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General Construction of Physical States in 2D Quantum Gravity

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ABSTRACT

The BRST cohomology of two dimensional gravity coupled to $0 \leq c < 1$ minimal conformal matter is investigated in the conformal gauge. We formulate a general procedure to write down the physical states via the technique of spectral sequences. Paying attention to the ghost sectors, we show that the physical states corresponding to microscopic loops, and macroscopic loops are described by b-ghosts, and c-ghosts, respectively. For pure gravity, first several physical states are calculated explicitly, and in addition, the general formula for macroscopic loops is obtained by relating it to a simpler Lie algebra cohomology. We also discuss the local operator algebra of microscopic loops in relation to the Schwinger-Dyson equation in the matrix model.

1. Introduction

Recent progress in the study of 2D gravity has led to a number of remarkable results. By the discovery of the double scaling limit in the matrix model[1], the partition function as well as the correlation functions has been computed nonperturbatively[2][3][4]. Throughout this study it has been revealed that the matrix model has unexpected close relations with various subjects, such as the integrable KdV hierarchies[2], the intersection theory on the moduli space of punctured Riemann surface[5][6], and the topological gravity[7][8]. These remarkable successes are expected to play important roles in our complete understandings of more realistic string theories and quantum gravity. For this purpose it might be necessary to confirm these results within the conventional continuum field theories. The continuum theory in the conformal gauge, i.e., Liouville theory is one of the most important approaches[9]. Since this approach is perturbative in string coupling constant and the Liouville dynamics are governed by the non-linear interactions it seems so far rather difficult to treat it. However by the techniques of CFT we have already had some successes, for example construction of the correlation functions by analytic continuation methods[10][11][12] and the structure of physical spectrums[13][14]. In particular for the latter case Lian and Zuckermann classified the physical spectrums of 2D gravity coupled to $c < 1$ minimal conformal models in the framework of BRST quantization. They found that there exist an infinite number of physical states with non vanishing ghost numbers which appear corresponding to the singular vectors in the Virasoro Verma module of a conformal matter. These infinite physical states have been also found in the matrix model and from the view point of the ordinary bosonic critical string theory they can be interpreted as the remnants of higher massive modes. However the unusual situation is the existence of ghost numbers of the physical states and it is important to reveal the role of such ghost fields. For this end it seems necessary to know the explicit operator contents of physical states, especially on ghost sectors.

In this paper we present a systematic prescription to calculate the physical states of 2D gravity coupled to $0 \leq c < 1$ minimal conformal matter. In the conformal gauge the system reduces to the Liouville, the matter and the ghost sectors

coupled by the BRST constraint. Then the physical states are defined to be the cohomology of the BRST charge Q_B which acts on the tensor product of the Liouville Fock space, the irreducible Virasoro module and the ghost Fock space. On the other hand we can replace the irreducible Virasoro module by Felder's cohomology with an appropriate choice of resolutions[15]. Through this Fock space representation we can naturally consider the double complex graded by the ghost number and Felder's degree with the coboundary operator $D \sim \delta + Q_B$. The key observation is that there exist the following isomorphisms between the cohomologies: $H_D \cong H_{Q_B} H_\delta \cong H_\delta H_{Q_B}$. Indeed starting from the $H_\delta H_{Q_B}$ which we can easily obtain [14], it is possible to arrange the general procedures, "collating formula", to write down the cohomology $H_{Q_B} H_\delta$ (,i.e, physical states) by the explicit constructions of the isomorphism between them. This procedure is equivalent to the use of the the simplest case of spectral sequences. If we inspect the construction of the isomorphism we will find the following general structures of physical states: The physical states are divided into two types by the difference of ghost fields ,i.e., the microscopic loops (Seiberg state) contain the b-ghosts only and the macroscopic loops (anti-Seiberg state) contain c-ghosts only[16].

This paper is organized as follows. In section 2 we briefly review the results by Lian-Zuckermann and establish the notations. In section 3 we arrange the general procedure to write down the physical states. Then we clarify how the ghost fields appear in the physical states. In section 4 we apply our construction to the case of pure gravity (c=0). In this case the general formula for macroscopic loops is obtained and is identified with a Lie algebra cohomology[17]. Section 5 is devoted to discussions. In particular, we refer to the relationship between the local operator algebras of the microscopic loops in Liouville spectrums and those in the Schwinger-Dyson equations of the matrix models from the view point of the conservation of ghost numbers.

2. BRST ANALYSIS OF PHYSICAL STATES

In this section we briefly summarizes the BRST formalism for 2D gravity and the recent results by Lian and Zuckermann. Let us consider the 2D gravity coupled to $0 \leq c < 1$ minimal conformal matter. Upon the conformal gauge fixing the theory factorizes into three parts, i.e., Liouville, matter and ghost sectors. We first establish the notations of these sectors respectively.

Liouville sector: Under the zero cosmological constant limit the Liouville theory is described by a free boson with a background charge. We take the following convention for the stress energy tensor

$$K(z) = -\frac{1}{2} : \partial\phi_L(z)\partial\phi_L(z) : +iQ_L\partial^2\phi_L(z) \quad , \tag{2.1}$$

where the two point function is $\langle\phi_L(z)\phi_L(z)\rangle = -\ln(z-w)$. In terms of the mode expansion the Liouville field ϕ_L is expressed as follows: $i\partial\phi_L(z) = \sum_{n\in Z} a_n^{(L)} z^{-n-1}$ and $p_L = a_0^{(L)}$ with the commutation relation $[a_n^{(L)}, a_m^{(L)}] = n\delta_{n+m,0}$. The mode expansion of stress energy tensor

$$K_n = \frac{1}{2}\sum_{m\in Z} : a_{n+m}^{(L)} a_{-m}^{(L)} : -(n+1)Q_L a_n^{(L)} \quad , \tag{2.2}$$

satisfies the Virasoro algebra with the central charge $c_L = 1 - 12Q_L^2$. We denote by $\mathcal{F}_L(p_L)$ the Fock space which is built on the vacuum state $|p_L\rangle$ with the momentum p_L and the conformal dimension $\Delta_L(p_L) = \frac{1}{2}p_L(p_L - 2Q_L)$.

Matter sector: The minimal conformal model[18] with the central charge

$$c_{pq} = 1 - 6\frac{(p-q)^2}{pq} \ , (p < q, \ gcd(p,q) = 1) \quad , \tag{2.3}$$

comprises the irreducible Virasoro module $\mathrm{Vir}(c_{pq}, \Delta_{rs})$ $(1 \leq r \leq q-1, \ 1 \leq s \leq p-1, pr > qs)$ with

$$\Delta_{rs} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \quad . \tag{2.4}$$

The convention of the free field representation of the stress energy tensor and the

Virasoro generators is the same as that of the Liouville field

$$T(z) = -\frac{1}{2} : \partial\phi_M(z)\partial\phi_M(z) : +iQ_M\partial^2\phi_M(z) \quad , \tag{2.5}$$

$$L_n = \frac{1}{2}\sum_{m\in Z} : a^{(M)}_{n+m}a^{(M)}_{-m} : -(n+1)Q_M a^{(M)}_n \quad , \tag{2.6}$$

with $c_{p,q} = 1 - 12Q_M^2$ and $Q_M = \frac{\alpha_+ + \alpha_-}{2}$ $(\alpha_+ = \sqrt{\frac{2q}{p}}, \alpha_- = -\sqrt{\frac{2p}{q}})$.

Ghost sector: Upon gauge fixing, the ordinary reparametrization ghosts (b, c) appear with the conformal dimension $(2, -1)$. The ghost system is also a CFT with central central charge $c_{bc} = -26$ and its stress energy tensor is given by

$$T^{bc} = -2b(z)\partial c(z) - \partial b(z)c(z) \quad . \tag{2.7}$$

Let Λ^*_{bc} be the ghost Fock space with $*$ representing the ghost number. The ghost Fock space has both the SL(2,R) invariant vacuum $|0\rangle_{gh}$ and the physical vacuum $c_1|0\rangle_{gh}$. The physical vacuum is annihilated by the mode $c_n(n \geq 1)$ and $b_n(n \geq 0)$. Finally we take a convention that (b, c) has the ghost number $(-1, 1)$ and the physical vacuum has ghost number zero.

Under the above preparations we are now ready to review the BRST analysis and the result by Lian and Zuckerman. In the BRST formalism physical states are identified with the BRST cohomology[19][20]. The BRST operator is defined to be

$$Q_B = \oint \frac{dz}{2\pi i} : (K(z) + T(z) + \frac{1}{2}T^{bc}(z))c(z) : \quad , \tag{2.8}$$

which acts on the tensor product space

$$C^*_{abs} = \mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda^*_{bc} \quad . \tag{2.9}$$

The nilpotency of the BRST operator is equivalent to the anomaly cancellation condition

$$c_{tot} \equiv c_L + c_{pq} - 26 = 0 \quad , \tag{2.10}$$

in which case we can study the cohomology.

