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by

S. R. Beane and Charles B. Chiu

Center for Particle Physics
University of Texas at Austin
Austin, Texas 78712

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S. R. Beane and C. B. Chiu

Center for Particle Physics
and Department of Physics
The University of Texas at Austin,
Austin, TX 78712

ABSTRACT

We solve a relativistic Hamiltonian theory where we treat the amplitudes of definite isospin and angular momentum obtained from a chiral $SU(2) \times SU(2)$ invariant Lagrangian as separable potentials. We find that the constraint of quantum mechanical completeness, which is built into the Hamiltonian model, finds a natural expression in well known Lorentz-invariant and crossing-symmetric sum rules. In particular, by way of the Adler sum rule, we show that the lowest order effective potentials yield a solution for π-π scattering that treats the I=0 amplitude in an inconsistent way. By including the next-to-leading order contributions in the effective potentials, we find solutions that satisfy the Adler sum rule and in the process fix the crossed-channel renormalization constants to values that are consistent with experiment. As a more speculative application, we consider the line shape of the Higgs resonance in the standard model, and the modifications that result from the presence of a a techni-rho meson. This formalism provides a powerful parametrization of Goldstone boson scattering in the presence of resonances. In a more general sense, the constraints implied by completeness sum rules serve to delimit the class of models that yield a consistent parametrization of Goldstone Boson scattering in the resonance region. We consider the method of Padé approximants as an example of a model that fails to satisfy these constraints.

1. Introduction

In the limit where the u and d quark masses are set equal to zero, the Q. C. D. Lagrangian is invariant under the chiral group $SU(2) \times SU(2)$. This symmetry is assumed to be spontaneously broken by the Q. C. D. vacuum to the diagonal subgroup of isospin transformations. The resulting three pseudoscalar Goldstone bosons are identified with the pions. Fortunately, the strong interaction dynamics of the pions is accessible to an expansion in powers of momenta and quark masses. This expansion can be obtained in a systematic and elegant fashion from an effective Lagrangian involving only the pion fields [1]. The non-renormalizable character of the Lagrangian is manifest in the need to introduce new undetermined parameters at each non-leading order in the expansion. These scale dependent parameters are unconstrained by chiral symmetry and represent the low energy effect of the underlying theory. Presumably they depend on the quark masses and the Q. C. D. scale, and in principle can be derived from the Q. C. D. Lagrangian. However, no one knows how to do this and so these parameters are obtained phenomenologically near threshold [2,3]. This method constitutes a powerful test of chiral symmetry.

In the resonance region, for obvious reasons, it is no longer meaningful to keep only the first few terms in the chiral expansion. This poses no apparent setback since one can couple any fields to the pions in a chiral invariant way [4]. However, these fields appear in the scattering amplitudes as zero width particles, and the couplings of these fields to the pions, which are related to the decay width, are in general unknown. Furthermore, it is not clear how to generate imaginary parts of propagators associated with higher order loop diagrams within chiral perturbation theory. These difficulties can be circumvented by using a simple unitarization scheme, however, as will be seen below, this will not always yield the correct width. If the resonance is integrated out, its effect is felt at low energies through the renormalization constants associated with loop diagrams involving Goldstone bosons alone. In the case of π-π scattering, the low energy constants are completely saturated by ρ exchange [2,5,6]. This is reminiscent of vector meson dominance [7] and has been refered to as "chiral duality" in ref. [6]. In general, one expects the resonance with the lowest mass to have the greatest effect near threshold. Of course, the resonance spectrum itself is

not fixed by chiral symmetry; although the Lagrangian incorporates all of the symmetries that the S-matrix is known to satisfy, the particle content is unconstrained.

There have been many attempts to go beyond chiral perturbation theory and make definite statements regarding the resonance spectrum [8-13]. Recent interest in this approach is in part due to the "equivalence theorem" which relates the Goldstone boson self-interactions with the interactions of the longitudinal components of the standard model gauge bosons [14]. In particular, the Higgs boson should occur in the custodial-isospin $I = 0$ S-wave channel of longitudinal gauge boson scattering. The complete lack of experimental data in this energy region renders the effective Lagrangian approach academic. Moreover, there is reason to believe that, even if there is an observed enhancement in the yield of longitudinal gauge boson pairs in the energy region that will be probed by the SSC, it will prove difficult to make use of chiral Lagrangians in relating the data to an underlying theory [15].

An alternative approach to the Goldstone boson interactions in the resonance region involves using a chiral Lagrangian to calculate an effective potential, rather than the S-matrix itself. The effective potential is subjected to a model dependent prescription which generates the S-matrix. In this paper we treat the partial wave amplitudes of definite isospin, obtained from a chiral Lagrangian, as effective potentials. We then obtain the scattering amplitudes by solving a Schrödinger equation for a relativistic Hamiltonian (R.H.) [16]. In particular, we consider a separable potential model [17] where we replace the energy variable by the center-of-mass energy-squared, s. The positions of the poles of the scattering amplitudes are determined by the renormalization constants associated with subtracted dispersion integrals. We assume the absence of exotic resonances and therefore focus our attention on the I=0 and I=1 channels. We work in the chiral limit since we are particularly interested in physics at the ρ scale.

For a tree-level effective potential, the $I = 1$ amplitude that we derive is identical to the Brown and Goble amplitude [8] obtained by considering an effective range expansion about the current algebra low-energy theorem. It automatically yields the KSFR relation [18] for the ρ width which is in good agreement with experiment. We find that there are distant tachyon poles in both the I=0 and I=1 amplitudes, however we show that these occur beyond the scale at which our effective theory breaks down [19].

In relativistic field theories, the use of completeness sum rules is complicated by the fact that the asymptotic behaviour of cross sections renders most dispersion integrals divergent, thus requiring the introduction of undetermined subtraction constants. However, arguments based on Regge pole theory allow the derivation of a small number of sum rules [20,21] for a given process, based on the properties of the leading Regge trajectories that participate in that process. In particular, we find that for the R.H. amplitudes with tree-level effective potentials, the Adler sum rule [22] is not satisfied with the ρ and ϵ contributions in the resonance-saturation approximation. We show why this is the case by rearranging the Adler sum rule into sum rules for the direct-channel isospin amplitudes. These sum rules display the direct- and crossed-channel resonance-exchange contributions separately. We find that if one assumes, as in the current algebra derivation of the KSFR relation, that crossed-channel contributions to the $I = 1$ amplitude are absent, then it is inconsistent to assume the same for $I = 0$. This suggests that the tree-level $I = 0$ effective potential is inadequate.

In order to overcome the difficulties inherent to the tree-level case, we construct the R.H. amplitudes with one loop effective potentials [16]. We find that the ρ and the ϵ widths are modified by the crossed-channel corrections. We fix the renormalization constants that determine the strength of these corrections by requiring the Adler sum rule to be satisfied. The resulting parametrization of the I=0 and I=1 π-π scattering amplitudes depends only on the ρ and ϵ masses, and is in good agreement with experiment.

The sum rule analysis is completely general, and can be applied to any $SU(2) \times SU(2)$ symmetric theory. Of course, the analysis must be supplemented with assumptions concerning the resonance spectrum. The familiar isomorphism between the π-π interactions, and the interactions of the longitudinal components of the standard model gauge bosons [14] suggests an interesting application for the sum rule analysis. Specifically, we consider the deviations from the Higgs line shape due to the presence of a techni-rho resonance, which presumably would exist in Q.C.D.-like theories such as Technicolor.

