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Theory of the Neutral Kaon System

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Abstract

We investigate the neutral kaon system based on an extended Lee model. An exact solution is obtained. By deforming the unitarity cut to be along a contour Γ, we obtain an analytically continued theory, referred to as the "Γ theory". Within this continued theory, the orthogonality and completeness relations are verified. The probability amplitude is defined as the scalar product between the discrete right eigenstates: $|\alpha\rangle$, and the discrete dual states: $\langle\beta^*|$. For instance, orthogonality property implies $\langle K_\ell^*|K_s\rangle = 0$. A corresponding Γ^* theory can be defined with the unitarity cut deformed along the contour Γ^* which is the mirror image of Γ in the real axis.

Since probability involves the absolute square of the probability amplitude, any probability quantity involves the multiplication of two scalar products, one defined in "Γ theory" and the other in the "Γ^* theory". The conventional "overlap function" $\langle K_\ell|K_s\rangle$ in the Lee-Oehme-Yang model is, strictly speaking, ill-defined. This is because the conventional hermitean conjugate state $\langle K_\ell|$ belongs to the spectrum of the "Γ^* theory" and $|K_s\rangle$ to the spectrum of the "Γ theory". An inner product should be defined among vectors in the same space.

Based on our theory, we derive a refined version of the Bell-Steinberger relation. For the neutral kaon system, numerically, this new version in a good approximation can be reduced to the original relation.

1 Introduction

The study of the decay of a metastable quantum system began with Gamow's theory [1] of an α decay of atomic nuclei and Dirac's theory [2] of spontaneous emission of radiation by an excited atom. A general treatment of decaying system was given by Weisskopf and Wigner [3] which leads to the familiar exponential decay law. Three decades ago, Khalfin observed [4] that the decay could not be strictly exponential for a quantum system with an energy spectrum bounded from below. The exact solution of decay in a system of the kind that Dirac considered was investigated by Friedrichs [5] and in an elegant form by Lee [6], by Glaser and Källen [7], and others [8].

In the Friedrichs-Lee model, one may identify unstable quantum state as the bare V particle. There the survival amplitude of the V particle with a bare mass m_0 is given by

$$\langle V|e^{-iHt}|V\rangle = \frac{1}{2\pi i}\int_0^\infty d\lambda e^{i\lambda t}\langle V|\lambda\rangle\langle\lambda|V\rangle = \frac{1}{2\pi i}\int_C d\lambda \frac{1}{\alpha(\lambda)}\, e^{-i\lambda t}, \tag{1.1}$$

where

$$\alpha(\lambda) = \lambda - m_0 - \frac{1}{\pi}\int_0^\infty dz\, \frac{\langle V|z\rangle\langle z|V\rangle}{\lambda - z + i\epsilon}. \tag{1.2}$$

One finds that the time derivative of the survival probability vanishes, at $t = 0$, which leads to the Zeno effect [9], occurring in the small time domain $t \lesssim 1/E_0$, where E_0 is the difference between the resonance mass and the threshold energy [10]. Beyond the Zeno region, there is the exponential decay region, which is dominated by the contribution of the zero of the $\alpha(E)$ function, which is at the resonance pole on the second sheet. In the large t region, the survival amplitude is suppressed by a power law in $1/t$.

Since stable particles correspond to poles in the S matrix, assuming the S-matrix is an analytic function of the complex variable E, it appears natural to continue to identify the pole in the S-matrix as the resonance states [11]. How could a complex energy be an eigenvalue of a hermitean hamiltonian? This is not possible. However, one may invoke the notion of analytic continuation, to analytically continue the original theory, through the deformation of the unitarity cut along the real axis, to an analytically continued theory

where the branch cut is along a contour Γ in the fourth quadrant. Here the resonance pole may be exposed. Many authors act as if poles in the analytically continued theory are the only relevant singularities [12]. On the contrary [13], it is the resonance poles together with the complex branch cut along Γ which are the relevant singularities on the physical sheet. Correspondingly, the spectrum of the continued theory consists of both the discrete spectrum as well as the continuum. They together span the state space $\mathcal{G}$ of the analytically continued theory. Recently we have also applied this analytic continuation approach to consider the spectrum of the cascade model which involves three particle channels [14].