We denote this cohomology as $H^*_{abs}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq},\, \Delta_{r,s}) \otimes \Lambda_{bc})$ and it is called the absolute BRST cohomology. Moreover it is convenient to introduce the relative BRST cohomology $H^*_{rel}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc})$ which is defined as the cohomology of Q_B on the subcomplex $C^*_{rel} \equiv C^*_{abs} \cap Ker L_0^{tot} \cap Ker b_0$. If we decompose Q_B with respect to the ghost zero modes, we can easily see that the BRST charge reduces on the relative complex into

$$\hat{Q}_B = \sum_{n\neq 0} : c_{-n}(K_n + L_n) : -\frac{1}{2} \sum_{\substack{n,m\neq 0 \\ n+m\neq 0}} (n-m) : c_{-n} c_{-m} b_{n+m} : \quad . \tag{2.11}$$

For a moment we will focus on the relative cohomology H^*_{rel}. In the present case it is straightforward to compute the absolute cohomology from the relative one. We will refer to this relation in Section 5.

Let $E_{r,s}(p,q)$ denote the conformal dimensions of the singular vectors in the embedding diagram of Virasoro Verma module $M(c_{pq}, \Delta_{r,s})$

$$E_{r,s}(p,q) \equiv \{a_t, b_t \mid t \in \mathbf{Z}\}$$

where

$$a_t = \frac{(2pqt + pr + qs)^2 - (p-q)^2}{4pq}, \qquad b_t = \frac{(2pqt - pr + qs)^2 - (p-q)^2}{4pq} \quad , \tag{2.12}$$

and furthermore we define a set $\widetilde{E}_{r,s} \equiv \{1 - a_t, 1 - b_t \mid t \in \mathbf{Z}\}$. Note that $\widetilde{E}_{r,s}$ is a set of weights of the singular vectors appearing in the dual embedding diagram of $M(c_{p,q}, \Delta_{r,s})$, that is, it is also a embedding diagram with $26 - c_{p,q}$ obtained by replacing each conformal weights $E_{r,s}$ by $\widetilde{E}_{r,s}$ with all arrows reversed. Finally we can summarize the result by Lian and Zuckerman [13] as follows:

For any irreducible Virasoro module the relative BRST cohomology $H^*_{rel}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc})$ is non-vanishing only for the discrete values of the Liouville momentum p_L s.t. $\Delta_L(p_L) \in \widetilde{E}_{r,s}$ in which case dim $H^n_{rel}(\mathcal{L}(p_L) \otimes Vir(c_{pq},\, \Delta_{r,s}) \otimes \Lambda_{bc}) = \delta_{n,\eta(p_L)d(\Delta(p_L))}$,

where $\eta(p_L) \equiv sign(i(p_L - Q_L))$ *and*

$$d(\Delta) \equiv \begin{cases} \mid 2t+1 \mid & \text{if } \Delta = 1 - a_t \quad , \\ \mid 2t \mid & \text{if } \Delta = 1 - b_t \quad , \end{cases} \tag{2.13}$$

is the number of arrows leading from the selected singular vector to the tip in the embedding diagram.

3. COMPUTATION OF THE COHOMOLOGY

In this section we present a general procedure to compute the elements of the relative BRST cohomology. Our procedure is similar to the one when we prove the isomorphism between the de Rham cohomology and the Čech cohomology (see e.g.[21]). However there is a crucial difference between them because the cohomology considered here is defined on the space of the semi-infinite forms, i.e., the semi-infinite cohomology theory[22]. Therefore some careful treatments are needed as we will see below.

3.1 Double complex

For any irreducible Virasoro module ($0 \leq c < 1$), we can replace it by Felder's free field resolution [15]

$$Vir(c_{pq}, \Delta_{r,s}) \cong H^0(\sum_{t \in Z} \mathcal{F}_M(p_t(r,s), \delta)) \quad , \tag{3.1}$$

where $p_{2t}(r,s) = \alpha_{r,s-2tp}$, $p_{2t+1}(r,s) = \alpha_{r,-s-2tp}$ with $\alpha_{k,l} \equiv \frac{1-k}{2}\alpha_- + \frac{1-l}{2}\alpha_+$. Therefore the computation of the cohomology can be done on the product of the Fock spaces due to the isomorphism,

$$H^*_{rel}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc}) \cong H^*_{rel,\hat{Q}_B}(H^0_\delta(\mathcal{F}_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc}) \quad . \tag{3.2}$$

Now let us consider the double complex $\oplus_{i,j} C^{i,j}$ graded by the ghost number and Felder's degree

$$C^{i,j} = [\mathcal{F}_L(p_L) \otimes \mathcal{F}_M(p_j(r,s)) \otimes \Lambda^i_{bc}]_{rel} \quad , \tag{3.3}$$

and define the cohomology on it. This complex is equipped with two coboundary

operators, i.e., the BRST operator and Felder's coboundary operator. Since operations Q_B and δ commute each other, we can introduce a new coboundary operator $D = \delta + (-1)^j \hat{Q}_B$ which acts on the singly graded complex $\oplus_{i+j=n} C^{i,j}$ as

$$D : \bigoplus_{i+j=n} C^{i,j} \longrightarrow \bigoplus_{i+j=n+1} C^{i,j} \quad . \tag{3.4}$$

Here we should note that, for a given n, the two gradings i and j are bounded from both above and below. This is due to the fact that the total conformal dimension of the double complex is always restricted to zero. We denote the cohomology on the double complex $\{\oplus C^{i,j}, D\}$ as $H^*_{rel,D}(F_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc})$.

The following result by Bouwknegt *et al* [14] will play an important role in our construction:

> *If either the condition $P^+(n) \neq 0$ for all $n \in \mathbf{Z}, n \neq 0$ or the condition $P^-(m) \neq 0$ for all $m \in \mathbf{Z}, m \neq 0$ are fulfilled and, in addition, the on-shell condition $\Delta_L(p_L) + \Delta_M(p_M) = 1$ is satisfied then*

$$H^n_{rel}(\mathcal{F}_L(p_L) \otimes \mathcal{F}_M(p_M) \otimes \Lambda_{bc}, \hat{Q}_B) = \begin{cases} |p_L\rangle \otimes |p_M\rangle \otimes c_1|0\rangle_{gh} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} \tag{3.5}$$

> *where the generalized momenta $P_\pm(n)$ are defined by*

$$P_\pm(n) \equiv \frac{1}{\sqrt{2}}((p_M - (n+1)Q_M) \pm i(p_L - (n+1)Q_L)) \quad . \tag{3.6}$$

Before going into the detailed discussions let us briefly outline our general construction. As will be confirmed in the subsequent section there exist the following important isomorphisms.

$$\begin{aligned} &H^n_{rel,Q_B}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc}) \\ &\quad \cong H^n_{rel,\hat{Q}_B}(H^0_\delta(\mathcal{F}_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc})) \\ &\quad \cong H^n_{rel,D}(\mathcal{F}_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc}) \\ &\quad \cong H^n_\delta(H^0_{rel,\hat{Q}_B}(\mathcal{F}_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc})) \quad . \end{aligned} \tag{3.7}$$

Fortunately the cohomology $H^n_\delta H^0_{rel,\hat{Q}_B}$ is more tractable and can be obtained through the result (3.5). Therefore our strategy to compute the physical states

in $H^n_{rel,Q_B}H_\delta$ can be summarized to prove the existence of such isomorphisms and to construct them explicitly. Then starting from any given cohomology element of $H^n_\delta H^0_{rel,\hat{Q}_B}$ we can construct the corresponding cocycle in $H^n_{rel,D}$ and finally obtain the cocycle in $H^n_{rel,Q_B}H_\delta$ through the isomorphisms.

In view of the application of (3.5), we should choose Felder's resolution appropriately. By closer examination the appropriate choice of the momenta in the resolution turns out to be

$$p_t = \begin{cases} p_t(r,s) & \begin{array}{l} \text{if } \ \eta(p_L) > 0 \text{ and } \Delta_L(p_L) \in \widetilde{E}_{rs}(p,q)_+ \\ \text{or } \ \eta(p_L) < 0 \text{ and } \Delta_L(p_L) \in \widetilde{E}_{rs}(p,q)_- \end{array} \\ \\ \tilde{p}_t(r,s) & \begin{array}{l} \text{if } \ \eta(p_L) > 0 \text{ and } \Delta_L(p_L) \in \widetilde{E}_{rs}(p,q)_- \\ \text{or } \ \eta(p_L) < 0 \text{ and } \Delta_L(p_L) \in \widetilde{E}_{rs}(p,q)_+ \end{array} \end{cases} \tag{3.8}$$

where $\tilde{p}_t(r,s) = 2Q_M - p_{-t}(r,s)$, $\tilde{E}_{r,s}(p,q)_+ \equiv \{1-a_t, 1-b_t | t \in \mathbf{Z}_{\geq 0}\}$ and $\tilde{E}_{r,s}(p,q)_- \equiv \{1-a_t, 1-b_{-t-1} | t \in \mathbf{Z}_{<0}\}$.

3.2 Construction of the isomorphisms

We will now give an explicit description of the isomorphisms and formulate the procedure to obtain the physical states with non-vanishing ghost numbers. The key ingredient is the fact that as we have summarized in the previous section both the δ cohomology and the $\hat{Q}_B$ cohomology are trivial except the degree $n=0$. We will investigate the $n>0$ case and the $n \leq 0$ case separately because there is a crucial difference between them as addressed at the beginning of this section.