Several recent papers have critiqued the Brown and Goble parametrization and claimed that the method of Padé approximants is a superior unitarization scheme since it includes crossed-channel contributions [12]. We show, by way of our sum rule analysis, that the Padé amplitudes do not include the crossed-channel contributions in the proper way.

The paper is organized as follows. In sect. 2 we obtain the effective potentials from a chiral Lagrangian. In sect. 3 we introduce the separable potential model, and explicitly calculate the scattering amplitude for an arbitrary potential. We then consider the special case of a tree-level effective potential for π-π scattering in sect. 4. In sect. 5 we construct completeness sum rules for fully crossing symmetric amplitudes and consider the constraints that these sum rules impose on the R.H. model π-π amplitudes. In sect. 6 we construct the amplitudes with 1-loop order effective potentials and again consider the special case of π-π scattering. Sect. 7 applies the results of sect. 6 to the scenario of a strongly interacting Higgs sector. In sect. 8 we make use of our sum rule analysis to critique the method of Padé approximants. Finally, in sect. 9 we summarize our results and state what has been learned by way of this analysis. Our π-π conventions are given in an appendix.

2. Effective Lagrangian

In the chiral limit, the lowest order term in the effective Lagrangian is the non-linear sigma model

$$\mathcal{L}^{(2)} = \frac{{F_\pi}^2}{4}\mathrm{Tr}\left(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right), \tag{2.1}$$

where

$$\Sigma = exp\left(\frac{i\vec{\tau}\cdot\vec{\pi}}{F_\pi}\right) \tag{2.2}$$

and has the transformation property

$$\Sigma \rightarrow g_L \Sigma g_R^\dagger \quad ; \quad g_{L,R} \in SU_{L,R}(2) \tag{2.3}$$

under $SU_L(2) \times SU_R(2)$. This lowest order term reproduces the current algebra low-energy theorem [4]. There are no undetermined parameters at this order since all theories invariant under $SU_L(2) \times SU_R(2)$ must posses this term. At next-to-leading order there are two additional terms,

$$\mathcal{L}^{(4)} = C_1 \operatorname{Tr}\left(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right) \operatorname{Tr}\left(\partial_\nu \Sigma \partial^\nu \Sigma^\dagger\right) + C_2 \operatorname{Tr}\left(\partial_\mu \Sigma \partial_\nu \Sigma^\dagger\right) \operatorname{Tr}\left(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger\right), \tag{2.4}$$

where C_1 and C_2 are unknown constants which characterize the underlying theory at low energy. According to the usual power counting rules [1], the contributions of order E^4 are obtained from loop graphs using $\mathcal{L}^{(2)}$ and tree graphs using $\mathcal{L}^{(4)}$. However, since the unknown constants are associated with the real part of the amplitude, we can just as easily dispense with $\mathcal{L}^{(4)}$ and use $\mathcal{L}^{(2)}$ together with unitarity and subtracted dispersion relations to obtain $A^{(4)}(s,t,u)$ [9]. In that case, C_1 and C_2 take the form of subtraction constants. A direct Feynman diagram calculation using $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ yields [1] (see Appendix for conventions)

$$\begin{aligned} A(s,t,u) = \frac{s}{{F_\pi}^2} + \frac{4}{{F_\pi}^4}[2C_1(\mu^2)\, s^2 + C_2(\mu^2)\,(t^2+u^2)] \quad + \frac{1}{(4\pi)^2\, {F_\pi}^4}[\frac{-s^2}{2}\, log\left(\frac{-s}{\mu^2}\right) \\ - \frac{1}{12}\left(3t^2+u^2-s^2\right)\log\left(\frac{-t}{\mu^2}\right) - \frac{1}{12}\left(3u^2+t^2-s^2\right)\log\left(\frac{-u}{\mu^2}\right)]. \end{aligned} \tag{2.5}$$

The logarithmic divergences resulting from loop diagrams involving $\mathcal{L}^{(2)}$ have been absorbed into a scale dependence μ for C_1 and C_2. In particular,

$$C_1(\mu^2) = C_1(\nu^2) - \frac{1}{24\,(4\pi^2)}\log\left(\frac{\mu^2}{\nu^2}\right) \tag{2.6}$$

$$C_2(\mu^2) = C_2(\nu^2) - \frac{1}{12\,(4\pi^2)}\log\left(\frac{\mu^2}{\nu^2}\right). \tag{2.7}$$

For later use we project out the amplitudes of definite isospin and angular momentum, a_{IJ}. We find,

$$\begin{aligned} a_{00}(s) &= \frac{s}{16\pi {F_\pi}^2} + \frac{s^2}{64\pi {F_\pi}^4}\left[24\,(2C_1+C_2) - \frac{1}{4\pi^2}\log\left(\frac{-s}{\mu^2}\right)\right] \\ &\quad + \frac{s^2}{64\pi {F_\pi}^4}\left[\frac{8}{3}\,(4C_1+5C_2) - \frac{1}{4\pi^2}\left(\frac{7}{18}\log\left(\frac{s}{\mu^2}\right) - \frac{11}{108}\right)\right]; \quad (2.8a) \\ a_{11}(s) &= \frac{s}{96\pi {F_\pi}^2} + \frac{s^2}{96\pi {F_\pi}^4}\left[2C_2 - \frac{1}{6\,(4\pi)^2}\log\left(\frac{-s}{\mu^2}\right)\right] \end{aligned}$$

$$+\frac{s^2}{96\pi {F_\pi}^4}\left[2\left(C_2-4C_1\right)+\frac{1}{4\pi^2}\left(\frac{1}{24}\log\left(\frac{s}{\mu^2}\right)-\frac{1}{72}\right)\right]; \qquad (2.8b)$$

$$a_{20}\left(s\right)=\frac{-s}{32\pi {F_\pi}^2}+\frac{s^2}{64\pi {F_\pi}^4}\left[12C_2-\frac{1}{\left(4\pi\right)^2}\log\left(\frac{-s}{\mu^2}\right)\right]$$
$$+\frac{s^2}{64\pi {F_\pi}^4}\left[\frac{4}{3}\left(8C_1+7C_2\right)-\frac{1}{4\pi^2}\left(\frac{11}{36}\log\left(\frac{s}{\mu^2}\right)-\frac{25}{216}\right)\right], \qquad (2.8c)$$

where we have decomposed the amplitudes according to the following prescription :

$$a_{IJ}\left(s\right)=a_{IJ}^{(2)}\left(s\right)+a_{IJ}^{(4)D}\left(s\right)+a_{IJ}^{(4)C}\left(s\right). \qquad (2.9)$$

The second and third terms are contributions from direct- and crossed-channel loop diagrams respectively. We note that due to a cancellation between the direct- and crossed-channel loop contributions, no logarithm appears in the I=J=1 amplitude to this order.