The neutral kaon system presents new features of unstable quantum system. It involves two resonance poles, which correspond to the K_ℓ and K_s "particles". Lee, Oehme and Yang (LOY) [15] formulated the necessary generalization of the Weisskopf-Wigner formalism which has been the basis for the discussion of the empirical data [16]. This phenomenological theory has the some shortcomings of the Weisskopf-Wigner theory discussed earlier.

In ref. 17, we considered an extended Lee model which contains the dynamical ingredients of the neutral kaon system. The exact solution was obtained. The corresponding matrix elements of the survival amplitude matrix are found to have the analogous three time domains.

In this paper we consider the analytic continuation of this extended Lee model and identify the appropriate resonance poles as the K_ℓ and K_s resonances in the space $\mathcal{G}$. Within the present framework, we derive a refined version of the Bell-Steinger relation, which differs from the original Bell-Steinberger relation [18] in the order of $|m_\ell^* - m_s|/m_s$.

The plan of the remaining paper is as follows. In sec. 2, part of the LOY theory relevant to our present discussion is reviewed. In sec. 3, based on the exact solution of the extended Lee model, we investigate the resonance states as the discrete states in the space $\mathcal{G}$. In sec. 4, the orthogonality relations and the completeness relations of the space $\mathcal{G}$ states are presented. In sec. 5, the difference between the identity operator defined in the space $\mathcal{G}$ the intertwining operator, which maps states in $\mathcal{G}$ to states in $\mathcal{G}^*$ is discussed. In sec. 6, the improved Bell-Steinberger relation is derived. A summary is given in section 7. Detailed calculations for the orthogonality relations and the completeness relations for the eigenstates

in the space $\mathcal{G}$ are given in three Appendices.

2 The Lee, Oehme and Yang Theory [15]

Consider the neutral kaon system. In the K^0 and $\overline{K^0}$ basis, the eigenvalue equation obtained from the perturbation theory is given by

$$E\psi = \lambda\psi, \text{ or } \begin{pmatrix} A & B \\ C & A \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \lambda \begin{pmatrix} r \\ s \end{pmatrix}. \tag{2.1}$$

In Weisskopf-Wigner approximation,

$$A = M_{11} - \frac{i\Gamma_{11}}{2},\ B = M_{12} - \frac{i\Gamma_{12}}{2} \text{ and } C = M_{21} - \frac{i\Gamma_{21}}{2} \tag{2.2}$$

with M_{ij} and Γ_j being constant elements of the real part and the imaginary part of the complex mass matrix. Here CPT invariance is invoked, so the 11-element and the 22-element are identified, *i.e.*$M_{11} = M_{22}$ and $\Gamma_{11} = \Gamma_{22}$. In the Weisskopf-Wigner approximation, E is a constant matrix. There are two roots to (2.1). They are complex eigenvalues, which are the complex masses of the K_ℓ and K_s states. Denote them as

$$\lambda_\ell = m_\ell - \frac{i\Gamma_\ell}{2} \text{ and } \lambda_s = m_s - \frac{i\Gamma_s}{2}. \tag{2.3}$$

We proceed to get the eigenvectors. The trace of E,

$$\mathrm{Tr}\, E = 2A = \lambda_\ell + \lambda_s, \text{ or } A = \frac{1}{2}(\lambda_\ell + \lambda_s). \tag{2.4}$$