Case (I) $n > 0$

We explain how the following isomorphism arises:

$$H^n_{rel,D}(\mathcal{F}_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc}) \cong H^n_{rel,\hat{Q}_B}(H^0_\delta(\mathcal{F}_L(p_L) \otimes \sum_t \mathcal{F}_M(p_t) \otimes \Lambda_{bc})) \quad . \tag{3.9}$$

Let $\psi = \sum_{i+j=n} \psi_{i,j}$ be a D-cocycle with $\psi_{i,j}$ belonging to $C^{i,j}$. As we have already pointed out both gradings (i,j) are bounded from above and below. Then there

exist two integers (k, l) such that ψ is expressed by

$$\psi = \psi_{n+l,-l} + \psi_{n+l-1,-l+1} + \cdots + \psi_{n,0} + \cdots + \psi_{0,n} + \cdots + \psi_{n-k,k} \quad . \tag{3.10}$$

Henceforth we assume both k and l to be positive in order to consider the most general situation. Substituting the r.h.s. of (3.10) into the cocycle condition $D\psi = 0$ we obtain

$$\begin{cases} \delta\psi_{n-k,k} = 0 \quad , \\ \hat{Q}_B\psi_{n+l,-l} = 0 \quad , \\ D'\psi_{n+l-m,m-l} + \delta\psi_{n+l-m+1,m-l-1} = 0 \quad (1 \leq m \leq l+k) \quad , \end{cases} \tag{3.11}$$

where we set the notation $D' \equiv (-1)^j \hat{Q}_B$ and hence $D = \delta + D'$ for simplicity. It is helpful to remember that this cocycle condition is often pictured as a "zig-zag" in the (j, i) plane. The condition $\delta\psi_{n-k,k} = 0$ and the δ-exactness for $k \neq 0$ tell us that $\psi_{n-k,k}$ is written as δ of something, which we denote as $\delta^{-1}\psi_{n-k,k}$. Then by subtracting $D\delta^{-1}\psi_{n-k,k}$ from ψ, we can remove the component $\psi_{n-k,k}$ and still stay in the same D-cohomology class as ψ

$$\begin{aligned} \psi - D\delta^{-1}\psi_{n-k,k} =& \psi_{n+l,-l} + \cdots + \psi_{n,0} \\ &+ \cdots + \psi_{0,n} + \cdots + (\psi_{n-k+1,k-1} - D'\delta^{-1}\psi_{n-k,k}) \end{aligned} \quad , \tag{3.12}$$

where $\psi_{n-k+1,k-1} - D'\delta^{-1}\psi_{n-k,k} \in C_{n-k+1,k-1}$. Substituting again the r.h.s. of (3.12) into the D-cocycle condition, we obtain $\delta(\psi_{n-k+1,k-1} - D'\delta^{-1}\psi_{n-k,k}) = 0$. Therefore the same procedure as above can be applied and we can proceed to the next step. After iterating this procedure until the non trivial cohomology element (0-th cohomology of δ) appears, we will find the following D-cocycle

$$\psi' \equiv \psi_{n+l,-l} + \cdots + \psi_{n,0} + \sum_{j=1}^{k} (-S)^j \psi_{n-j,j} \quad , \tag{3.13}$$

where we define $S \equiv D'\delta^{-1}$. Notice the fact that both $\sum_{j=1}^{k}(-S)^j\psi_{n-j,j}$ and $\psi_{n,0}$ belong to the same space $C^{n,0}$ and ψ' is D-cohomologous to ψ.

Next let us consider another condition $\hat{Q}_B\psi_{n+l,-l} = 0$. Because of the $\hat{Q}_B$-exactness for a non-zero ghost number, we can develop the same argument as in the case of δ-cohomology and move ψ' to a D-cocycle $\tilde{\psi}_{n,0}$ with only $C^{n,0}$ component,

$$\tilde{\psi}_{n,0} \equiv \psi_{n,0} + \sum_{i=1}^{l}(-\tilde{S})^i\psi_{n+i,-i} + \sum_{j=1}^{k}(-S)^j\psi_{n-k,k} \quad , \tag{3.14}$$

where $\tilde{S} \equiv \delta D'^{-1}$. Since $\tilde{\psi}_{n,0}$ is D-cohomologous to ψ, the cocycle condition

$$D\tilde{\psi}_{n,0} = \delta\tilde{\psi}_{n,0} + \hat{Q}_B\tilde{\psi}_{n,0} = 0 \quad , \tag{3.15}$$

leads to $\delta\tilde{\psi}_{n,0} = \hat{Q}_B\tilde{\psi}_{n,0} = 0$ which implies $\tilde{\psi}_{n,0} \in H^*_{rel,Q}H^0_\delta$.

In this way we can always take the representative of the form $\tilde{\psi}_{n,0}$ (,i.e., $\tilde{\psi}_{i,n-i} = 0$ for $i \neq n$), which is not uniquely determined for a D-cohomology element $[\psi]_D \in H^n_D$. The remaining ambiguity of adding the coboundary term $D\phi = D(\sum_{i+j=n-1}\phi_{ij})$ is the one which preserves the condition of $\tilde{\psi}_{i,n-i} = 0$ for $i \neq n$, and can always be put into the $\hat{Q}_B$-coboundary term due to the similar argument to (3.11)-(3.14):

$$D(\sum_{i+j=n-1}\phi_{ij}) = D(\tilde{\phi}_{n,0} + D\chi) = \hat{Q}_B\tilde{\phi}_{n,0} \quad .$$

Therefore the map $[\psi]_D \mapsto [\tilde{\psi}_{n,0}]_{2'}$ constitutes a well defined map from $H^n_{rel,D} \to H^n_{rel,\hat{Q}_B}H^0_\delta$. This map turns out to be the desired isomorphism. In fact, if $\tilde{\psi}_{n,0} \in [\hat{Q}\phi_{n-1,0}]_{1'}$ with $[\phi_{n-1,0}]_{1'} \in C^{n-1}_{rel,\hat{Q}_B}(H^0_\delta)$, then for arbitrary choice of the representative $\phi_{n-1,0}$ of $[\phi_{n-1,0}]_{1'}$, there exists $\phi_{n,-1} \in C^{n,-1}$ such that

$$\tilde{\psi}_{n,0} = \hat{Q}_B\phi_{n-1,0} + \delta\phi_{n,-1} \ , \ \hat{Q}_B\delta\phi_{n,-1} = 0 \quad . \tag{3.16}$$

Owing to the δ-exactness, $H^n_\delta = 0$ $(n \neq 0)$, the relations (3.16) means that there exist $\phi_{n-i-1,i}(i < 0)$ so that $\tilde{\psi}_{n,0} = D(\sum_{i=k}^{0}\phi_{n-i-1,i})$ for $k < 0$. Thus we see that the map $[\psi]_D \mapsto [\tilde{\psi}_{n,0}]_{2'}$ is injective. On the other hand the surjectiveness follows rather easily from the same argument as above, and thus we arrive at the proof of the isomorphism(3.9).

The next isomorphism

$$H^n_{rel,D}(\mathcal{F}_L(p_L)\otimes\sum_t\mathcal{F}_M(p_t)\otimes\Lambda_{bc})\cong H^n_\delta(H^0_{rel,\hat{Q}_B}(\mathcal{F}_L(p_L)\otimes\sum_t\mathcal{F}_M(p_t)\otimes\Lambda_{bc}))\ , \tag{3.17}$$

follows in the same way: for each D-cocycle ψ, we can construct this time

$$\tilde{\psi}_{0,n}\equiv\psi_{0,n}+\sum_{i=1}^{n+l}(-\tilde{S})^i\psi_{i,n-i}+\sum_{j=1}^{k-n}(-S)^j\psi_{-j,n+j}\quad . \tag{3.18}$$

Then the map $[\psi]_D\mapsto[\tilde{\psi}_{0,n}]_2$ constitutes the desired isomorphism.

Now we can present how to construct the physical state utilizing the isomorphism $H_\delta H_{\hat{Q}_B}\cong H_{\hat{Q}_B}H_\delta$. According to the result (3.5), $\mathcal{O}_\delta(p_L,p_n)=|p_L\rangle\otimes|p_n\rangle\otimes c_1|0\rangle_{gh}$ is a cocycle in $H^n_\delta H^0_{rel,\hat{Q}_B}$ and furthermore, in the present case $n>0$, the state $\mathcal{O}_\delta(p_L,p_n)$ is a D-cocycle at the same time. Therefore we obtain the cocycle $\mathcal{O}_{\hat{Q}_B}(p_L,p_n)$ in $H^n_{rel,\hat{Q}_B}H^0_\delta$ through the isomorphism (3.9)

$$\mathcal{O}_{\hat{Q}_B}(p_L,p_n)=(-S)^n\mathcal{O}_\delta(p_L,p_n)\quad . \tag{3.19}$$

Case (II) $n\leq 0$

Due to the exactness of both δ- and $\hat{Q}_B$-sequences, we can follow the same processes as in case (I), but the final form of the cocycle is quite different from the former. In this case, we will find that the D-cocycle ψ (3.10) is D-cohomologous to $\tilde{\psi}$,

$$\tilde{\psi}=\tilde{\psi}_{0,n}+\psi_{-1,n+1}+\cdots+\psi_{n+1,-1}+\tilde{\psi}_{n,0}\quad , \tag{3.20}$$

where we define $\tilde{\psi}_{n,0}\equiv\psi_{n,0}+\sum_{j=1}^k(-S)^j\psi_{n-j,j}$ and $\tilde{\psi}_{0,n}\equiv\psi_{0,n}+\sum_{i=1}^{n+l}(-\tilde{S})^i\psi_{i,n-i}$. It is possible to show through the similar argument to the case $n>0$, that the map $[\psi]_D\to[\tilde{\psi}_{0,n}]_2$ ($[\psi]_D\to[\tilde{\psi}_{n,0}]_{2'}$) realizes the isomorphism $H_{rel,D}\cong H_\delta H_{rel,\hat{Q}_B}$ ($H_{rel,D}\cong H_{rel,\hat{Q}_B}H_\delta$).