3. A Relativistic Hamiltonian Model with Separable Potentials

In this section we recall the Relativistic Hamiltonian model of ref. [16] for the two-Goldstone boson interaction. Since total angular momentum and isospin are conserved, the various spin-isospin channels do not mix with each other. For the two-Goldstone Boson system the problem thus reduces to a single channel problem which can be treated in terms of a series of "Hamiltonia" with the states labeled by the center-of-mass energy-squared, s. We will employ a separable potential with an interaction obtained from chiral perturbation theory. We choose the separable potential for two reasons : first, the structure of the wave functions and the scattering amplitudes is relatively simple [17,23] even with the natural constraint of unitarity; second, no generality is lost since any one channel scattering amplitude can be reproduced by a suitable separable potential [24].

The system consists of two fields associated with N and θ particles. The Hamiltonian is given by

$$H=\int dssa^\dagger\left(s\right)a\left(s\right)\mp\int dsg\left(s\right)N^\dagger a^\dagger\left(s\right)\int ds'g\left(s'\right)Na\left(s'\right) \qquad (3.1)$$

where without loss of generality, $g(s)$ may be chosen real and positive and the $-$ or $+$ signs designate the potential to be respectively attractive or repulsive. The theory decomposes

into a countable set of distinct sectors labeled by the $N-\theta$ content. The matrix element of the Hamiltonian between the one N-one θ sector is given by

$$\langle N\theta_s|H|N\theta_{s'}\rangle = s\delta\left(s-s'\right) \mp g\left(s\right) g\left(s'\right). \tag{3.2}$$

The continuum wavefunctions are determined through the eigenvalue equation

$$\langle N\theta_s|H|\gamma\rangle = \gamma\langle N\theta_s|\gamma\rangle \equiv \gamma\phi_\gamma\left(s\right). \tag{3.3}$$

Explicitly evaluating this matrix element yields

$$\left(\gamma-s\right)\phi_\gamma\left(s\right) = \mp g\left(s\right)\int ds' g(s')\phi_\gamma(s'). \tag{3.4}$$

The "in" solutions are

$$\phi_\gamma\left(s\right) = \delta\left(\gamma-s\right) \mp \frac{g\left(s\right)}{\gamma-s+i\epsilon}\int ds' g(s')\phi_\gamma(s'). \tag{3.5}$$

A series of algebraic manipulations determines the integral on the right hand side so that

$$\phi_\gamma\left(s\right) = \delta\left(\gamma-s\right) \mp \frac{g\left(s\right)g\left(\gamma\right)}{\gamma-s+i\epsilon}\cdot\frac{1}{1\mp\int\frac{g^2(s')ds'}{\gamma-s'+i\epsilon}}. \tag{3.6}$$

These scattering-state wavefunctions, together with the wavefunction of a bound state, if present, satisfy an appropriate orthonormality relation and are complete. The scattering amplitude is given by [17]

$$t\left(s\right) = -\pi\left(\gamma-s\right)\phi_\gamma\left(s\right)|_{s=\gamma} = \frac{\pm\pi g^2\left(s\right)}{1\pm\int\frac{g^2(s')ds'}{s-s'+i\epsilon}}. \tag{3.7}$$

4. Renormalization and Particle Content

We define our effective potential as

$$\tilde{V}(s) \equiv \pm\pi g^2(s). \tag{4.1}$$

For a given $\tilde{V}(s)$, we obtain an "exact" scattering amplitude $t(s)$. In order to avoid double counting, our effective potential consists of all direct-channel two-particle irreducible diagrams in chiral perturbation theory. The effective potential, or numerator function, includes all contributions associated with the left-hand discontinuity. To lowest order, the effective potential is given by the current algebra result. Say

$$\tilde{V}(s) = a_{IJ}^{(2)}(s) = \alpha_{IJ} s \tag{4.2}$$

with

$$\alpha_{00} \equiv \frac{1}{16\pi {F_\pi}^2}, \quad and \quad \alpha_{11} \equiv \frac{1}{96\pi {F_\pi}^2}. \tag{4.3}$$

Then

$$t_{IJ}(s) = \frac{\alpha_{IJ} s}{1 - \frac{\alpha_{IJ}}{\pi}\int_0^\infty \frac{ds' s'}{s'-s-i\epsilon}}. \tag{4.4}$$

The dispersion integral is clearly undefined. It requires some form of regularization. Define

$$I_{IJ}(s) = \frac{\alpha_{IJ}}{\pi}\int_0^\infty \frac{ds' s'}{s'-s-i\epsilon} + counterterms. \tag{4.5}$$

With two subtractions[1] we obtain

$$I_{IJ}(s) = \frac{\alpha_{IJ}}{\pi}\left(\int_0^\infty \frac{ds' s'(s-s_0)^2}{(s'-s)(s'-s_0)^2} + P(s)\right) \tag{4.6}$$

where $P(s)$ is a polynomial of the first degree. Integrating and noting that s_0 is arbitrary and not independent of $P(s)$ yields

1 The subtractions take place at the same point s_0 for the amplitude and its first derivative.

$$I_{IJ}(s) = -\frac{\alpha_{IJ}}{\pi}\left(s\left(\log(\frac{-s}{\mu^2})\right) + \tilde{P}\left(s;\mu^2\right)\right) \tag{4.7}$$

where

$$\tilde{P}\left(s;\mu^2\right) \equiv R_{IJ}\left(\mu^2\right)s + b_{IJ}. \tag{4.8}$$

We can absorb b_{IJ} into a redefinition of the pion decay constant,

$$F_{\pi R}^2 = F_\pi^2\left(1 + \frac{\alpha_{IJ}b_{IJ}}{\pi}\right). \tag{4.9}$$

However, we set $b_{IJ} = 0$ since quadratically divergent loop integrals involving massless particles vanish in dimensional regularization. That is, in chiral perturbation theory with the Goldstone bosons alone, the pion decay constant is not renormalized. It follows that

$$t_{IJ}(s) = \frac{\alpha_{IJ}s}{1 + \frac{\alpha_{IJ}s}{\pi}\left[\log(\frac{-s}{\mu^2}) + R_{IJ}\left(\mu^2\right)\right]} \tag{4.10}$$

where independence of the amplitude on the choice of scale requires

$$R_{IJ}\left(\mu^2\right) = R_{IJ}\left(\nu^2\right) + \log\left(\frac{\mu^2}{\nu^2}\right). \tag{4.11}$$

The appearance of an undetermined parameter is a reflection of the non-renormalizability of the chiral Lagrangian from which we derived our effective potential. We note that the absence of the left-hand discontinuity in the tree-level effective potential implies a violation of crossing symmetry [25].