From (2.1) and (2.2), the eigenvalue equations are:

$$Ar + Bs = \lambda_\ell r \tag{2.5}$$

$$Cr + As = \lambda_\ell s. \tag{2.6}$$

Equations (2.4) and (2.5) lead to

$$B = \frac{r}{2s}(\lambda_\ell - \lambda_s), \tag{2.7}$$

while (2.4) and (2.6) lead to

$$C = \frac{s}{2r}\left(\lambda_\ell - \lambda_s\right). \tag{2.8}$$

Combining (2.7) and (2.8) gives

$$\left(\frac{r}{s}\right)^2 = \frac{B}{C}, \text{ or } \frac{r}{s} = \pm\sqrt{\frac{B}{C}}. \tag{2.9}$$

Making the correspondence between the "±" choice with the K_ℓ and K_s states, for the K_ℓ state:

$$\frac{r}{s} = \sqrt{\frac{B}{C}} = \frac{p}{q}, \text{ or } \psi_\ell = N\begin{pmatrix} p \\ q \end{pmatrix} \tag{2.10}$$

and for the K_s state,

$$\frac{r}{s} = -\sqrt{\frac{B}{C}} = -\frac{p}{q}, \text{ or } \psi_s = N\begin{pmatrix} p \\ -q \end{pmatrix}. \tag{2.11}$$

Define the bra-state, as usual, the hermitean conjugate of the ket state, *i.e.* $\langle K_\alpha| = |K_\alpha\rangle^+$. Then

$$\langle K_s|K_\ell\rangle = N^2\left(|p|^2 + |q|^2\right) = \langle K_\ell|K_\ell\rangle = 1. \tag{2.12}$$

This completes our review of the part of the LOY theory relevant to the present discussions. Note that the complex energies and the nonself-adjoint Hamiltonian specified by (2.1) and (2.2) are there by virtue of the fact that the LOY theory, like its predecessor the Weisskopf-Wigner-theory, is a phenomenological approximation. We now ask how unstable states get described in an exact theory.

3 Discrete solutions to an extended Lee Model

3.1 The extended Lee model

In ref. 17, we obtained exact solutions of an extended Lee model, which has the essence of the decay of $K^0\overline{K}^0$ complex into arbitrary number of decay products. Here the basis states

are:

$$|V_1\rangle,\ |V_2,\rangle\ |N\theta_1\rangle, \cdots, |N\theta_p\rangle \cdots$$

The hamiltonian is given by

$$H = \begin{pmatrix} m_{ij} & g^{+}_{iq}(\omega) \\ g_{pj}(\omega) & \omega\delta(\omega-\omega')\delta_{pq} \end{pmatrix} . \tag{3.1}$$

We label the continuum eigenstates by $|\lambda, r\rangle$, where the index r indicates the presence of a plane wave "in"state in the rth channel. The continuum eigen wave function may be represented by

$$\psi^r_\lambda = \begin{pmatrix} \langle V_i|\lambda, r\rangle \\ \langle N\theta_p(\omega)|\lambda\, r\rangle \end{pmatrix} = \begin{pmatrix} a^r_{i\lambda} \\ b^r_{p\lambda}(\omega) \end{pmatrix} \tag{3.2}$$

where

$$b^r_{p\lambda}(\omega) = \delta(\lambda-\omega)\delta_{pr} + \cdots . \tag{3.3}$$

The function a is independent of r, or $a_{i\lambda} = a^r_{i\lambda}$. When the context is unambiguous, we will often suppress the λ label and sometimes also the ω-dependence, *i.e.*

$$a_i = a^r_{i\lambda},\ b^r_p = b^r_{p\lambda} = b^r_{p\lambda}(\omega) . \tag{3.4}$$

Expanding the eigenvalue equation $H\psi^r_\lambda = \lambda\psi^r_\lambda$ gives

$$(\lambda\delta_{ij} - m_{ij})a_j = \int_0^\infty g^{+}_{iq}(\omega)b^r_q(\omega)dw, \tag{3.5}$$

and

$$(\lambda-\omega)b^r_p(\omega) = g_{pj}(\omega)a_j . \tag{3.6}$$

From (3.2) and (3.5), we write

$$b^r_p(\omega) = \delta(\lambda-\omega)\delta_{pr} + \frac{g_{pj}(\omega)a_j}{\lambda-\omega+i\epsilon} . \tag{3.7}$$