In contrast to the case (I), when constructing the physical states, we must first find the D-cocycle which corresponds to the cocycle $\mathcal{O}_\delta(p_L, p_n)$ in $H_\delta H_{\hat{Q}_B}$. The D-cocycle corresponding to (3.20) turns out to be

$$\mathcal{O}_D(p_L, p_n) = \{1 + (-S') + (-S')^2 + \cdots + (-S')^{-n}\}\mathcal{O}_\delta(p_L, p_n) \quad , \tag{3.21}$$

where $S' \equiv (-1)^j \hat{Q}_B^{-1}\delta$. Therefore we can obtain the corresponding physical state in $H_{\hat{Q}_B} H_\delta$ as

$$\mathcal{O}_{\hat{Q}_B}(p_L, p_n) = (-S')^{-n}\mathcal{O}_\delta(p_L, p_n) \quad . \tag{3.22}$$

3.3 The general structure of physical states

In the previous subsection we have obtained the general formula which entails the physical states in (3.19) and (3.22). By examining carefully the operations of $(-S)^n$ and $(-S')^{-n}$ there, we can confirm the following forms of physical states.

$$\mathcal{O}_{\hat{Q}_B}(p_L, p_n) = \begin{cases} P(\{L_{-m}\}, \{c_{-m}\}) \mid p_L\rangle \otimes \mid p_0\rangle \otimes c_1|0\rangle_{gh} & if\ n > 0 \\ P(\{K_{-m}\}, \{L_{-m}\}, \{b_{-m}\}) \mid p_L\rangle \otimes \mid p_0\rangle \otimes c_1|0\rangle_{gh} & if\ n \leq 0 \end{cases} \tag{3.23}$$

where P means some polynomial of the arguments and $p_0 = p_0(r, s)$ or $2Q_M - p_0(r, s)$ depending on the value p_L. As it is clear now, ***physical states with positive ghost number*** $(n > 0)$ ***contain only c-ghost excitations and those with non-positive ghost number*** $(n \leq 0)$ ***contain only b-ghost excitations from the physical vacuum.*** Let us explain the above proposition.

(i) $n > 0$: We first rewrite (3.19) to the following sequence of maps and investigate each operation $\hat{Q}_B\delta^{-1}$, "zig-zag", in detail,

$$\mathcal{O}_{0,n} \xrightarrow{\delta^{-1}} \mathcal{O}_{0,n-1} \xrightarrow{\hat{Q}_B} \mathcal{O}_{1,n-1} \xrightarrow{\delta^{-1}} \mathcal{O}_{1,n-2} \xrightarrow{\hat{Q}_B} \mathcal{O}_{2,n-2} \xrightarrow{\delta^{-1}} \cdots \xrightarrow{\hat{Q}_B} \mathcal{O}_{n,0} \tag{3.24}$$

where $\mathcal{O}_{0,n} \equiv \mathcal{O}_\delta(p_L, p_n) = \mid p_L\rangle \otimes \mid p_n\rangle \otimes c_1|0\rangle_{gh}$ and $\mathcal{O}_{n,0} \equiv \mathcal{O}_{\hat{Q}_B}(p_L, p_n)$. Owing to the property of Felder's resolution, the first action δ^{-1} results in the co-singular

vector in $\mathcal{F}(p_{n-1})$, which can not be written via L_{-n}'s and is determined up to the action δ of something,

$$\begin{aligned}\mathcal{O}_{0,n-1} &= \delta^{-1}\mathcal{O}_{0,n} \\ &= \mid p_L\rangle \otimes P(\{a_{-m}\}) \mid p_{n-1}\rangle \otimes c_1|0\rangle_{gh} \quad .\end{aligned} \tag{3.25}$$

Now introduce the following notations:
$\hat{Q}_B = \hat{Q}_L + \hat{Q}_M + \hat{Q}_{gh}$ with $\hat{Q}_L = \sum_{n\neq 0} c_{-n}K_n$, $\hat{Q}_M = \sum_{n\neq 0} c_{-n}L_n$ and $\hat{Q}_{gh} = -\frac{1}{2}\sum(n-m) : c_{-n}c_{-m}b_{n+m} :$. Then due to the fact that $\hat{Q}_L \mid p_L\rangle \otimes c_1|0\rangle_{gh} = 0$ and $\hat{Q}_{gh}c_1|0\rangle_{gh} = 0$, only the $\hat{Q}_M$-term acts on the state $\mathcal{O}_{0,n-1}$ as

$$\begin{aligned}\mathcal{O}_{1,n-1} &= \hat{Q}_B\mathcal{O}_{0,n-1} \\ &= \mid p_L\rangle \otimes \hat{Q}_M P(\{a_{-m}\}) \mid p_{n-1}\rangle \otimes c_1|0\rangle_{gh} \\ &= P(\{L_{-m}\},\{c_{-m}\}) \mid p_L\rangle \otimes \mid p_{n-1}\rangle \otimes c_1|0\rangle_{gh} \quad ,\end{aligned} \tag{3.26}$$

where in the last line all the oscillators $\{a_{-m}\}$ can be replaced safely with the Virasoro generators. This is indeed possible because the conformal dimension of the matter sector is reduced to the amount that the ghost excitations have used, and there is no longer the degeneracy between the basis a_n's and L_n's.

The subsequent operations are essentially the same as (3.25) and (3.26). In fact since the coboundary operator δ commutes with Virasoro generators it follows that

$$\begin{aligned}\mathcal{O}_{1,n-2} &= \delta^{-1}\mathcal{O}_{1,n-1} \\ &= P(\{L_{-m}\},\{c_{-m}\}) \mid p_L\rangle \otimes P(\{a_{-m}\}) \mid p_{n-2}\rangle \otimes c_1|0\rangle_{gh} \quad ,\end{aligned} \tag{3.27}$$

and again the same structure as (3.26) can be found in $\mathcal{O}_{2,n-2} = \hat{Q}_B\mathcal{O}_{1,n-2}$. Iterating these processes, we can deduce that the b-ghosts never appear in the final result. Another important point in the process is related to the ambiguity associated with the action δ^{-1}. Although each action of the zig-zag suffers from this ambiguity, it can be seen that only the one arising from the final step is relevant to the form of $\mathcal{O}_{n,0}$. And this ambiguity is not important when we view the Virasoro irreducible module as $Vir \cong H^0_\delta(\sum \mathcal{F}_t)$. In this sense, we arrive at the general form (3.23).

(ii) $n \leq 0$: Let us write the operations of $(-S')^{-n}$ on $\mathcal{O}_\delta(p_L, p_n)$ as

$$\mathcal{O}_{0,n} \xrightarrow{\delta} \mathcal{O}_{0,n+1} \xrightarrow{\hat{Q}_B^{-1}} \mathcal{O}_{-1,n+1} \xrightarrow{\delta} \mathcal{O}_{-1,n+2} \xrightarrow{\hat{Q}_B^{-1}} \mathcal{O}_{-2,n+2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{O}_{n+1,0} \xrightarrow{\hat{Q}_B^{-1}} \mathcal{O}_{n,0} \tag{3.28}$$

where $\mathcal{O}_{0,n} \equiv \mathcal{O}_\delta(p_L, p_n) =| p_L\rangle \otimes | p_n\rangle \otimes c_1|0\rangle_{gh}$ and $\mathcal{O}_{n,0} \equiv \mathcal{O}_{\hat{Q}_B}(p_L, p_n)$. Since the state $| p_n\rangle$ is mapped under δ to the Virasoro singular vector in $\mathcal{F}(p_{n+1})$, it follows that

$$\begin{aligned} \mathcal{O}_{0,n+1} &= \delta \mathcal{O}_{0,n} \\ &=| p_L\rangle \otimes P(\{L_{-m}\}) \mid p_{n+1}\rangle \otimes c_1|0\rangle_{gh} \quad . \end{aligned} \tag{3.29}$$

