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by

E. C. G. Sudarshan

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Center for Particle Physics
University of Texas at Austin
Austin, Texas 78712

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UNCERTAINTY RELATIONS, ZERO POINT ENERGY AND THE LINEAR CANONICAL GROUP

E.C.G. SUDARSHAN
Physics Department and the Center for Particle Physics
University of Texas, Austin, Texas 78712

Abstract

The close relationship between the zero point energy, the uncertainty relations, coherent states, squeezed states and correlated states for one mode is investigated. This group - theoretic perspective enables the parametrization and identification of their multimode generalization. In particular the generalized Schrödinger-Robertson uncertainty relations are analyzed. An elementary method of determining the canonical structure of the generalized correlated states is presented.

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1 Introduction

Advances in atomic physics and quantum optics have made it possible to examine and verify many of the immediate predictions of quantum mechanics. The most celebrated of these is the Heisenberg [?] uncertainty relation

$$(\Delta q)^2 \ (\Delta p)^2 \geq \left(\frac{\hbar}{2}\right)^2 \tag{1}$$

where

$$(\Delta q)^2 = \langle q^2 \rangle - \langle q \rangle^2, \tag{2}$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \tag{3}$$

are the dispersons in the coordinate and momentum variable. The Heisenberg uncertainty relation in the form

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2} \tag{4}$$

has been verified in gedanken experiments like the Heisenberg microscope and in the simple pictures of de Broglie waves.

Since Δq and Δp have different dimensions their individual magnitudes cannot be compared without choosing units for length and momentum. By a suitable scale change we could scale them inversely as long as the unit of action is fixed; in this case the change is in the unit of $\{\text{mass}^2/(\text{time})^2\}$ or equally well in the unit of length since action has the dimensions of $\{\text{mass} \times (\text{length}))^2/\text{time}\}$. Having fixed any such choice we can talk of the numerical values of Δp and Δq. Another and earlier result of quantum theory is the existence of zero point energy [?]. If p and q are canonical operators satisfying the commutation relations

$$q\,p - p\,q = i\hbar \tag{5}$$

then the "energy" $\frac{1}{2}(p^2 + \omega^2 q^2)$ has a nonzero minimum value:

$$\frac{1}{2}(p^2 + \omega^2 q^2) = \omega \left\{ \frac{\omega q - ip}{\sqrt{2\omega}} \cdot \frac{\omega q + ip}{\sqrt{2\omega}} \right\} + \frac{\hbar\omega}{2} \geq \hbar\omega/2\,. \tag{6}$$

Since the first term is non negative, $\omega\, a^\dagger a$, there is the zeropoint energy $\hbar\omega/2$ for the ground state which is annihilated by the operator

$$a = (\omega q + ip)/\sqrt{2\omega}\,. \tag{7}$$

While the notation is new, the zeropoint energy is as old as quantum theory!

It is well known that there is an immediate connection between the two relations. For every ω, $-\infty < \omega < \infty$

$$E(\omega) = (\omega q - ip)(\omega q + ip) \geq 0 \tag{8}$$

but this implies

$$\omega^2 \langle q^2 \rangle + \langle p \rangle^2 + i\omega \langle qp - pq \rangle \tag{9}$$

$$= \omega^2 \langle q^2 \rangle - \omega\hbar + \langle p^2 \rangle \geq 0\,. \tag{10}$$

Hence the discriminant of this quadratic form should be negative: that is,

$$4\,\langle q^2 \rangle \langle p^2 \rangle \geq \hbar^2\,. \tag{11}$$

Noting that the deviations from the mean

$$Q = q - \langle q \rangle\,, \qquad P = p - \langle p \rangle \tag{12}$$

also satisfy the canonical commutation relations we, derive

$$\langle Q^2 \rangle \langle P^2 \rangle \geq \frac{1}{4}\hbar^2 \tag{13}$$

which is Heisenberg's uncertainty relation.

We may therefore say that the zeropoint energy relation was not invariant under the linear canonical transformation

$$q \longrightarrow Q = q - \langle q \rangle \tag{14}$$

$$p \longrightarrow P = p - \langle p \rangle \tag{15}$$

nor under

$$q \longrightarrow Q = \omega^{\frac{1}{2}}\, q \tag{16}$$

$$p \longrightarrow P = \omega^{-\frac{1}{2}}\, p\,. \tag{17}$$

Imposition of these canonical transformations on the Planck zeropoint energy inequality gives the Heisenberg uncertainty relation.