Substituting (3.7) into (3.5) gives

$$(\lambda\delta_{ij} - m_{ij})a_j = g^{+}_{ir}(\lambda) + \int_0^\infty dw\, \frac{g^{+}_{iq}(\omega)g_{qj}(\omega)a_j}{\lambda-\omega+i\epsilon} . \tag{3.8}$$

The discrete eigenstate equation is

$$K_{ij}(\lambda+i\epsilon)a_j = 0 , \tag{3.9}$$

where

$$K_{ij}(\lambda+i\epsilon) = \lambda\delta_{ij} - m_{ij} - \int_0^\infty dw\, \frac{g^{+}_{iq}(\omega)g_{qj}(\omega)}{\lambda-\omega+i\epsilon} . \tag{3.10}$$

In ref. 17, we considered the conventional theory where the eigenvalues are all real. Now we proceed to analytically continue the original theory to the complex energy plane.

3.2 Analytic continuation of the model

To analytically continue the variable ω, we first replace the hermitean conjugate operation by the "dual" operation, *i.e.*

$$[F(\omega)]^+ \longrightarrow \widetilde{F}(\omega) = F^+(\omega^*)\,. \tag{3.11}$$

The bra state is to be replaced by

$$\langle M_\alpha| \longrightarrow \langle M_\alpha^*| \tag{3.12}$$

where $\langle M_\alpha^*|$ is the dual state with eigenvalue $\lambda = M_\alpha$. The analytically continued operator K of (3.10) is given by

$$K_{ij}(\lambda + i\epsilon) = \lambda\delta_{ij} - m_{ij} - \int_\Gamma dw\, \frac{\widetilde{g}_{iq}(\omega) g_{qj}(\omega)}{\lambda - \omega + i\epsilon}\,. \tag{3.13}$$

The corresponding discrete state wave function based on (3.9) is given by

$$\Psi_\alpha = \begin{pmatrix} \eta_\alpha \\ \phi_{p\alpha} \end{pmatrix} = N_\alpha \begin{pmatrix} c_{i\alpha} \\ \frac{g_{pj}(\omega) c_{j\alpha}}{M_\alpha - \omega + i\epsilon} \end{pmatrix} \tag{3.14}$$

where $+i\epsilon$ serves as a reminder that M_α is above the Γ-cut. (See Fig. 1). Correspondingly, the dual wave function of another discrete state at $\lambda = M_\beta$ is

$$\Phi_\beta(\chi_\beta, \zeta_\beta) = N_\beta \left(d_{\beta i},\ \frac{d_{\beta i}\widetilde{g}_{ip}(\omega)}{M_\beta + \omega + i\epsilon} \right)\,. \tag{3.15}$$

Again, here M_β is also above the Γ-cut. (See Fig. 2.) We label the discrete eigenstate by K_ℓ and K_s. Similar to (2.10) and 2.11),

$$c_\ell = N_\ell \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \quad \text{and} \quad c_s = N_s \begin{pmatrix} p_s \\ q_s \end{pmatrix}\,, \tag{3.16}$$

except that p <u>and</u> q <u>now depend on</u> λ, which are evaluated at $\lambda = M_\ell$ and M_s. The N's are the normalization factors yet to be determined.