By noting the fact

$$\hat{Q}_B \mathcal{O}_{0,n+1} = \hat{Q}_B \delta \mathcal{O}_{0,n} = \delta \hat{Q}_B \mathcal{O}_{0,n} = 0 \quad , \tag{3.30}$$

and $\mathcal{O}_{0,n+1}$ is trivial in $H^0_{rel,\hat{Q}_B}$, we can write $\mathcal{O}_{0,n+1}$ as a $\hat{Q}_B$-exact form:

$$\mathcal{O}_{0,n+1} \equiv \hat{Q}_B \mathcal{O}_{-1,n+1} \quad . \tag{3.31}$$

Our assertion is that in general $\mathcal{O}_{-1,n+1}$ can be taken as the following form:

$$\mathcal{O}_{-1,n+1} = P_1(\{K_{-m}\}, \{L_{-m}\}, \{b_{-m}\})|p_L\rangle \otimes | p_{n+1}\rangle \otimes c_1|0\rangle_{gh} \quad , \tag{3.32}$$

where P_k means some homogeneous polynomial of b-ghosts with degree k.
This fact is confirmed in the following way. From (3.29) $\mathcal{O}_{-1,n+1}$ is defined so that it satisfies

$$P(\{L_{-m}\}) \mid p_L\rangle \otimes | p_{n+1}\rangle \otimes c_1|0\rangle_{gh} = \hat{Q}_B \mathcal{O}_{-1,n+1} \quad . \tag{3.33}$$

The constraint from the ghost number restricts the general form of $\mathcal{O}_{-1,n+1}$. In our case, the following types of monomials exhaust the possibility;

$$\{K\}\{L\}b, \quad \{K\}\{L\}bbc, \quad \{K\}\{L\}bbbcc, \cdots \quad \{K\}\{L\}\underbrace{bb\cdots b}_{n}\underbrace{cc\cdots c}_{n-1} \quad ,$$

where for simplicity we omitted the subscripts of each modes and $\{*\}$ means some products of $*$. If we inspect the way of operation $\hat{Q}_B$ on these monomials, excluding

the trivial operation arising from $\hat{Q}_B^2 = 0$, we will find that only the first type of monomial satisfies the relation (3.33). In other words any monomials except the first one and their arbitrary linear combination never give rise to single $\{L\}$- term under $\hat{Q}_B$. Thus follows our assertion.

We can apply the similar argument to the subsequent steps and will find the following result,

$$\mathcal{O}_{-k,n+k} = P_k(\{K_{-m}\},\{L_{-m}\},\{b_{-m}\})|p_L\rangle \otimes |\, p_{n+k}\rangle \otimes c_1|0\rangle_{gh} \quad , \quad (1 \leq k \leq -n) \tag{3.34}$$

up to $\hat{Q}_B$-exact forms. Generally, as has been the case for $n > 0$, the ambiguities associated with $\hat{Q}_B$-exact forms at each zig-zag are irrelevant to the final expression of $\mathcal{O}_{n,0}$ except the last two zig-zag operations. One of the two ambiguities which are remaining will disappear again when we view the Virasoro irreducible module as $Vir \cong H_\delta^0(\sum \mathcal{F}_t)$, and the other is the one which is always associated with the BRST analysis.

4. Pure Gravity

In this section, we will apply our general procedure to the case of $c = 0$, i.e., the pure gravity. Then it will be found that there is a close interplay between the embedding structure of the Virasoro Verma modules in the matter Fock module and that in the Liouville Fock module. As our general formula shows, the construction of the physical states is different depending on $n > 0$ or $n \leq 0$; the former corresponds to the macroscopic loops ($\eta(p_L) > 0$) and the latter corresponds to the microscopic loops ($\eta(p_L) < 0$) which have direct relation to the observables found in the matrix model.

Let us first recall some results on Felder's free field resolution and the structure of the Liouville Fock space $\mathcal{F}_L(p_L)$ as the Virasoro module. In the case of $c = 0$, where $(p,q) = (2,3), (r,s) = (1,1)$ and $Q_M = \frac{1}{2\sqrt{3}}$, Felder's resolution can be obtained via coboundary operator expressed through a single contour integration

$$\delta = \int dz : \mathrm{e}^{i\sqrt{3}\phi_M} : \quad . \tag{4.1}$$

The conformal dimensions of the (co-)singular vectors are given by so-called Euler's

pentagon numbers,

$$a_t = (2t+1)(3t+1) \quad , \quad b_t = t(6t+1) \quad (t \in \mathbf{Z}) \quad . \tag{4.2}$$

Under the on-shell condition, $\Delta_L(p_L) + a_s = 1$ or $\Delta_L(p_L) + b_s = 1$ for some $s \in \mathbf{Z}$, the Liouville Fock space $\mathcal{F}_L(p_L)$ belongs to the type $\mathrm{III}_+(+)$ or $\mathrm{III}_+(-)$ in ref.[23] depending on $\eta(p_L) < 0$ or $\eta(p_L) > 0$, respectively. The embedding structure of the Virasoro Verma modules in $\mathrm{III}_+(\pm)$ are drawn in Fig.1, in conformity with the recent correction by E.Frenkel[24], where the degree of the (co-)singular vectors are given by $1 - a_t$ or $1 - b_t$ ($|t| \leq |s|$).

The on-shell condition $\Delta_L(p_L) = 1 - a_t$ or $1 - b_t$ $(t \in \mathbf{Z})$, under which the physical states exist, can be solved with respect to the Liouville momentum as

$$\begin{cases} p_L(2t+1)_{\mp} = (-i)\frac{5\mp|12t+5|}{2\sqrt{3}} & \text{for the choice of } a_t \ ; \\ p_L(2t)_{\mp} = (-i)\frac{5\mp|12t-1|}{2\sqrt{3}} & \text{for the choice of } b_{-t} \ , \end{cases} \tag{4.3}$$

with $-$ sign for $\eta(p_L) < 0$ and $+$ sign for $\eta(p_L) > 0$. The general form of the cocycles $\mathcal{O}_{\hat{Q}_B}(p_L, p_n)$ which has been revealed in the previous section simplifies further in this specific case as

$$\mathcal{O}_{\hat{Q}_B}(p_L, p_n) = \begin{cases} P(\{c_{-m}\})|p_L\rangle \otimes |p_0\rangle \otimes c_1|0\rangle_{gh} & \text{for } \eta(p_L) > 0 \ ; \\ P(\{K_{-m}\}, \{b_{-m}\})|p_L\rangle \otimes |p_0\rangle \otimes c_1|0\rangle_{gh} & \text{for } \eta(p_L) < 0, \end{cases} \tag{4.4}$$

with $p_0 = 0$ or $2Q_M$ depending on the value of p_L (see (3.8)).

Now let us look into the details how the concrete expressions (4.4) can be obtained through our general formula (3.19) and (3.22).

The case of $\eta(p_L) < 0$:

The first simple but non-trivial example is the construction of the physical state $|\mathcal{O}_7\rangle = (b_{-2} + \frac{3}{2}K_{-1}b_{-1})|\frac{i}{\sqrt{3}}\rangle_L \otimes |0\rangle_M \otimes c_1|0\rangle_{\mathrm{gh}}$, where and hereafter the subscript of observables $\mathcal{O}$ indicates the KdV flow which the observables are assumed to

generate under the correspondence to the matrix model. According to our general formula, $|\mathcal{O}_7\rangle$ is represented as

$$\begin{aligned}\mathcal{O}_{\hat{Q}_B}(p_L(-1)_-, p_{-1}(1,1)) &= (-S')\mathcal{O}_\delta(p_L(-1)_-, p_{-1}(1,1)) \\ &= -\hat{Q}_B^{-1}\delta|\frac{i}{\sqrt{3}}\rangle_L \otimes |-\sqrt{3}\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad ,\end{aligned} \tag{4.5}$$

and is evaluated as follows: The first action of δ is

$$\delta|-\sqrt{3}\rangle_M = L_{-2}|0\rangle_M \quad . \tag{4.6}$$

In contrast, we must be a little involved in evaluating the action $\hat{Q}_B^{-1}$; we first find the following relation

$$\begin{aligned}&\hat{Q}_B(b_{-2} + \frac{3}{2}K_{-1}b_{-1})|\frac{i}{\sqrt{3}}\rangle_L \otimes |0\rangle_M \otimes c_1|0\rangle_{\rm gh} \\ &\qquad = \{L_{-2} + (K_{-2} + \frac{3}{2}K_{-1}^2)\}|\frac{i}{\sqrt{3}}\rangle_L \otimes |0\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad .\end{aligned} \tag{4.7}$$

Here we recognize that the operators which have appeared in the right hand side coincide with the singular vectors in the Virasoro Verma modules $M(c=0, h=0)$ and $M(c=26, h=-1)$. However, in the Fock space representation, both of them do not constitute the singular vector at the same time; in the present case $L_{-2}|0\rangle_M$ is a singular vector in $\mathcal{F}_M(p_M=0)$ whereas $(K_{-2}+\frac{3}{2}K_{-1}^2)|\frac{i}{\sqrt{3}}\rangle_L$ vanishes identically in $\mathcal{F}_L(p_L=\frac{i}{\sqrt{3}})$ because of the existence of the co-singular vector. Due to these facts, we obtain an representative of the physical state as

$$|\mathcal{O}_7\rangle = -(b_{-2} + \frac{3}{2}K_{-1}b_{-1})|\frac{i}{\sqrt{3}}\rangle_L \otimes |0\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad . \tag{4.8}$$