At this point we must say something about the spectrum of the specific system that we are considering. We specialize to the π-π system, and we assume the existence of a ρ-meson in the I=1 P-wave channel, and an ϵ-meson in the I=0 S-wave channel. We choose our renormalization point such that the resonance occurs "near" $s = \mu^2$ (i.e. when the narrow width approximation is valid). Defining

$$D_{IJ}(s) \equiv 1 + \frac{\alpha_{IJ}s}{\pi}\left[\log(\frac{-s}{\mu^2}) + R_{IJ}\left(\mu^2\right)\right], \tag{4.12}$$

we assume that at $s = \mu^2$, $ReD_{IJ}(s) = 0$, which gives

$$R_{IJ}\left(\mu^2\right) = \frac{-\pi}{\alpha_{IJ}\mu^2}, \tag{4.13}$$

and

$$D_{IJ}\left(s\right) = 1 - \frac{s}{\mu^2} + \frac{\alpha_{IJ}s}{\pi}\log(\frac{-s}{\mu^2}). \tag{4.14}$$

Finally, we obtain

$$D_{00}\left(s\right) = 1 - \frac{s}{{m_\epsilon}^2} + \frac{s}{16\pi^2{F_\pi}^2}\log\left(\frac{-s}{{m_\epsilon}^2}\right), \tag{4.15}$$

and

$$D_{11}\left(s\right) = 1 - \frac{s}{{m_\rho}^2} + \frac{s}{96\pi^2{F_\pi}^2}\log\left(\frac{-s}{{m_\rho}^2}\right). \tag{4.16}$$

We define the resonance widths as

$$\frac{d\delta_{IJ}\left(s\right)}{ds}|_{s=m^2} = \frac{1}{\Gamma m} \tag{4.17}$$

from which we obtain

$$\Gamma = \alpha_{IJ}m^3\left(1 - \frac{\alpha_{IJ}m^2}{\pi}\right)^{-1} \simeq \alpha_{IJ}m^3 \tag{4.18}$$

where the last approximation follows when the narrow-width approximation is valid. For the ρ we obtain

$$\Gamma_\rho = \frac{m_\rho^3}{96\pi{F_\pi}^2}. \tag{4.19}$$

This relation, known as the KSFR relation for the ρ width, was first derived using current algebra methods [18] in a calculation that implicitly assumes that left-hand cut contributions to the I=J=1 amplitude are negligible [26].

If we also treat the ϵ in the narrow-width approximation, we find

$$\Gamma_\epsilon = \frac{m_\epsilon^3}{16\pi{F_\pi}^2}. \tag{4.20}$$

There is a broad enhancement in the I=J=0 phase shift in the vicinity of the ρ. However, an ϵ this broad should not be amenable to a narrow-width treatment. Nevertheless, we

continue to use the narrow-width approximation and hope that at least our qualitative conclusions remain unaffected. This treatment is in the spirit of Q. C. D. in the large-N_c-limit [27], where elastic meson-meson scattering is given by a sum of tree diagrams involving the exchange of zero width mesons [28,29].

An interesting feature of our amplitudes is the existence of poles at euclidean momenta, which we refer to as tachyons. There are distant tachyon poles in both amplitudes. Following the conventional approach [19], we define a cutoff Λ, which we interpret as the scale at which our amplitudes cease to make sense as an effective theory. From Eq. (4.11) and Eq. (4.13), we obtain a renormalization group equation for the I=0 renormalization constant,

$$\frac{1}{m^2(\mu^2)} = \frac{1}{{m_\epsilon}^2} - \frac{1}{16\pi^2 {F_\pi}^2} \log\left(\frac{\mu^2}{{m_\epsilon}^2}\right). \tag{4.21}$$

Defining our cutoff as the scale at which $m^2(\mu^2)$ blows up, we find

$$\Lambda^2 = {m_\epsilon}^2 \exp\left(\frac{16\pi^2 {F_\pi}^2}{{m_\epsilon}^2}\right) \simeq (2.4 \quad GeV)^2 \tag{4.22}$$

where in the last approximation we have assumed that $m_\epsilon = m_\rho = 770 \quad MeV$, and $F_\pi = 93 \quad MeV$. On the other hand, from Eq. (4.15) we find that the position of the distant tachyon is given by

$$s_T = -{m_T}^2 \simeq -\Lambda^2. \tag{4.23}$$

Therefore, we regard the distant tachyon as outside the range of validity of our effective theory. A similar analysis follows for the I=1 amplitude. In that case we find

$$\Lambda^2 = {m_\rho}^2 \exp\left(\frac{96\pi^2 {F_\pi}^2}{{m_\rho}^2}\right) \simeq (770 \quad GeV)^2. \tag{4.24}$$

This difference of two orders of magnitude between the two cutoffs is related to physics near the threshold point: the I=1 low-energy theorem has a larger domain of validity than the I=0 low-energy theorem, as is made clear by the saturation of the unitarity bound [3].

5. Completeness Sum Rules

We recall that assuming the leading Regge trajectory for I=1 t-channel exchange has $\alpha_1(0) < 1$ leads to an unsubtracted dispersion relation for the I=1 t-channel amplitude. Evaluating at threshold and using the optical theorem yields the Adler sum rule [22],

$$\frac{1}{{F_\pi}^2} = \frac{1}{\pi}\int_0^\infty \frac{ds}{s}\left[\frac{2}{3}\sigma^0_{tot} + \sigma^1_{tot} - \frac{5}{3}\sigma^2_{tot}\right]. \tag{5.1}$$

In the resonance-saturation approximation [30], where we assume the absence of "exotic" I=2 resonances we find

$$\frac{1}{{F_\pi}^2} = \frac{32\pi\Gamma_\epsilon}{3m_\epsilon^3} + \frac{48\pi\Gamma_\rho}{m_\rho^3}. \tag{5.2}$$

This relation is not satisfied by the ρ and ϵ widths given by Eq. (4.19) and Eq. (4.20) respectively. The only other resonance with a significant coupling to the π-π channel is the $f_2(1270)$ which contributes about one third as much as the ρ to the Adler sum rule. However, inclusion of this contribution only serves to increase the discrepancy, so for now we will neglect it.

We can understand the failure of Eq. (4.19) and Eq. (4.20) to satisfy Eq. (5.2) if we decompose the Adler sum rule into algebraically equivalent sum rules for the direct-channel isospin amplitudes which display the direct- and crossed-channel contributions separately. With the help of the crossing matrix given in the appendix, we find for $I^{(s)}$=0 and $I^{(s)}$=1 respectively[2],

$$\frac{1}{{F_\pi}^2} = \frac{1}{\pi}\int_0^\infty \frac{ds}{s}\left[\sigma^0_{tot}\right] - \frac{1}{\pi}\int_0^\infty \frac{ds}{s}\left[\frac{1}{3}\sigma^0_{tot} - \sigma^1_{tot} + \frac{5}{3}\sigma^2_{tot}\right]; \tag{5.3}$$

$$\frac{1}{2{F_\pi}^2} = \frac{1}{\pi}\int_0^\infty \frac{ds}{s}\left[\sigma^1_{tot}\right] - \frac{1}{\pi}\int_0^\infty \frac{ds}{s}\left[-\frac{1}{3}\sigma^0_{tot} + \frac{1}{2}\sigma^1_{tot} + \frac{5}{6}\sigma^2_{tot}\right]. \tag{5.4}$$

Saturating these sum rules with ρ and ϵ yields

2 It is interesting to note that, within the resonance-saturation approximation, the same results follow if we assume that we can write an unsubtracted dispersion relation for $\frac{A^{(s)}(s,0)}{s}$.

$$\frac{1}{{F_\pi}^2} = \left[\left(\frac{16\pi\Gamma_\epsilon}{m_\epsilon^3}\right)\right]^{(s)} - \left[\frac{1}{3}\left(\frac{16\pi\Gamma_\epsilon}{3m_\epsilon^3}\right) - \left(\frac{48\pi\Gamma_\rho}{m_\rho^3}\right)\right]^{(u)}; \tag{5.5}$$

$$\frac{1}{2{F_\pi}^2} = \left[\left(\frac{48\pi\Gamma_\rho}{m_\rho^3}\right)\right]^{(s)} - \left[-\frac{1}{3}\left(\frac{16\pi\Gamma_\epsilon}{3m_\epsilon^3}\right) + \frac{1}{2}\left(\frac{48\pi\Gamma_\rho}{m_\rho^3}\right)\right]^{(u)}. \tag{5.6}$$