For the dual wave function, we proceed to solve for $d_\beta = (r, s)$ based on

$$(r, s) \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & A_\lambda \end{pmatrix} = \lambda(r, s)\,. \tag{3.17}$$

Taking the transpose we have

$$\begin{pmatrix} A_\lambda & C_\lambda \\ B_\lambda & A_\lambda \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \lambda \begin{pmatrix} r \\ s \end{pmatrix}. \tag{3.18}$$

Comparison with (2.1) reveals that analogous to (2.9), here

$$\left(\frac{r}{s}\right)^2 = \frac{C_\lambda}{B_\lambda}, \ \frac{r}{s} = \pm\sqrt{\frac{C_\lambda}{B_\lambda}}. \tag{3.19}$$

In other words, for the K_ℓ and K_s dual states,

$$d_\ell = N_\ell(q_\ell, p_\ell), \ d_s = N_s(q_s, -p_s). \tag{3.20}$$

3.3 Orthnormality relations

The inner product of a discrete state labelled by α with another dual discrete state $\langle\beta|_D \equiv \langle\beta^*|$, is given by

$$\begin{aligned} \langle\beta^*|\alpha\rangle &= N_\alpha N_\beta \left(d_{\beta i}, \frac{d_{\beta j}\tilde{g}_{jp}(\omega)}{M_\beta - \omega + i\epsilon}\right) \begin{pmatrix} c_{i\alpha} \\ \frac{g_{pj}(\omega)c_{j\alpha}}{M_\alpha - \omega + i\epsilon} \end{pmatrix} \\ &= N_\alpha N_\beta \, d_{\beta i} \left[\delta_{ij} + \int_\Gamma dw \, \frac{\tilde{g}_{ip}(\omega) g_{pj}(\omega)}{(M_\beta - \omega)(M_\alpha - \omega)}\right] c_{j\alpha} \end{aligned} \tag{3.21}$$

The discontinuity of K across the Γ-cut can be read off from (3.13), giving

$$K(\lambda + i\epsilon) - K(\lambda - i\epsilon) = 2\pi i \tilde{g}(\lambda) g(\lambda) \tag{3.22}$$

Thus the integral in (3.21) can be deformed in the following manner: (see Fig. 2.)

$$\begin{aligned} \int_\Gamma dw \, \tilde{g}_{ip}(\omega) g_{pj}(\omega) \cdots &= \frac{1}{2\pi i} \int_\Gamma dw \, [K_{ij}(\omega + i\epsilon) - K_{ij}(\omega - i\epsilon)] \cdots \\ &= \frac{-1}{2\pi i} \int_C dw \, K(\omega) \cdots \end{aligned} \tag{3.23}$$

where as indicated in Fig. 2 the contour C wraps around the Γ-cut in a counter-clockwise manner. The equations for the discrete solutions are

$$\begin{aligned} &\text{at } \lambda = M_\alpha, \quad K_{ij}(M_\alpha) c_{j\alpha} = 0, \quad \text{and} \\ &\text{at } \lambda = M_\beta, \quad d_{\beta i} K_{ij}(M_\beta) = 0. \end{aligned} \tag{3.24}$$

Inspection on the contours in Fig. 2 reveals

$$C_\infty = C_\alpha + C_\beta + C\,. \tag{3.25}$$

The corresponding integrals are related by

$$I_C = -I_\alpha - I_\beta + I_\infty, \text{ with } I_C = \frac{-d_\beta}{2\pi i}\int_C \frac{dwK(\omega)c_\alpha}{(M_\beta-\omega)(M_\alpha-\omega)}\,. \tag{3.26}$$

Consider first the case $\beta \neq \alpha$. On account of

$$I_\alpha = \frac{-1}{2\pi i}\int_{C_\alpha} \frac{dw}{(M_\beta-\omega)(M_\alpha-\omega)}\, d_\beta\, K(\omega)c_\alpha = 0\,. \tag{3.27}$$

Similarily,

$$I_\beta = 0\,. \tag{3.28}$$

From (3.10), the asymptotic behavior of $K(\omega)$ is

$$K_{ij}(\omega) \xrightarrow[\omega\to\infty]{} \omega\delta_{ij} \tag{3.29}$$

$$\begin{aligned}(I_\infty)_{\beta\alpha} &= -\frac{1}{2\pi i}\int_{C_\infty} \frac{dw}{(M_\beta-\omega)(M_\alpha-\omega)}\, d_{\beta i}(\omega\delta_{ij})c_{j\alpha} \\ &= -d_{\beta i}\,\delta_{ij}\,c_{i\alpha}\,. \end{aligned} \tag{3.30}$$