Now let us consider the next example $|\mathcal{O}_{11}\rangle$ whose construction is more complicated. The observable $|\mathcal{O}_{11}\rangle$ is the one which arises from the on-shell condition $\Delta_L(p_L) = 1 - b_{-1}$, and can be expressed via our general formula as

$$\begin{aligned}\mathcal{O}_{\hat{Q}_B}(p_L(2)_-, \tilde{p}_{-2}(1,1)) &= (-S')^2\mathcal{O}_\delta(p_L(2)_-, \tilde{p}_{-2}(1,1)) \\ &= -\hat{Q}_B^{-1}\delta\hat{Q}_B^{-1}\delta|i\sqrt{3}\rangle_L \otimes |-\frac{5}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad .\end{aligned} \tag{4.9}$$

As has been the case for $|\mathcal{O}_7\rangle$, we can see again an interplay between the singular vectors in $\mathcal{F}_M(p_M)$ and the existence of the corresponding co-singular vectors in

$\mathcal{F}_L(p_L)$ in each step of the "zig-zag" operation, $\hat{Q}_B^{-1}\delta$. The first zig-zag operation can be solved by noting the following relation;

$$\delta \ |i\sqrt{3}\rangle_L \otimes |-\frac{5}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh} = \hat{Q}_B|\Psi_{(-1,-1)}\rangle \quad , \tag{4.10}$$

where the state $|\Psi_{(-1,-1)}\rangle$ has an complicated expression with arbitrary constants A and B (, see appendix for its expression). From the construction of Felder's resolution for the l.h.s. of (4.10) and the evaluation of the commutation relations for the r.h.s. of (4.10), we obtain

$$(\text{l.h.s. of (4.10)}) = V_s|i\sqrt{3}\rangle_L \otimes |-\frac{2}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh} \tag{4.11}$$

$$(\text{r.h.s. of (4.10)}) = (V_s + \frac{19}{3}\chi_4)|i\sqrt{3}\rangle_L \otimes |-\frac{2}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh} \quad , \tag{4.12}$$

where V_s and χ_4 are the expression of the singular vectors in the Verma modules $M(c=0,\Delta=0)$ and $M(c=26,\Delta=-4)$, respectively, and are given by

$$\begin{aligned} V_s &= L_{-4} - L_{-3}L_{-1} - L_{-2}^2 + \frac{5}{3}L_{-2}L_{-1}^2 - \frac{1}{4}L_{-1}^4 \quad , \\ \chi_4 &= K_{-4} + \frac{13}{19}K_{-3}K_{-1} + \frac{5}{19}K_{-2}K_{-1}^2 + \frac{3}{19}K_{-2}^2 + \frac{3}{76}K_{-1}^4 \quad . \end{aligned} \tag{4.13}$$

Then, the equality (4.10) is attained due to the fact that $\chi_4|i\sqrt{3}\rangle_L \equiv 0$. The evaluation of the second zig-zag proceeds in the same way but is more complicated. The fundamental relation we note is again

$$\delta|\Psi_{(-1,-1)}\rangle = -\hat{Q}_B|\mathcal{O}_{11}\rangle \quad , \tag{4.14}$$

with setting $A = -\frac{3}{2}$, $B = -\frac{1}{6}$ in $|\Psi_{(-1,-1)}\rangle$ and defining

$$\begin{aligned} |\mathcal{O}_{11}\rangle =& \{b_{-2}b_{-1}(K_{-2} + \frac{1}{6}K_{-1}^2) - \frac{1}{6}b_{-3}b_{-1}K_{-1} \\ &+ \frac{1}{3}b_{-4}b_{-1} - b_{-3}b_{-2}\}|i\sqrt{3}\rangle_L \otimes |\frac{1}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh} \quad . \end{aligned} \tag{4.15}$$

If we note $(L_{-2} - \frac{3}{2}L_{-1}^2)|\frac{1}{\sqrt{3}}\rangle_M \equiv 0$ in the matter Fock space, we will find for the

l.h.s. of (4.14)

$$\delta|\Psi_{(-1,-1)}\rangle$$
$$= \{\frac{1}{3}b_{-4}L_{-1} - b_{-3}(\frac{1}{6}K_{-1}L_{-1} + L_{-2}) + b_{-2}(K_{-2}L_{-1} + \frac{1}{6}K_{-1}^2L_{-1} + L_{-3})$$
$$- b_{-1}(\frac{1}{6}K_{-1}^2L_{-2} - \frac{1}{6}K_{-1}L_{-3} - K_{-2}L_{-2} - \frac{1}{3}L_{-4})\}\ |i\sqrt{3}\rangle_L \otimes |\frac{1}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh}\ . \tag{4.16}$$

On the other hand, for the r.h.s. of (4.14), we obtain

$$-\hat{Q}_B|\mathcal{O}_{11}\rangle = (\text{r.h.s. of (4.16)}) + (2b_{-2}\chi_3 - \frac{1}{3}b_{-1}\tilde{\chi}_4)|i\sqrt{3}\rangle_L \otimes |\frac{1}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh}\ , \tag{4.17}$$

where $\chi_3 = K_{-3} + \frac{1}{2}K_{-2}K_{-1} + \frac{1}{12}K_{-1}^3$ and $\tilde{\chi}_4 = 19\chi_4 - 9K_{-1}\chi_3$. Again the facts that $\chi_3|i\sqrt{3}\rangle_L \equiv 0$ and $\tilde{\chi}_4|i\sqrt{3}\rangle_L \equiv 0$ entail the equality (4.14). Thus we finally arrive at the expression for the observable $|\mathcal{O}_{11}\rangle$ of the general form (4.4).

Here two general remarks are in order; 1) the action $\hat{Q}_B^{-1}$ in each zig-zag is always accompanied with the ambiguity of the term $\hat{Q}_B(*)$ where $(*)$ is an arbitrary operator with a definite ghost number and a definite conformal dimension. However, in the successive actions of the zig-zag; $(\hat{Q}_B^{-1}\delta)(\hat{Q}_B^{-1}\delta)\cdots(\hat{Q}_B^{-1}\delta)$, all the ambiguities except those from the last two zig-zags are irrelevant to the final expression of the observables because ,e.g., $(\hat{Q}_B^{-1}\delta)(\hat{Q}_B^{-1}\delta)\hat{Q}_B(*) = 0$. One of the remaining ambiguities are fixed uniquely by demanding the general form (4.4) of the observables. 2) As is clear in the above examples, each zig-zag operation $\hat{Q}_B^{-1}\delta$ is solved under the deep interplay between the singular vectors and the existence of the corresponding co-singular vectors in the matter and the Liouville Fock spaces. This structure in our procedure stems from the duality in the representation theory of the Virasoro algebra with (c, Δ) and $(26-c, 1-\Delta)$, and can be extended to other Kac-Moody Lie algebras [24]. Physically such an extension means to construct a G/G-topological field theory[25] and seems interesting in its own right.

We list the results for some other examples in table 1.

The case of $\eta(p_L) > 0$:

Although we will find later a general formula for all physical states in this case, it seems still instructive to see how our formula works. As an example let

us consider the physical state $|\tilde{\mathcal{O}}_{11}\rangle$ which corresponds to the on-shell condition $\Delta_L(p_L) = 1 - b_{-1}$ with $\eta(p_L) > 0$. Our general formula leads us to

$$\begin{aligned}\mathcal{O}_{\hat{Q}_B}(p_L(2)_+, p_2(1,1)) &= (-S)^2 \mathcal{O}_\delta(p_L(2)_+, p_2(1,1)) \\ &= -\hat{Q}_B \delta^{-1} \hat{Q}_B \delta^{-1} | - i\frac{8}{\sqrt{3}}\rangle_L \otimes |2\sqrt{3}\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad .\end{aligned} \tag{4.18}$$

The first action δ^{-1} on $|2\sqrt{3}\rangle_M$ brings us the co-singular vector in $\mathcal{F}_M(p_M = \sqrt{3})$ which can not be represented via L_{-n}'s,

$$\delta^{-1}|2\sqrt{3}\rangle_M = (a_{-4} - \sqrt{3}a_{-3}a_{-1} - \frac{\sqrt{3}}{8}a_{-2}^2 + \frac{3}{2}a_{-2}a_{-1}^2 - \frac{\sqrt{3}}{8}a_{-1}^4)|\sqrt{3}\rangle_M \quad , \tag{4.19}$$

where the right hand side is determined up to the action δ of something. The next action $\hat{Q}_B$ entails a rather complicated expression

$$\begin{aligned}&\hat{Q}_B \delta^{-1} | - i\frac{8}{\sqrt{3}}\rangle_L \otimes |2\sqrt{3}\rangle_M \otimes c_1|0\rangle_{\rm gh} \\ &= \frac{1}{\sqrt{3}}\{\frac{17}{9}c_{-1}L_{-1}^3 - \frac{22}{3}c_{-1}L_{-2}L_{-1} + \frac{25}{6}c_{-1}L_{-3} - \frac{41}{12}c_{-2}L_{-1}^2 \\ &\quad + 7c_{-2}L_{-2} + 7c_{-3}L_{-1} - \frac{17}{2}c_{-4}\}| - i\frac{8}{\sqrt{3}}\rangle_L \otimes |\sqrt{3}\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad .\end{aligned} \tag{4.20}$$