We note that if the u-channel contributions are neglected in Eq. (5.5) and Eq. (5.6), we obtain Eq. (4.19) and Eq. (4.20). This is reasonable since the R.H. amplitudes with tree-level effective potentials explicitly neglect the u-channel resonance-exchange contributions. However, we see from Eq. (5.5) and Eq. (5.6) that if it is a good approximation to assume that the u-channel contributions cancel in the $I^{(s)}$=1 sum rule, then *strictly speaking* it is inconsistent to assume that the u-channel contributions cancel in the $I^{(s)}$=0 sum rule. Since the KSFR relation is well satisfied experimentally, this implies that a tree-level effective potential might not describe the I=0 channel in a satisfactory manner. This clearly does not mean that a tree-level effective potential cannot serve as a good approximation in the I=0 channel, since the missing contributions are associated with the crossed-channel and are therefore necessarily small in the resonance region.

We can also consider a superconvergent sum rule for the I=2 t-channel amplitude which follows if the leading Regge trajectory for I=2 exchange has $\alpha_2(0) < 0$ [31]. We obtain

$$0 = \int_0^\infty ds \left[\frac{1}{3}\sigma_{tot}^0 - \frac{1}{2}\sigma_{tot}^1 + \frac{1}{6}\sigma_{tot}^2\right]. \tag{5.7}$$

In the resonance saturation approximation, if we assume that the Adler sum rule and the KSFR relation are satisfied, we find $m_\epsilon = m_\rho$.

In fig. 1 we display the I=0 and I=1 phase shifts for the tree-level effective potentials in the special case where $m_\epsilon = m_\rho$, in accord with the superconvergence relation.

6. 1-Loop Effective Potential

In this section we construct the R.H. amplitudes with 1-loop order effective potentials [16]. Appealing to Eq. (2.9) we find

$$\tilde{V}(s) = a_{IJ}^{(2)}(s) + a_{IJ}^{(4)C}(s). \tag{6.1}$$

That is, in accord with our prescription given in sect. 4, we include all two-particle irreducible diagrams at order s^2 in chiral perturbation theory. As explicitly displayed in sect. 2, this includes loop diagrams in the t and u channels as well as tree diagrams which are the low-energy manifestation of resonance exchange in the t and u channels.

In order to obtain the amplitude from this effective potential, we must evaluate another dispersion integral. Define

$$I_1(s) = \frac{1}{\pi}\int_0^\infty \frac{ds' a_{IJ}^{(4)C}(s)}{s' - s - i\epsilon} + counterterms \tag{6.2}$$

where

$$a_{IJ}^{(4)C}(s) \equiv \beta_{IJ} s^2 \left[\log\left(\frac{s}{\mu^2}\right) + \gamma_{IJ}\right], \tag{6.3}$$

with

$$\beta_{00} \equiv \frac{-1}{\pi}\left(\frac{7}{18}\right)(\alpha_{00})^2, \quad and \quad \beta_{11} \equiv \frac{1}{\pi}(\alpha_{11})^2. \tag{6.4}$$

The γ_{IJ}'s are the renormalization constants associated with crossed-channel resonance exchange. With three subtractions[3] we can write

$$I_1(s) = \frac{\beta_{IJ}}{\pi}\int_0^\infty \frac{ds' s'^2 \log\left(\frac{s'}{\mu^2}\right)(s-s_0)^3}{(s'-s)(s'-s_0)^3} + \frac{\beta_{IJ}\gamma_{IJ}}{\pi}\int_0^\infty \frac{ds' s'^2 (s-s_0)^3}{(s'-s)(s'-s_0)^3} + P_1(s), \tag{6.5}$$

where

$$P_1(s) \equiv M_{IJ} s^2 + N_{IJ} s + O_{IJ}. \tag{6.6}$$

3 The subtractions take place at the same point s_0 for the amplitude, its first derivative, and its second derivative.

Evaluating the integrals, we obtain

$$I_1(s) = -\frac{\beta_{IJ}}{2\pi}s^2\left(\log(\frac{-s}{\mu^2})\right)^2 - \frac{\beta_{IJ}\gamma_{IJ}}{\pi}s^2\log(\frac{-s}{\mu^2}) + P_1(s). \tag{6.7}$$

N_{IJ}=0 follows from our definition of R_{IJ}. O_{IJ}=0 for the same reason that b_{IJ}=0 in sect. 4. M_{IJ} consists of a new set of undetermined parameters which appear at order s^3 in the effective potential. For the full amplitude we have

$$t_{IJ}(s) = \frac{\alpha_{IJ}s + \beta_{IJ}s^2\left[\log\left(\frac{s}{\mu^2}\right) + \gamma_{IJ}\right]}{1 + \frac{\alpha_{IJ}s}{\pi}\left[\log(\frac{-s}{\mu^2}) + R_{IJ}(\mu^2)\right] + \frac{\beta_{IJ}}{2\pi}s^2\left[\left(\log(\frac{-s}{\mu^2})\right)^2 + 2\gamma_{IJ}(\mu^2)\log(\frac{-s}{\mu^2}) + M_{IJ}(\mu^2)\right]}. \tag{6.8}$$

One can easily verify that this amplitude is scale invariant if

$$\gamma(\mu^2) = \gamma(\nu^2) + \log\left(\frac{\mu^2}{\nu^2}\right) \tag{6.9}$$

and

$$M(\mu^2) = M(\nu^2) + 2\gamma(\nu^2)\log\left(\frac{\nu^2}{\mu^2}\right) - \log\left(\frac{\nu^2}{\mu^2}\right). \tag{6.10}$$

For I=0 and I=1 we continue to choose our renormalization point such that a resonance occurs "near" $s = \mu^2$. For the denominator function we find

$$D_{IJ} = 1 - \frac{s}{\mu^2} + \frac{\alpha_{IJ}s}{\pi}\log(\frac{-s}{\mu^2}) + \frac{\beta_{IJ}}{2\pi}s^2\left[\left(\log(\frac{-s}{\mu^2})\right)^2 + 2\gamma_{IJ}(\mu^2)\log(\frac{-s}{\mu^2}) + \pi^2\right]. \tag{6.11}$$

An interesting new feature that arises from a 1-loop effective potential is the shift in the resonance widths. In the narrow-width approximation we find

$$\Gamma = \alpha_{IJ}\mu^3 + \beta_{IJ}\gamma_{IJ}\mu^5. \tag{6.12}$$

The width is shifted from the tree-level case by an amount whose strength is determined by the crossed-channel renormalization constant, γ_{IJ}. This is completely consistent with our sum rule analysis of sect. 5; there should be contributions to particle widths associated with crossed-channel resonance exchange.