Putting (3.27), (3.28) and (3.30) into (3.26), using the definition (3.23), the inner product (3.21) becomes

$$\langle M_\beta^*|M_\alpha\rangle = N_\alpha N_\beta d_{\beta i}\,[\delta_{ij} + (-0-0-\delta_{ij})]\,c_{j\alpha} = 0\,. \tag{3.31}$$

For the case $\beta = \alpha$,

$$\begin{aligned}\langle M_\alpha^*|M_\alpha\rangle &= 1 = N_\alpha^2\, d_{\alpha i}\left[\delta_{ij} + \int \frac{\tilde{g}_{iq}(\omega)g_{qj}(\omega)dw}{(M_\alpha-\omega)^2}\right] c_{\alpha j} \\ &= N_\alpha^2\, d_{\alpha i}K'_{ij}\,c_{j\alpha} \end{aligned} \tag{3.32}$$

with $K' = \left[\frac{dK}{d\lambda}\right]_{\lambda=M_\alpha}$. Thus the normalization is given by

$$N_\alpha = [d_\alpha\, K' c_\alpha]^{-1/2}\,. \tag{3.33}$$

3.4 The "overlap function"

The contribution to the overlap function from V_1 and V_2 componets alone is given by

$$\sum_i \langle M_\beta^* | V_i \rangle \langle V_i | M_\alpha \rangle = N_\beta N_\alpha \, d_\beta c_\alpha . \tag{3.34}$$

For $\alpha = K_\ell$ and $\beta = K_s$, using (3.20) the right hand side of (3.34) becomes

$$\begin{aligned} RHS &= N_\ell N_s \left(q_s - p_s \right) \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \\ &= N_\alpha N_s \left(p_\ell q_s - q_\ell p_s \right) . \end{aligned} \tag{3.35}$$

Strictly speaking since (p_ℓ, q_ℓ) and (p_s, q_s) are evaluated at M_ℓ and M_s, $RHS \neq 0$. However, to the extent that the energy dependence of the coupling function $g(\omega)$ in the analytically continued Hamiltonian can be neglected, $p_\ell \approx p_s$, $q_\ell \approx q_s$, or $RHS \approx 0$.

4 Analytically continued wave functions and their properties

So far we have looked at the discrete solutions in the analytically continued theory with the continuum states defined along the contour Γ. Hereafter, we refer to it as the "Γ-theory". The continuum states and the dual states are defined along the same contour Γ. We proceed to display the complete set of wave functions including both the discrete states and the continuum states and to investigate their orthonormality properties and the completeness relations. Some of the calculations were given in the previous sections and the details of the remainder can be found in Appendices A, B and C.

4.1 The complete set of the wave-functions

<u>The discrete states</u>

From (3.14) and (3.15), the wave functions of the discrete states and corresponding dual

wave functions are given by

$$\Psi_\alpha \equiv \begin{pmatrix} \langle V_i|\alpha\rangle \\ \langle N\theta_p|\alpha\rangle \end{pmatrix} \equiv \begin{pmatrix} \eta_\alpha \\ \phi_{p\alpha} \end{pmatrix} = N_\alpha \begin{pmatrix} c_{i\alpha} \\ \frac{g_{pj}(\omega)c_{j\alpha}}{M_\alpha-\omega} \end{pmatrix} \tag{4.1}$$

$$\begin{aligned} \Phi_\beta &\equiv (\langle\beta^*|V_i\rangle, \langle\beta^*|N\theta_p\rangle) \equiv (\chi_{\beta i}, \zeta_{\beta p}) \\ &= N_\beta \left(d_{\beta i}, \frac{d_{\beta i}\tilde{g}_{ip}(\omega)}{M_\beta - \omega} \right) \end{aligned} \tag{4.2}$$