The following action δ^{-1} has the effect of simply replacing $|\sqrt{3}\rangle_M$ with $a_{-1}|0\rangle_M$ again with the ambiguity δ of something. Then the final action $\hat{Q}_B$ results in

$$\begin{aligned}|\tilde{\mathcal{O}}_{11}\rangle &= \mathcal{O}_{\hat{Q}_B}(p_L(2)_+, p_2(1,1)) \\ &= \{\frac{19}{3}c_{-2}c_{-1}L_{-2} + 9c_{-3}c_{-2} - \frac{19}{3}c_{-4}c_{-1}\}| - i\frac{8}{\sqrt{3}}\rangle_L \otimes |0\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad .\end{aligned} \tag{4.21}$$

Here we should note two important facts in the above process; the first is that the excitation of the Liouville fields does not appear at any steps and the second is that among the ambiguities associated with the action δ^{-1} only the final one is relevant to the expression of the physical states. In the case of (4.21) we can choose a representative for $|\tilde{\mathcal{O}}_{11}\rangle$ utilizing this ambiguity as

$$|\tilde{\mathcal{O}}_{11}\rangle = (c_{-3}c_{-2} - \frac{1}{3}c_{-4}c_{-1})| - i\frac{8}{\sqrt{3}}\rangle_L \otimes |0\rangle_M \otimes c_1|0\rangle_{\rm gh} \quad . \tag{4.22}$$

As apparent from the above example, all physical states can be described by the

polynomials $P(\{c_{-m}\})$. Therefore the physical states live in the subspace $|p_L\rangle \otimes |p_0\rangle \otimes \wedge_c$ where $\wedge_c \equiv \sum_{\{n_1,\cdots,n_k\}} \mathbf{C} c_{-n_1} \cdots c_{-n_k}(c_1|0\rangle_{gh})$. Since the BAST operator turns out to act in this subspace effectively as an operator

$$d = -\frac{1}{2} \sum_{n,m>0} (n-m) c_{-n} c_{-m} b_{n+m} \quad , \tag{4.23}$$

the cohomology is naturally embedded into a simpler cohomology $H^*(\wedge_c, d)$. One should note that the cohomology $H^*(\wedge_c, d)$ is nothing but the standard Lie algebra cohomology [17] $H^*(\wedge\mathcal{L}_+, \mathbf{C})$ with trivial coefficient $\mathbf{C}$. Where $\mathcal{L}_+$ means a subalgebra, spanned by the generators with positive conformal dimensions of the Virasoro algebra $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-$. The embedding is in fact an isomorphism.

All physical states with $\eta(p_L) > 0$ (, anti-Seiberg states,) have representatives in $H^*(\wedge\mathcal{L}_+, \mathbf{C})$ and thus

$$\sum_{t\in\mathbf{Z}} H^*(\mathcal{F}_L(p_L(t)_+) \otimes \mathbf{C} \otimes \wedge_{bc}, \hat{Q}_B) \cong H^*(\wedge\mathcal{L}_+, \mathbf{C}) \quad . \tag{4.24}$$

The problem to calculate the cohomology $H^*(\wedge\mathcal{L}_+, \mathbf{C})$ was first posed by I.M. Gel'fand at the 1970 mathematical congress, at the early stage of the representation theory of the Virasoro algebra, and subsequently it was solved by Goncharova and Gel'fand, Feigin and Fuchs[26]. To describe their results, let us define the coherent state representation for a state $|\Psi\rangle = \sum a(n_1, \cdots, n_s) c_{-n_1} \cdots c_{-n_s}(c_1|0\rangle_{gh})$ with a fixed ghost number s by

$$\Psi(z_1, \cdots, z_s) \equiv \langle\tilde{0}|(\sum_{n_1=0}^{\infty} z_1^{n_1} b_{n_1}) \cdots (\sum_{n_s=0}^{\infty} z_s^{n_s} b_{n_s})|\Psi\rangle \quad , \tag{4.25}$$

where $\langle\tilde{0}|$ means $(c_1|0\rangle_{gh})^\dagger = \langle 0|b_{-1}$ (, note that the definition of the dagger operation for our harmonic theory is unconventional here). It is an easy task to recover the form of $|\Psi\rangle$ from the totally anti-symmetric function $\Psi(z_1, \cdots, z_s)$.

Within this representation of states Gel'fand *et al* obtained the all harmonic cocycles in $H^*(\wedge\mathcal{L}_+, \mathbf{C})$ as $z_1 \cdots z_q \Pi_q^3$ and $z_1^2 \cdots z_q^2 \Pi_q^3$ where $\Pi_q \equiv \Pi_{1\leq i<j\leq q}(z_i - z_j)$ and $q \in \mathbf{Z}_{>0}$. Using their results, we can express all physical states by

$$\tilde{\mathcal{O}} = \begin{cases} |p_L(s)_+\rangle \otimes z_1 \cdots z_s \Pi_s^3 & (s \geq 0) \ ; \\ |p_L(s)_+\rangle \otimes z_1^2 \cdots z_{-s}^2 \Pi_{-s}^3 & (s < 0) \ . \end{cases} \tag{4.26}$$

In table 1, we list some of the observables in the form of operators using the state and the operator correspondence.

5. DISCUSSIONS

BRST analysis of two dimensional gravity tells us that there are infinitely many physical states with non-vanishing ghost numbers. On these non- standard ghost numbers, we have shown in this paper that c-ghosts and b-ghosts are attached to the macroscopic loops and microscopic loops, respectively. To reveal the role of such ghost fields must be important to our deeper insight into the whole theory. As such a possible attempt within our approach we discuss the consistency of local operator algebra of microscopic loops with those in the Schwinger-Dyson equation of the matrix model paying attention to the ghost numbers and the gravitational scaling dimensions of physical states. We will study this for two examples ,i.e., pure gravity (c=0) and Ising model coupled to gravity ($c = \frac{1}{2}$). As a result a part of correspondence of both theories is indeed observed although some appropriate choices of cohomology elements will be required.

So far we have only considered the relative BRST cohomology. In the present case the relation between the relative cohomology and the absolute one is simple and is given by [14]

$$\begin{aligned} H^*_{abs}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc}) &= H^*_{rel}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc}) \\ &\oplus c_0 H^{*-1}_{rel}(\mathcal{F}_L(p_L) \otimes Vir(c_{pq}, \Delta_{r,s}) \otimes \Lambda_{bc}) \ . \end{aligned} \tag{5.1}$$

In order to relate the matrix model to Liouville theory, let us recall the definition of the gravitational scaling dimension [9] of some spinless operator Φ_Δ with conformal

dimension Δ

$$\text{gdim } \Phi_\Delta \equiv 1 - \frac{p+q-|2pqt \pm pr + qs|}{p+q-1} \quad , \tag{5.2}$$

where + sign for a_t and − sign for b_t. Now define $A_t \equiv |2pqt + pr + qs|$, $B_t \equiv |2pqt - pr + qs|$ and denote the local operators of Liouville theory as $\mathcal{O}_{A_t}$, $\mathcal{O}_{B_t}$ which just correspond to the microscopic operators associated to the generalized KdV flows [2] in the matrix model.

(i) pure gravity (c=0)

In this case there are one puncture operator P and its gravitational descendants whose scaling dimensions are $A_t = |12t + 5|$ and $B_t = |12t + 1|$. It is convenient to divide the local operators $\{\mathcal{O}_{A_t}, \mathcal{O}_{B_t}\}$ which belong to the relative cohomology into two sequences: α-sequence $\{\mathcal{O}_{6k+1}\}_{k\geq 0}$ and β-sequence $\{\mathcal{O}_{6k-1}\}_{k\geq 1}$ where both $\mathcal{O}_{6k+1}$ and $\mathcal{O}_{6k-1}$ have the same ghost number $-k$. If we assume the local operator algebra $\mathcal{O}_p\mathcal{O}_q \sim \mathcal{O}_{p+q-3}$ from the Schwinger-Dyson equation, then the following relations are expected to hold

$$\alpha \times \alpha \sim \beta \,, \quad \beta \times \beta \sim \alpha \quad \alpha \times \beta \sim 0 \quad . \tag{5.3}$$

where the last relation means that the products of operators in α- sequence and those in β-sequence produce the the boundary operators[27] which do not appear in our present approach. The first relation is consistent with the ghost number conservation whereas the second one is not consistent with it. In fact for $\beta \in \mathcal{O}_{6k-1}$, $\mathcal{O}_{6k'-1}$

$$\mathcal{O}_{6k-1}\mathcal{O}_{6k'-1} \sim \mathcal{O}_{6(k+k'-1)+1} \quad , \tag{5.4}$$

and the ghost number of l.h.s. $(-k-k')$ does not coincides with that of r.h.s. $(-k-k'+1)$. However we note that, in the sense of the absolute cohomology (5.1), $\partial c\mathcal{O}_l$ also has a meaning as a BRST invariant operator with the same scaling dimension as $\mathcal{O}_l$. Therefore if we assign c_0 to either of the two operators in the l.h.s. of (5.4), then the ghost number conservation can be recovered.