For the ρ width we find

$$\Gamma_\rho = \frac{m_\rho^3}{96\pi {F_\pi}^2}\kappa_{11} \tag{6.13}$$

where

$$\kappa_{11} \equiv 1 + \frac{{m_\rho}^2}{96\pi^2 {F_\pi}^2}\gamma_{11} \tag{6.14}$$

which yields

$$\gamma_{11} = \frac{(\kappa_{11} - 1)\, 96\pi^2 {F_\pi}^2}{{m_\rho}^2}. \tag{6.15}$$

Since the KSFR relation is well satisfied experimentally, we set $\kappa_{11} = 1$ or $\gamma_{11} \sim 0$ which is consistent with the fact that the tree-level effective potential is a good approximation for I=1.

In turn, the modified ϵ width is given by

$$\Gamma_\epsilon = \frac{m_\epsilon^3}{16\pi {F_\pi}^2}\kappa_{00} \tag{6.16}$$

where

$$\kappa_{00} \equiv 1 - \frac{7{m_\epsilon}^2}{288\pi^2 {F_\pi}^2}\gamma_{00} \tag{6.17}$$

or inverting,

$$\gamma_{00} = \frac{(1 - \kappa_{00})\, 288\pi^2 {F_\pi}^2}{7{m_\epsilon}^2}. \tag{6.18}$$

We can fix the parameter κ_{00} by requiring that the Adler sum rule be satisfied. If we saturate with the ρ and ϵ alone, this determines $\kappa_{00} = \frac{3}{4}$. Inclusion of $f_2(1270)$ reduces κ_{00} to $\frac{1}{2}$. In fig. 2 we plot the I=0 phase shift for these two values of κ_{00} where we choose $m_\epsilon = m_\rho$ in accord with the superconvergent sum rule.

7. The Higgs Boson Line Shape

In the standard model Higgs sector, the Adler sum rule no longer follows from the properties of the I=1 Regge trajectory, but rather from the good high energy behaviour associated with renormalizable gauge theories [32]. The statement of renormalizability is closely related to the fact that only the Higgs particle is necessary to effect the cancellations among Feynman diagrams which ensure that physical amplitudes are well behaved at high energies. In fact, the tree-level Higgs width, obtained from the minimal standard model couplings, trivially satisfies the Adler sum rule [29,33]. On the other hand, in Q. C. D.-like theories such as technicolor, one would expect a spectrum of low lying resonances as in Q. C. D. With the replacement of the pion decay constant with v, the weak scale, we can make use of the formulae in the previous section. With a Higgs alone, the Adler sum rule is satisfied with $\kappa_{00} = \frac{3}{2}$. Whereas if we assume the existence of a techni-rho with a width that satisfies a KSFR-type relation, we find $\kappa_{00} = \frac{3}{4}$ as in π-π scattering. In fig. 3 we exhibit the scalar meson line shapes corresponding to these two alternatives.

8. Critique of The Method of Padé Approximants

Beyond giving a powerful parametrization of Goldstone boson scattering in the resonance region, the Adler sum rule serves to delimit the class of models which yield a consistent unitary parametrization. For example, it is interesting to note that the [1,1] Padé approximant, applied to the chiral perturbation series [9-12], includes crossed-channel contributions, yet in the narrow width approximation the ϵ and ρ widths are the same as those obtained from the R.H. model with tree-level effective potentials. The [1,1] Padé approximant is given by

$$a_{IJ}^{[1,1]}(s) = \frac{a_{IJ}^{(2)}(s)}{1 - \frac{a_{IJ}^{(4)}(s)}{a_{IJ}^{(2)}(s)}} \tag{8.1}$$

where we appeal to the formulae of sect. 2. It is clear that the [1,1] Padé approximant suffers from the same ailment as the R.H. amplitudes with tree-level effective potentials: the denominator function has no imaginary part in the right-half plane whose strength is determined by crossed-channel contributions. Our sum rule analysis suggests that it

is reasonable to require that the lowest order contributions associated with resonance exchange in the crossed-channel contribute to the width in such a way as to allow the Adler sum rule to be satisfied. From this point of view, the method of Padé approximants is essentially equivalent to the Brown and Goble parametrization since neither unitarization scheme properly accounts for the crossed-channel contributions, and as a result, the particle widths are completely determined by unitarity.

As was mentioned in sect. 2, the logarithmic term does not appear in the I=J=1 amplitude to order E^4 in chiral perturbation theory because of a cancellation between direct- and crossed-channel loop diagrams. It has been noted recently [12] that it is a positive feature of the [1,1] Padé approximant that it also lacks this logarithmic term in the I=J=1 denominator function (as can be seen from Eq. (8.1)). We find it difficult to understand why this should be a desirable feature, since the one-to-one correspondence between a resonance and a pole on the second Riemann sheet is lost. Clearly the R.H. amplitudes with 1-loop effective potentials do not possess this feature since they are of $\frac{N}{D}$ form and therefore allow no cancellation between direct- and crossed-channel contributions. In fact, it is precisely this structure which allows the Adler sum rule to be satisfied. However, if we expand the I=1 P-wave amplitude, the logarithm is absent at order s^2.

9. Summary and Conclusion

We solved a relativistic Hamiltonian theory with effective potentials given by the low-energy theorems of current algebra, and obtained the Brown and Goble parametrization of π-π scattering [8]. By considering suitable dispersion relations, particularly in the form of the Adler sum rule, we were able to understand that strictly speaking it is inconsistent to neglect the left-hand cut in the I=0 amplitude, *although and because* the experimental validity of the KSFR relation indicates that it is a good approximation to do so for I=1. This demonstrates that simple unitarization procedures do not necessarily fix resonance widths to the correct value. The case of the ρ is special because it turns out that in the I=1 p-wave channel, there happens to be a cancellation among the u-channel contributions.

We included crossed-channel contributions by going to the next order in the effective potential. We found that these contributions modify the imaginary parts of the denominator functions, as they should, in order that the tree-level difficulties be resolved. In particular, we found that we could shift the ϵ and ρ widths in order to achieve consistency with the Adler sum rule, and in the process fix the renormalization constants associated with the next-to-leading order terms in the effective potentials. The resulting parametrization of π-π scattering acts as a Lorentz-invariant and crossing-symmetric form of the Breit-Wigner approximation [21], and is in good agreement with experiment. As a more current application, we considered the implications of our analysis with regards to the line shape of the Higgs resonance. We found that the R.H. model amplitudes, constrained by the Adler sum rule, exhibit the modifications to the Higgs line shape due to the presence of a techni-rho. Finally, we considered wether the method of Padé approximants can be made consistent with completeness sum rules. We found that although the [1,1] Padé approximant includes the contributions that are neglected by Brown and Goble, the Adler sum rule implies that these contributions appear in the wrong place.

In general terms, our analysis appears to be a vindication of the $\frac{N}{D}$ form for the π-π scattering amplitude as first advocated by Chew and Mandelstam [34]. Of course, the $\frac{N}{D}$ form was motivated by analogy with potential theory where the N-function is said to provide the interaction force. The surprising thing here is that it proves fruitful to extend the analogy further by introducing the notion of completeness, in the form of well known sum rules, as a constraint on the form of the relativistic amplitudes.