But there are yet other linear canonical transformations: the simplest one is

$$q \longrightarrow q\cos\,\theta - \omega^{-1} p \sin\theta \tag{18}$$

$$p \longrightarrow \omega q \sin\,\theta + p\cos\theta\,. \tag{19}$$

While the Planck zeropoint inequality is invariant under this transformation, the Heisenberg uncertainty relation is not. We get, for any θ,

$$\left\{\langle q^2\rangle \cos^2\theta + \langle p^2\rangle \sin^2\theta - \langle qp + pq\rangle \cos\theta \sin\theta\right\}\,. \tag{20}$$

$$\left\{\langle q^2\rangle \sin^2\theta + \langle p^2\rangle \cos^2\theta + \langle qp + pq\rangle \cos\theta \sin\theta\right\} \geq \frac{\hbar^2}{4}\,. \tag{21}$$

By an elementary rearrangement this gives

$$\left\{\langle q^2\rangle + \langle p^2\rangle\right\}^2 - \left\{\left(\langle q^2\rangle - \langle p^2\rangle\right)\cos 2\theta - \left\langle\frac{qp+pq}{2}\right\rangle \sin^2\theta\right\}^2 \geq \hbar^2\,. \tag{22}$$

By choosing

$$\tan 2\theta = \frac{\langle qp + pq\rangle}{2} \Big/ \left\{\langle q^2\rangle - \langle p^2\rangle\right\} \tag{23}$$

we get the inequality

$$\langle q^2\rangle\langle p^2\rangle - \frac{\langle qp+pq\rangle^2}{2} \geq \frac{\hbar^2}{4}\,. \tag{24}$$

This is the Schrödinger uncertainty relation provided we replace q and p by $q - \langle q\rangle$ and $p - \langle p\rangle$. It was derived by Schrödinger and by Robertson[?]. It is stronger than the Heisenberg uncertainty relations and reduces to it in the special case of "uncorrelated states" for which

$$\langle (q - \langle q\rangle)\,(p - \langle p\rangle) + pq\rangle = 0 \tag{25}$$

or equivalently

$$\langle qp + pq\rangle = \langle q\rangle\langle p\rangle + \langle p\rangle\langle q\rangle\,. \tag{26}$$

Even for a harmonic oscillator of frequency ν this is not in general true and the correlation oscillates with twice the frequency. So a Heisenberg minimum uncertainty state is not canonically invariant. For the harmonic oscillator this has been known for decades. Dodunov and Mańko [?] have given a general systematics of such a derivation. The clue to the Schrödinger-Robertson generalization of the Heisenberg uncertainty relations is the requirement of invariance under the group of linear canonical transformations. The state of the minimum energy for the harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2) = \left(a^\dagger a + \frac{\hbar}{2}\right) \tag{27}$$

is the vacuum state $|\Psi\rangle$ satisfying

$$a|\Psi\rangle = 0 \tag{28}$$

with the associated wave function

$$\psi(x) = (\pi)^{-1/4} \exp(-x^2/2)\,. \tag{29}$$

This is a state of the minimum uncertainty. But the minimum uncertainty class is wider, among these are

$$a|z\rangle = z\,|z\rangle\,, z \text{ complex number} \tag{30}$$

with wave function

$$\psi(x) = (\pi)^{-1/4} \exp\left\{-(x-z)^2/2\right\}\,. \tag{31}$$

These are the "coherent states" introduced by Schrödinger [?] and rediscovered decades later in the context of quantum optics by Glauber [?] and by Sudarshan [?]. They constitute an overcomplete family of states in terms of which every state can be expressed in infinitely many ways; further in terms of them every density matrix can be exhibited as a sum of projectors $|z\rangle\langle z|$ to the coherent states with distribution valued weight [?] and [?].

But the coherent states are not a canonically invariant set. The scale transformation ("squeezing")

$$q \longrightarrow \omega^{\frac{1}{2}} q\,, \qquad p \longrightarrow \omega^{-\frac{1}{2}} p \tag{32}$$

takes a coherent state into a new class of [?] states which are now called squeezed states. In terms of $a, a^\dagger$ these are the Bogoliubov - Valatin transformations [?]. The unitary transformation

$$V = \exp\left\{-i\omega^{\frac{1}{2}}(qp + pq)/2\right\} \tag{33}$$

accomplishes the squeezing: and thus the one parameter family of overcomplete sets of squeezed coherent states with wave functions.

$$\psi(x) = (\pi)^{-\frac{1}{4}} \exp\left\{-\omega(x-z)^2/2\right\} \tag{34}$$

labelled by 3 parameters ω, $Re\ z$, Im z. For each ω we have an overcomplete family of states.