(ii) Ising model coupled to gravity ($c = \frac{1}{2}$)

There are two gravitational primaries P, and Q in this case[28]. For convenience we will arrange the local operators as follows:

$$\begin{aligned} &P_1 - \text{sequence } \{\mathcal{O}_{12k-5}(P)\}_{k\geq 1}, \quad P_2 - \text{sequence } \{\mathcal{O}_{12k+1}(P)\}_{k\geq 0} \\ &P_3 - \text{sequence } \{\mathcal{O}_{12k-2}(P)\}_{k\geq 1}, \quad Q_1 - \text{sequence } \{\mathcal{O}_{12k+5}(Q)\}_{k\geq 0} \\ &Q_2 - \text{sequence } \{\mathcal{O}_{12k-1}(Q)\}_{k\geq 1}, \quad Q_3 - \text{sequence } \{\mathcal{O}_{12k+2}(Q)\}_{k\geq 0} \end{aligned} \tag{5.5}$$

where each representative has the ghost number $-k$. We can easily confirm that these sequences satisfy the following relations from the Schwinger-Dyson equation,

$$\mathcal{O}_p(P)\mathcal{O}_q(P) \sim \mathcal{O}_{p+q-1}(P), \quad \mathcal{O}_p(P)\mathcal{O}_q(Q) \sim \mathcal{O}_{p+q-1}(Q) \quad , \tag{5.6}$$

and

$$\mathcal{O}_p(Q)\mathcal{O}_q(Q)\mathcal{O}_r(Q) \sim \mathcal{O}_{p+q+r-8}(P), \quad \mathcal{O}_p(Q)\mathcal{O}_q(P)\mathcal{O}_r(P) \sim \mathcal{O}_{p+q+r-8}(Q) \quad . \tag{5.7}$$

Note that we have the local operator algebras (5.7) when three operators coincide as well as the algebras (5.6). There are several cases in which the ghost number conservation is violated for the relative cohomology elements. However as in the case of pure gravity it is possible to recover the ghost number conservation if we replace some appropriate operators $\mathcal{O}_l$ by the corresponding operators $\partial c\mathcal{O}_l$ in the absolute cohomology $\partial c\mathcal{O}_l$. It is expected that the similar discussion is also applicable to the general matter case.

Finally let us briefly refer to the implication of our results in comparison with the Seiberg's analysis on Quantum Liouville theory[29]. Within the minisuperspace approximation Seiberg showed the existence of two kinds of states ,i.e., macroscopic loops ($\eta(p_L) > 0$) and microscopic loops ($\eta(p_L) < 0$)) by using Hartle-Hawking like constructions. Macroscopic loops are not local operators and correspond to normalizable wave functions while microscopic loops are local operators and correspond to non-normalizable wave functions. As we have seen in this paper the same classification by Seiberg naturally arises in our BRST analysis and that if restricted to ghost sector, macroscopic loops contain c-ghosts only and in contrast microscopic loops contain b-ghosts only. These correspondences might possibly suggest the new aspect of the different geometrical and physical roles of ghost systems.

APPENDIX:

The state which appears in eq.(4.10);

$$
\begin{aligned}
|\Psi_{(-1,-1)}\rangle = \{ & \frac{7}{6}b_{-4} + b_{-3}(\frac{1}{12}K_{-1} - \frac{31}{12}L_{-1}) + b_{-2}(K_{-2} - L_{-2} - \frac{5}{3}K_{-1}L_{-1}) \\
& + b_{-1}(\frac{31}{12}K_{-3} + \frac{5}{3}K_{-2}K_{-1} + \frac{1}{4}K_{-1}^3 - \frac{1}{12}L_{-3} \\
& + \frac{5}{3}L_{-2}L_{-1} - \frac{1}{4}L_{-1}^3 + \frac{1}{4}K_{-1}L_{-1}^2 - \frac{1}{4}K_{-1}^2L_{-1}) \\
& + A\hat{Q}_B(-\frac{1}{2}b_{-3}b_{-1} + b_{-2}b_{-1}L_{-1}) \\
& + B\hat{Q}_B(2b_{-3}b_{-1} + b_{-2}b_{-1}K_{-1})\}|i\sqrt{3}\rangle_L \otimes |-\frac{2}{\sqrt{3}}\rangle_M \otimes c_1|0\rangle_{gh} \quad ,
\end{aligned}
$$

where A and B represent arbitrary constants.

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a)	$\tilde{\mathcal{O}}_1 = c e^{\sqrt{3}\phi_L}$; $\tilde{\mathcal{O}}_5 = e^{\frac{5}{\sqrt{3}}\phi_L} c\partial^2 c$; $\tilde{\mathcal{O}}_7 = e^{2\sqrt{3}\phi_L} c\partial^3 c$
	$\tilde{\mathcal{O}}_{11} = e^{\frac{8}{\sqrt{3}}\phi_L}(c\partial^3 c\partial^4 c - \frac{1}{5}c\partial^2 c\partial^5 c)$; $\tilde{\mathcal{O}}_{13} = e^{3\sqrt{3}\phi_L}(c\partial^4 c\partial^5 c - \frac{2}{9}c\partial^3 c\partial^6 c)$

b)	$\mathcal{O}_1 = c e^{\frac{2}{\sqrt{3}}\phi_L}$; $\mathcal{O}_5 = 1$; $\mathcal{O}_7 =: (bc + \frac{3}{2}\mathcal{K}_{-1}) e^{-\frac{\sqrt{3}}{3}\phi_L} :$
	$\mathcal{O}_{11} =: \{b(\mathcal{K}_{-2} + \frac{1}{6}\mathcal{K}_{-1}^2) - \frac{1}{6}\partial b\mathcal{K}_{-1} + \frac{1}{6}\partial^2 b - \partial bbc\} e^{-\sqrt{3}\phi_L} :$
	$\mathcal{O}_{13} =: \{b(\mathcal{K}_{-4} + \frac{71}{184}\mathcal{K}_{-3}\mathcal{K}_{-1} + \frac{136}{483}\mathcal{K}_{-2}^2 + \frac{85}{644}\mathcal{K}_{-2}\mathcal{K}_{-1}^2 + \frac{51}{5152}\mathcal{K}_{-1}^4)$ $-\frac{17}{56}\partial b(\mathcal{K}_{-3} + \frac{106}{391}\mathcal{K}_{-2}\mathcal{K}_{-1} + \frac{3}{92}\mathcal{K}_{-1}^3) + \frac{3}{28}\partial^2 b(\mathcal{K}_{-2} + \frac{17}{184}\mathcal{K}_{-1}^2)$ $-\frac{25}{15456}\partial^3 b\mathcal{K}_{-1} + \frac{89}{15456}\partial^4 b - \frac{136}{483}\partial bbc(\mathcal{K}_{-2} + \frac{3}{17}\mathcal{K}_{-1}^2)$ $-\frac{4}{161}\partial^2 bbc\mathcal{K}_{-1} - \frac{349}{5796}\partial^3 bbc + \frac{11}{56}\partial^2 b\partial bc\} e^{-\frac{4\sqrt{3}}{3}\phi_L} :$

Table 1; List of the observables:

$\tilde{\mathcal{O}}_l$ and $\mathcal{O}_l$ correspond to the anti-Seiberg states and Seiberg states, respectively. The subscripts indicate the KdV flow which these operators are assumed to generate under the correspondence to the matrix model.

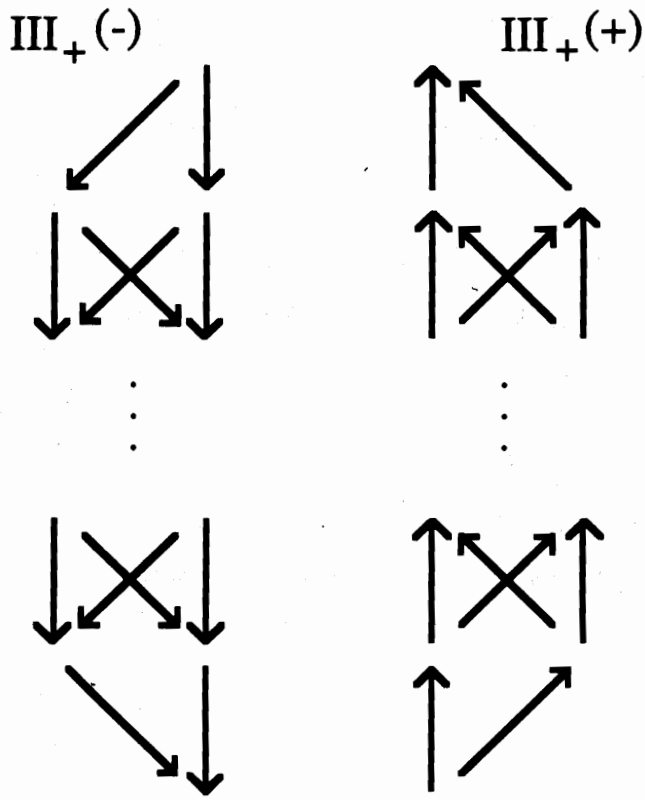

Fig.1. Structure of the Fock module for the Liouville part.