<u>Continuum states</u>

From (3.6) and (3.7), the continuum wave functions and their dual wave functions are given by

$$\begin{aligned} \Psi^r_\lambda &\equiv \begin{pmatrix} \langle V_i|\lambda r\rangle \\ \langle N\theta_p|\lambda r\rangle \end{pmatrix} \equiv \begin{pmatrix} \eta_{i\lambda} \\ \phi^r_{p\lambda} \end{pmatrix} \\ &= \begin{pmatrix} a_{i\lambda} \\ \delta(\lambda-\omega)\delta_{rp} + \frac{g_{pj}(\omega)a_{j\lambda}}{\lambda-\omega+i\epsilon} \end{pmatrix} \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Phi^r_\lambda &\equiv (\langle\lambda^* r|V_i\rangle, \langle\lambda^* r|N\theta_p\rangle) \equiv (\chi_{\lambda i},\ \zeta^r_{\lambda p}) \\ &= \left(\tilde{a}_{\lambda i},\ \delta(\lambda-\omega)\delta_{pr} + \frac{\tilde{a}_{\lambda j}\tilde{g}_{jp}(\omega)}{\lambda-\omega-i\epsilon} \right) \end{aligned} \tag{4.4}$$

From (3.8),

$$Ka = \tilde{g},\ a = K^{-1}\tilde{g},\ \tilde{a} = g\widetilde{K^{-1}}\,. \tag{4.5}$$

4.2 Orthonormality relations:

The identity operator in the bare basis is

$$I = |V_i\rangle\langle V_i| + \int_\Gamma dw |N\theta_p(\omega)\rangle\langle N\theta_p(\omega^*)| \tag{4.6}$$

where summations over i and p are understood. The expected orthonormality relations are:

$$\langle\beta^*|\alpha\rangle = \langle\beta^*|V_i\rangle\langle V_i|\alpha\rangle + \int_\Gamma dw\, \langle\beta^*|N\theta_p\rangle\langle N\theta_p|\alpha\rangle$$

$$= \chi_{\beta i}\eta_{i\alpha} + \int_\Gamma dw \zeta_{\beta p}(\omega)\phi_{p\alpha}(\omega) = \delta_{\alpha\beta}, \tag{4.7}$$

$$\langle \lambda^*, r|\alpha\rangle = \chi_{\lambda i}\eta_{i\alpha} + \int_\Gamma dw\, \zeta_{\lambda p}(\omega)\phi_{p\alpha}(\omega) = 0, \tag{4.8}$$

$$\langle \beta^*|\lambda, r\rangle = \chi_{\beta i}\eta_{i\lambda} + \int dw\, \zeta_{\beta p}(\omega)\phi^r_{p\lambda}(\omega) = 0, \tag{4.9}$$

$$\langle \lambda^*, r|\mu, s\rangle = \chi^r_{\lambda i}\eta^s_{i\mu} + \int dw\, \zeta^r_{\lambda p}(\omega)\phi^s_{p\mu}(\omega) = \delta(\lambda-\mu)\delta_{rs}\,. \tag{4.10}$$

The proof of (4.7) is given in the previous section (see (3.31) and (3.32)). The remaining relations are proven in Appendix A.

4.3 The completeness relations:

The spectrum in the analytic continued theory consists of the discrete states K_ℓ and K_s at the complex energies M_ℓ and M_s respectively. This defines a space "$\mathcal{G}$", where the identity operator is given by

$$I = |\alpha\rangle\langle\alpha^*| + \int_\Gamma d\lambda |\lambda r\rangle\langle\lambda^* r|\,. \tag{4.11}$$

