_p] = 0$, we obtain a relation between $u^r$ and $F_{\phi T}$ as follows:

$$\omega k_\theta U^r \left( \frac{1}{2M} - i\omega + \frac{d}{dr_\ast} \right) (-i\omega + \frac{d}{dr_\ast}) u^r$$

$$= -\frac{(2M)^2 K_F}{\Psi_\theta} U^r \left( 1 - \frac{2M}{r} \right)^{-1} \left( -\frac{3}{2M} - i\omega + \frac{d}{dr_\ast} \right) \left( -\frac{1}{M} - i\omega + \frac{d}{dr_\ast} \right) (-i\omega + \frac{d}{dr_\ast})^2 F_{\phi T}. \quad (A5)$$

Substituting (A2) into (A5), we obtain (21). On the other hand, inserting (A3) into (17), with the help of (18) and (A1), we have another independent relation between $u^r$ and $F_{\phi T}$,

$$\omega k_\theta U^r \left( \frac{1}{2M} - i\omega + \frac{d}{dr_\ast} \right) (-i\omega + \frac{d}{dr_\ast}) u^r$$

$$= \frac{(2M)^2 K_F}{\Psi_\theta} U_{PM}^2 \left( 1 - \frac{2M}{r} \right)^{-1} \left( \frac{1}{2M} - i\omega \right) \left( -\frac{1}{M} - i\omega + \frac{d}{dr_\ast} \right) (-i\omega + \frac{d}{dr_\ast})$$

$$+ \left( \frac{k_\theta}{2M} \right)^2 \left( \frac{1}{2M} - i\omega + \frac{d}{dr_\ast} \right) (-i\omega + \frac{d}{dr_\ast}) F_{\phi T}. \quad (A6)$$
where the fast-magnetosonic speed $U_{FM}$ is given by (24).

Equations (A5) and (A6) are further combined into a single differential equation. After some manipulation we obtain

$$\left(-\frac{1}{M} - i\omega + \frac{d}{dr_*}\right)\left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right)\left(-i\omega + \frac{d}{dr_*}\right)\Lambda \nu = 0,$$

(A7)

where the second-order differential operator $\Lambda$ is defined by (23). We note here that the operator $\Lambda$ contains both the ingoing and outgoing fast magnetosonic modes, because in the short-wavelength limit the equation $\Lambda \nu = 0$ gives the dispersion relation for the fast-magnetosonic mode. We must therefore look more carefully into the equation $\Lambda \nu = 0$, which is presented in section 2 as equation (22), because in the short-wavelength limit a magnetically dominated accretion suffers a large radial acceleration due to the MHD interactions near the fast-surface for the outgoing fast-magnetosonic mode (paper II).
References

Fig. 1. Schematic figure (side view) of a rotating magnetosphere around a black hole. Accretion starts from the equatorial disk with very low poloidal velocity, subsequently pass through the fast-surface denoted by the dashed curve, and finally reach the horizon. In the magnetically dominated limit, the fast-surface is located very close to the horizon.

Fig. 2. Schematic figure (side view) of a black hole magnetosphere in the close vicinity of the event horizon. The thick solid curves denote characteristics of equation (28) which originate from the point \((x_0,\theta_0)\). The maximum-amplitude surface \(z = x_F - (dF/d\theta)^2\), which is shown by the dashed line, coincides with the fast-magnetosonic separatrix surface. The poloidal magnetic field lines are not drawn for avoiding complication. The super-fast region is very thin in the magnetically dominated limit; however, the width of this region is exaggerated.