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Appendix : The π-π System

The π-π scattering amplitudes are determined by crossing symmetry in terms of a single analytic function A(s,t,u) as

$$T_{\alpha\beta;\gamma\delta} = A(s,t,u)\,\delta_{\alpha\beta}\delta_{\gamma\delta} + A(t,s,u)\,\delta_{\alpha\gamma}\delta_{\beta\delta} + A(u,t,s)\,\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{1}$$

where

$$s = (p_\alpha + p_\beta)^2\,, \tag{2a}$$

$$t = (p_\alpha - p_\gamma)^2\,, \tag{2b}$$

$$u = (p_\alpha - p_\delta)^2\,, \tag{2c}$$

are the usual Mandelstam variables. These amplitudes can be decomposed into amplitudes of definite isospin by using the standard techniques of projection operators. We find,

$$T_0(s,t,u) = 3A(s,t,u) + A(t,s,u) + A(u,t,s)\,, \tag{3a}$$

$$T_1(s,t,u) = A(t,s,u) - A(u,t,s)\,, \tag{3b}$$

$$T_2(s,t,u) = A(t,s,u) + A(u,t,s)\,, \tag{3c}$$

where the crossing matrix that relates the amplitudes in the s-channel to those in the u-channel is given by

$$\begin{pmatrix} T_0^s \\ T_1^s \\ T_2^s \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -1 & \frac{5}{3} \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} T_0^u \\ T_1^u \\ T_2^u \end{pmatrix}. \tag{4}$$

In turn, the partial waves can be projected out. The result is,

$$a_{IJ}(s) = \frac{1}{32\pi}\int_{-1}^{1} d(\cos\theta)\,P_J(\cos\theta)\,T_I(s,t,u) \tag{5}$$

where for elastic scattering (in the chiral limit),

$$a_{IJ} = e^{i\delta_{IJ}}\sin\delta_{IJ}. \tag{6}$$

The total cross section follows from the optical theorem,

$$\sigma_{IJ}^{tot}(s) = \frac{16\pi}{s} Im\left(a_{IJ}(s)\right) \tag{7}$$

and the phase shifts are obtained from

$$\cot \delta_{IJ}(s) = \frac{Re\left(a_{IJ}(s)\right)}{Im\left(a_{IJ}(s)\right)}. \tag{8}$$

The narrow width approximation for the total cross section is given by

$$\sigma_R^{tot}(s) = \frac{16\pi^2 m_R}{s}(2J+1)\Gamma_R \delta\left(s - m_R^2\right). \tag{9}$$

Figure Captions

Fig.(1) (a) I=0 s-wave phase shift from R.H. model amplitude with tree level effective potential compared with experimental data of ref. [35](points), ref. [36](open circles), and ref. [37](open squares). (b) I=1 p-wave phase shift from R.H. model amplitude with tree level effective potential compared with experimental data of ref. [35](points), ref. [36](circles), and ref. [38](crosses). All data points are extracted from ref. [5].

Fig.(2) I=0 s-wave phase shift from R.H. model amplitude with 1-loop effective potential compared with experimental data of ref. [35](points), ref. [36](open circles), and ref. [37](open squares). The Solid line corresponds to $\kappa_{00} = \frac{3}{4}$, and the dotted line corresponds to $\kappa_{00} = \frac{1}{2}$. All data points are extracted from ref. [5].

Fig.(3) The scalar meson line shape. The solid line corresponds to $\kappa_{00} = \frac{3}{4}$, and the dotted line corresponds to $\kappa_{00} = \frac{3}{2}$.

Postscript

We thank the referee for bringing to our attention a publication by T.N. Truong (Phys. Rev. Lett. 67, 2260 (1991)) which is related to this work. Truong compares various unitarization schemes by focusing on the properties of the I=1 P-wave amplitude. He considers the $\frac{N}{D}$ amplitude with tree-level effective potential (equivalent to Brown and Goble result). He notes that due to a neglect of the left-hand cut contribution, the expanded amplitude does not allow the cancellation of logarithms that is a property of the s^2 term in the chiral expansion. He further notes that the [1,1] Padé approximant does satisfy this property. However, we note that consistency requires that the Padé result be compared with the $\frac{N}{D}$ (RH) amplitude with 1-loop effective potential, since it is through the 1-loop term in the effective potential that the lowest order left-hand cut contributions make their appearance. The cancellation among logarithms at order s^2 then follows, as stated. We have also shown that due to a cancellation among crossed-channel contributions in the I=1 P-wave amplitude, its properties are not generic.

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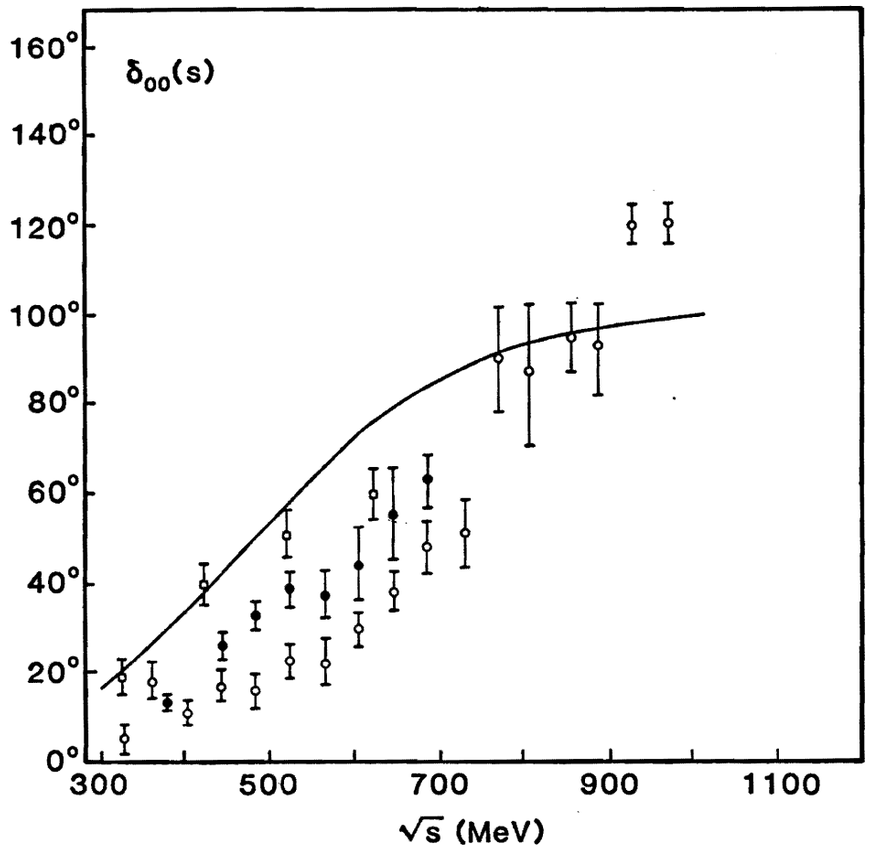
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140°
120°
100°
80°
60°
40°
20°
0°
300
500
700
900
1100
$\sqrt{s}$ (MeV)

Fig. 1(*a*)