This is still not general enough. There are still more canonical transformations that can be performed which will make the state no longer a minimum uncertainty state in the

Heisenberg sense but which would be minimum Schrödinger uncertainty states. These are the correlated states whose wave functions have been obtained by Dodunov, Kurmyshev and Mańko [?]

$$\psi(x) = (\pi)^{-\frac{1}{4}} \frac{1}{\sqrt{u-v}} \exp\left\{-\frac{u+v}{2(u-v)} \frac{x^2}{\ell^2}\right\} \exp\left\{-\frac{(\beta)^2}{2} + \frac{\sqrt{2}\beta x}{\ell(u-v)} + \frac{\beta^2}{2} \frac{v^* - u^*}{u-v}\right\} \quad (35)$$

where u, v are complex parameters satisfying $|u|^2 - |v|^2 = 1$ and ℓ a real length parameter, altogether a four parameter family. In terms of them

$$\begin{aligned} (\Delta q)^2 &= \ell^2 \left\{\frac{1}{2} + |v|^2 - Re\,(uv^*)\right\} \\ (\Delta p)^2 &= \ell^{-2}\hbar^2 \left\{\frac{1}{2} + |v|^2 - Re\,(uv^*)\right\} \\ \langle qp + pq\rangle &- \langle q\rangle\langle p - \langle p\rangle\rangle\langle q\rangle = \Delta q \cdot \Delta p Im(uv^*) \end{aligned} \quad (36)$$

Making use of the appealing phase space picture introduced by Planck [?] for the quantum oscillator, the ground state with the zeropoint energy (for $\omega = 1$) has a phase space patch which is a circle with unit radius and an area π which is (2π) times the uncertainty. The mean value of $\frac{1}{2}\,(p^2 + q^2)$ within this circular disc is $\frac{1}{2}$ which satisfied Planck. So his picture of the ground state is a circle of unit radius centered at the orgin. By

Fig.1. Planck's picture of the minimum energy state and the coherent states. The coherent states are centered at the point $\left(\sqrt{2}\ell z, \sqrt{2}lmz_\epsilon\right)$.

displacing the origin to $\sqrt{2}\,z$ we get the two parameter (one complex parameter) family of coherent states.

Squeezed states are obtained by area preserving deformations of the circles into ellipses

with major (minor) axis along the coordinate directions.

Fig.2. Planck pictures for squeezed states.

When the ellipse is tilted we get the more general family of correlated states discussed by Dodunov, Kurmyshev and Mańko. Of course this tilting alters things only for the squeezed states but not for the coherent states.

Fig.3. Planck pictures for correlated states.

2 The Group Theoretic Significance of the States Which Have Minimum Schrödinger Uncertainty.

The linear canonical transformations on a pair of canonical variables form a group $SL(2,R) \boxed{\sim} T(2)$, the semidirect product of the special linear group with translations. The minimum uncertainty state of Planck are invariant under the harmonic SO(2) subgroup of this group; this is its stability group. So the quotient of the canonical group by the harmonic stability group the correlated states are in one-to-one correspondence with the elements of the coset of dimension $5 - 1 = 4$.

These states are realized by single mode lasers and states with substantial squeezing and/or correlation have been generated and identified.

It is a natural question to ask whether these notions and correspondences can be generalized to n-degrees of freedom and multimode laser beams. Group theory can be invoked to get a general answer to the problem.

3 Multimode Correlated States and Their Group-Theoretic Relevance

Consider a system of n canonical pairs $\{q_r, p_r\}$, $1 \leq r, s \leq n$. The homogeneous linear transformations are $Sp(2n, R)$ and the translations are $T(2n)$. So the linear canonical group is the semidirect product $Sp(2n, R) \boxed{\sim} T(2n)$ with $n(2n+1) + 2n(2n+3)$ parameters. We seek canonical invariants bilinear in the $2n$ canonical variables and look for the appropriate conditions to get the minimum generalized Schrödinger uncertainty. We expect this to come from the ground state $|\Omega\rangle$ annililated by all annililation operators $(q_r + ip_r)/\sqrt{2}$ and states obtained from $|\Omega\rangle$ by the action of the linear canonical group. Since these involve individual harmonic SO(2) elements for each degree of freedom and any $O(n)$ rotation between the various degrees of freedom the stability group of $|\Omega\rangle$ has $n + \frac{n(n-1)}{2} = \frac{1}{2}n(n+1)$ parameters, we expect a family with $\frac{1}{2}n(3n+5)$ parameters corresponding to the dimension of the coset space.