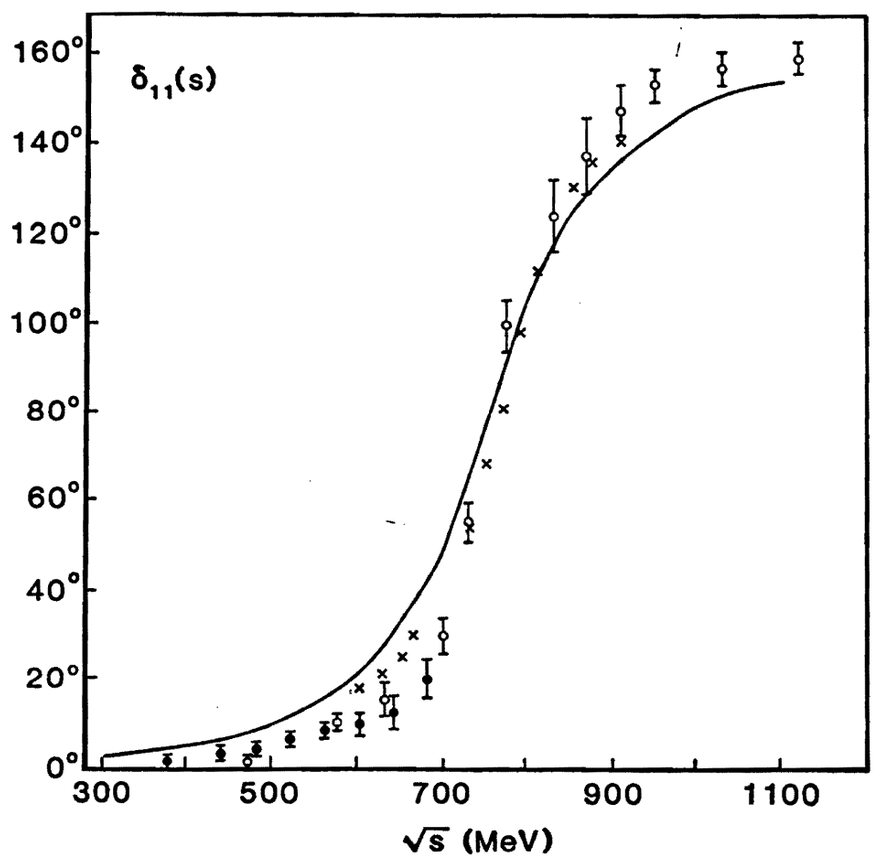
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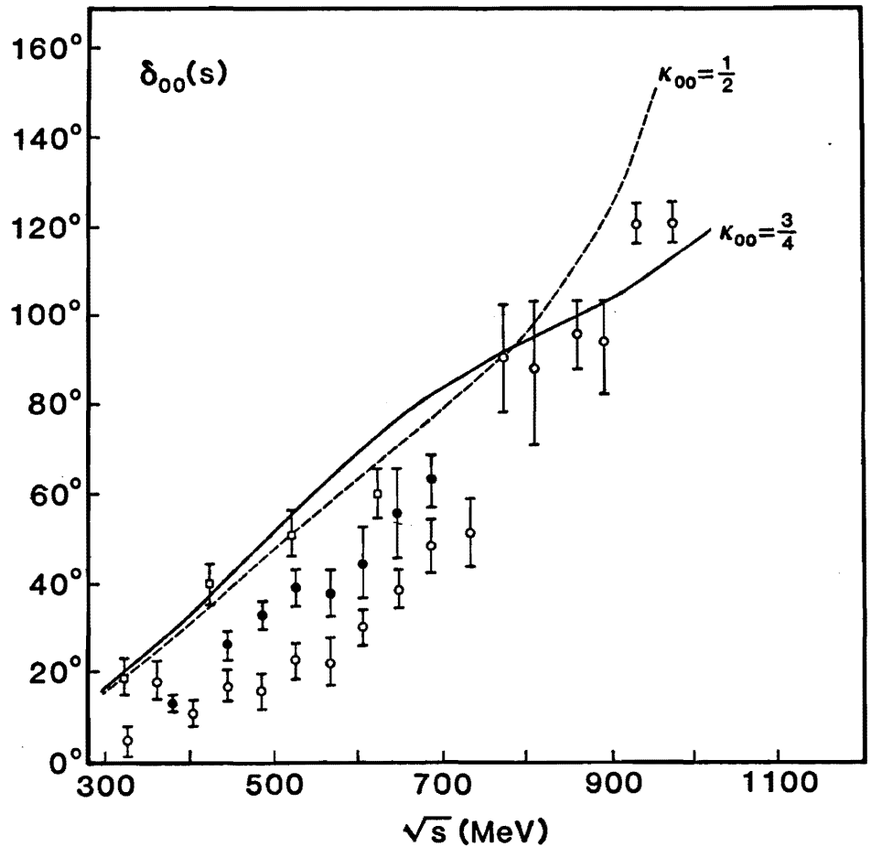
Fig. 1(*b*)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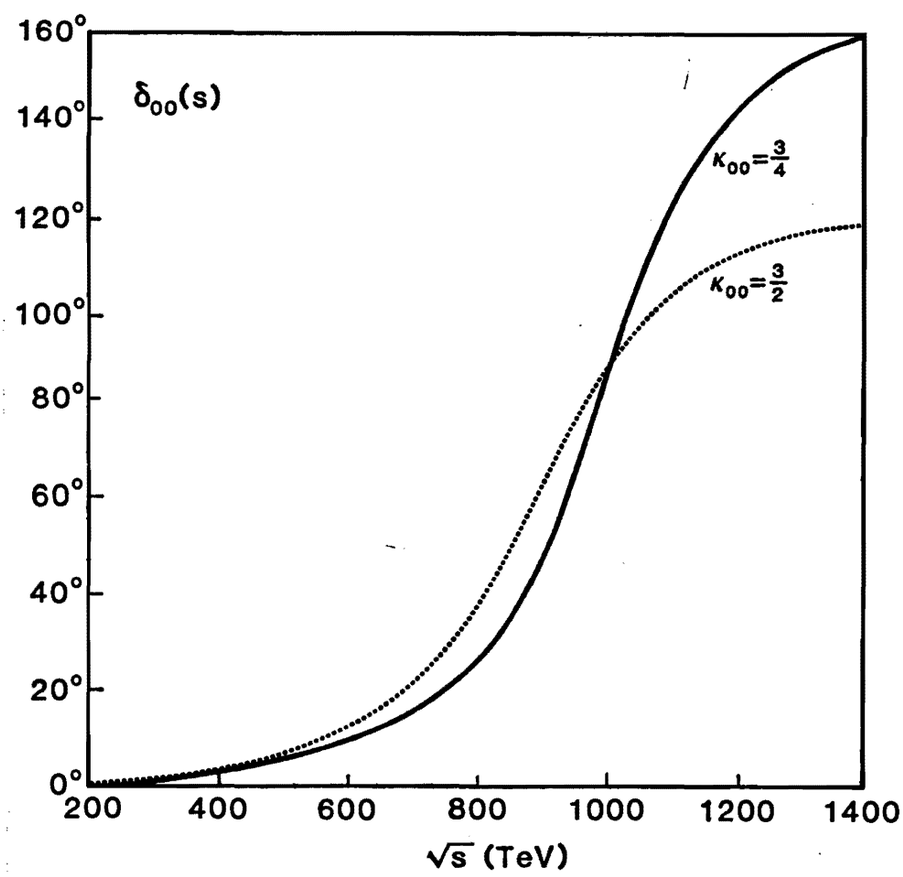
$\sqrt{s}$ (MeV)

Fig. 2


$\delta_{00}(s)$
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120°
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200
400
600
800
1000
1200
1400
$\sqrt{s}$ (TeV)

Fig. 3