Even for small values of n this dimension grows rapidly; we adopt a more elementary method to obtain the generalized correlated states. We describe in detail the case for $n = 2$ and remark that the method generalizes for arbitrary n. The multimode coherent states are $2n$ parameter states obtained by $T(2n)$ acting on $|\Omega\rangle$. Let us consider the group $Sp(4, R)$ which is a double covering of SO(3,2) and has the same Lie algebra of dimension ten. This algebra can be obtained by the three $(p_r p_s)$, the three $(q_r q_s)$ and the four $\frac{1}{2}(q_r p_s + p_s q_r)$ which close under commutation. The generic SO(3,2) algebra has two invariants, one of the second order and one of the fourth order. If we consider the expectation values of the ten quantities $(p_r p_s)$, $(q_r q_s)$, $\frac{1}{2}(q_r p_s + p_s q_r)$ they furnish a 4×4 symmetric non negative matrix which is bounded below by the zero point energy

1. Let this matrix be denoted by:

$$T_{\mu\nu} = \begin{pmatrix} e_{11} & e_{12} & a & b \\ e_{12} & e_{22} & c & d \\ a & c & f_{11} & f_{12} \\ b & d & f_{12} & f_{22} \end{pmatrix}. \tag{37}$$

By suitable harmonic SO(2) transformations in (q_1, p_1) and in (q_2, p_2) this can be reduced to the form

$$\begin{pmatrix} e_1 & 0 & a' & b' \\ 0 & e_2 & c' & d' \\ a' & e' & f_1 & 0 \\ b' & a' & 0 & f_2 \end{pmatrix}. \tag{38}$$

By scale transformations independently for the two degrees of freedom we can reduce this to the form

$$\begin{pmatrix} e & 0 & a'' & b'' \\ 0 & e & c'' & d'' \\ a'' & c'' & f & 0 \\ b'' & d' & 0 & f \end{pmatrix}. \tag{39}$$

Now harmonic SO(2) transformations in (q_1, p_1) and in (q_2, p_2) can be used to diagonalize the other diagonal blocks to get

$$\begin{pmatrix} e & 0 & a' & 0 \\ 0 & e & 0 & d' \\ a' & 0 & f & 0 \\ 0 & d' & 0 & f \end{pmatrix} . \tag{40}$$

Now the SO(2) rotation between the two degrees of freedom can be used to transform this into

$$\begin{pmatrix} e+a' & 0 & 0 & 0 \\ 0 & e+d' & 0 & 0 \\ 0 & 0 & f+a' & 0 \\ 0 & 0 & 0 & f+d' \end{pmatrix} . \tag{41}$$

Further scale transformations in the two degrees of freedom can render this to the final form

$$\begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & g_2 \end{pmatrix} . \tag{42}$$

Thus there are two invariant quantities g_1, g_2 which maybe recognized as the uncertainties in the two natural modes . Note that g_1, g_2 are both positive and not less than $\frac{1}{2}\hbar$.

Naturally the minimum uncertainty state must have degenerate structure with

$$g_1 = g_2 = \frac{1}{2}\hbar . \tag{43}$$

This is the vacuum state $|\Omega\rangle$ in the natural modes. The correlated states are obtained by the action of the group $Sp(4, R) \boxed{\sim} T(4)$. The $T(4)$ action demands that we replace q, p by $q - \langle q \rangle$, $p - \langle p \rangle$, after which we may ignore them. Since the state $|\Omega\rangle$ has a 3-parameter stability group we may restrict attention to the quotient manifold of cosets.

This construction can be immediately generalized. We take the 4×4 diagonal block of the $2n \times 2n$ matrix and carry out the transformations outlined in the previous scheme and then take the bordering 4×2, 2×4 and 2×2 blocks. Now make orthogonal transformations between the modes to make the 6×6 block diagonal with possibly unequal diagonal elelments. Scale transformations independently in the three modes will make them diagonal with pairs of values equal. Now the process can be repeated with the bordering 2×6, 6×2 and 2×2 blocks; and repeating the procedure we can diagonalize the 8×8 matrix with

$$\langle p_1^2 \rangle = \langle q_1^2 \rangle \ , \ \langle p_2^2 \rangle = \langle q_2^2 \rangle, \ldots, \langle p_4^2 \rangle = \langle q_4^2 \rangle . \tag{44}$$

This can be done with the $2n \times 2n$ has matrix is fully diagonalized with adjacent pairs of diagonal elements equal; that is the eigenvalues are

$$g_1, g_1, g_2, g_2, g_3, g_3, \ldots, g_n, g_n . \tag{45}$$

This is the canonical form with n invariants $g_1, g_2, \ldots, g_n$ with each $g_r \geq \frac{1}{2}\hbar$. The distinguished generalized correlated states have degenerate eigenvalues

$$g_1 = g_2 = \cdots = g_n = \frac{1}{2}\hbar\,. \tag{46}$$

This is the multimode vaccum state! We can get the multimode coherent states by displacements which are the real and imaginary parts of $z_1, z_2, \ldots, z_n$. Squeezed states are obtained by scale transformations in each mode independently so that the diagonal eigenvalues became

$$\lambda_1 g_1, \lambda_1^{-1} g_1, \ldots, \lambda_n g_n,\ \lambda_n^{-1} g_n\,. \tag{47}$$

The displacements and squeezings introduce $2n + n = 3n$ parameters. But the generalized correlated state is obtained by the full coset of the linear canonical group $Sp(2n, R) \boxed{\sim} T(2n)$ by the stability group of the N-mode vacuum state $|\Omega\rangle$.

These correlated states maybe displayed explicitly but are too cumbersome. The multimode correlated states have wave functions which are displaced Gaussians with phase factors. Depending upon the experimental requirements we may obtain intensity correlations, photocount statistics etc. directly. The number of parameters describing such correlated states are enormous and would be restricted by the method of generation of such states.

4 Discussion

Some remarks are in order about the correlated states in quantum field theory. As long as the number of excited modes is finite, however many, there exists a unitary transformation from the multimode4 facuum state to the multimode correlated state. These unitary transformations are generated by a quantity bilinear in the canonical variables. These operators are unbounded but do generate unitary transformations. When the number of modes became infinite, the generic correlated state cannot be obtained form the vacuum state they would be in a different Hilbert space from the Fock Vacuum. [?]

It was the purpose of this paper to demonstrate the close relation between the correlated states and the linear canonical group; and to show that the correlated states which minimize the Schrödinger uncertainties is related to the canonical multimode vacuum which is invariant under linear unitary transformations of the modes. The generic wave functions are Gaussians with a determined number of independent parameters.

The one and two-mode analysis is equally applicable to the propogation of the Gaussian Schell mode paraxial wave fronts through a system of thin lenses which are, respectively, isotropic and nonisotropic. This has been carried out elsewhere [?].

Correlated states are the generic family which include squeezed states and coherent states as special cases.

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References

[1] W. Heisenbeg, Z. Phys. **43**, 122 (1927).

[2] M. Planck, Ann. d. Physik **4**, 564 (1901); **31**, 758 (1910).

[3] E. Schrödinger, Ber. Kgl. Akad. Wiss 296 (1930);
H.P. Robertson, Phys. Rev. **A35**, 667 (1930).

[4] V.V. Dodunov and V.I. Mańko, Proc. of the Lebedev Institute Acad. of Sc. USSR **183**, 103 (1989).

[5] E. Schrödinger, Naturwissen schaften **14**, 664 (1926).

[6] R.J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).

[7] E.C.G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).

[8] C.L. Mehta and E.C.G. Sudarshan, Phys. Rev. **138**, B274 (1965)

[9] D.F. Wall, Nature **306**, 141 (1983).

[10] N.N. Bogoliubov, J.E.T.P. (USSR) **34**, 58 (1958), Nuovo Cim. **7**, 794 (1958);
J.G. Valatin, Nuovo Cim. **7**, 843 (1958).

[11] V.V. Dodunov, E.V. Kurmyshev, and V.I. Mańko, Phys. Lett. **A79**, 150 (1980).

[12] M. Planck, *Theory of Heat Radiation*, Dover New York (1959).

[13] V.I. Mańko, preprint (1991) Lebedev Institute.

[14] S.S. Schweber and A.S. Wightman, Phys. Rev. **98** 812 (1955);
E.C.G. Sudarshan, J. Math, Phys. **4**, 1029 (1963).

[15] R. Simon, E.C.G. Sudarshan and N. Mukunda, Phys. Rev. **A29**, 3273 (1984);
N. Mukunda, R. Simon and E.C.G. Sudarshan, Indian J. Pure and Applied Math, **19**, 91 (1988).