Again summation over the discrete labels α and r are understood. The identity operator leads to following set of completeness relations:

$$\langle V_i|V_j\rangle = \eta_{i\alpha}\chi_{\alpha j} + \int_\Gamma d\lambda \eta_{i\lambda}\chi_{\lambda j} = \delta_{ij} \tag{4.12}$$

$$\langle N\theta_p|V_i\rangle = \phi_{p\alpha}\chi_{\alpha i} + \int_\Gamma d\lambda \phi_{p\lambda}\chi_{\lambda i} = 0 \tag{4.13}$$

$$\langle V_i|N\theta_q\rangle = \eta_{i\alpha}\,\zeta_{\alpha q} + \int_\Gamma d\lambda \eta_{i\lambda}\zeta_{\lambda q} = 0 \tag{4.14}$$

$$\langle N\theta_p(\omega)|N\theta_q(\omega')\rangle = \phi_{p\alpha}\zeta_{\alpha q} + \int_\Gamma d\lambda \phi_{p\lambda}\zeta_{\lambda q} = \delta(\omega-\omega')\delta_{pq}. \tag{4.15}$$

The proofs of these relations are given in Appendices B and C.

5 Probabilities

5.1 Probability of $K_s \to K^0$.

Consider a K_s beam being prepared at $t = 0$. We are co-moving with the K_s particle. The amplitude for detecting the K^0-signal at t is given by

$$\langle K^0|M_s(t)\rangle = \langle K^0|e^{-iHt}|M_s\rangle = e^{-iM_s t}\langle K^0|M_s\rangle\,. \tag{5.1}$$

The detection probability at t is:

$$\begin{aligned}
\left|\langle K^0|M_s(t)\rangle\right|^2 &= \langle K^0|M_s(t)\rangle\langle K^0|M_s(t)\rangle^* \\
&= \langle K^0|e^{-iHt}|M_s\rangle\langle M_s|e^{+iHt}|K^0\rangle \\
&= e^{-i(M_s-M_s^*)t}\langle K^0|M_s\rangle\langle M_s|K^0\rangle \\
&= e^{-\Gamma_s t}\langle K^0|M_s\rangle\langle M_s|K^0\rangle\,.
\end{aligned} \tag{5.2}$$

5.2 Identity operator versus the intertwining operator.

Note that the inner product $\langle K^0|M_s(t)\rangle$ belongs to the space $\mathcal{G}$, while $\langle M_s|K^0\rangle$ belongs to the space $\mathcal{G}^*$. So the definition of probability involves the multiplication of *two linear products* each belonging to a distinct space. Since the scalar products themselves are complex numbers, they can be multiplied together without regard to the spaces in which they arose as scalar products!

For the bare state basis, in the $\mathcal{G}$ space, the identity operator is

$$I_{\mathcal{G}} = |V_i\rangle\langle V_i| + \int_\Gamma dw|N\theta_p(\omega)\rangle\langle N\theta_p(\omega^*)| \tag{5.3}$$

while in the $\mathcal{G}^*$ space, the identity operator is

$$I_{\mathcal{G}^*} = |V_i\rangle\langle V_i| + \int_{\Gamma^*} dw|N\theta_p(\omega)\rangle\langle N\theta_p(\omega^*)|\,. \tag{5.4}$$

Since the two spaces are distinct, the two identity operators are distinct also. For the physical spectrum the corresponding identity operators are

$$I_{\mathcal{G}} = |M_\alpha\rangle\langle M_\alpha^*| + \int_\Gamma d\lambda|\lambda r\rangle\langle\lambda^* r| \tag{5.5}$$

$$I_{\mathcal{G}^*} = |M_\alpha^*\rangle\langle M_\alpha| + \int_{\Gamma^*} d\lambda |\lambda r\rangle\langle\lambda^* r| \,. \tag{5.6}$$

Quantity such as $|M_s\rangle\langle M_s|$ which appears in (5.2), is not an "outer product", *i.e.*neither in $\mathcal{G}$ space nor in $\mathcal{G}^*$ space, but an intertwining operator between these spaces, acting on $\mathcal{G}^*$ and producing vectors in $\mathcal{G}$.

Experimentally, one sees a weaker and weaker K^0-signal as the time of travel t increases. This is what is predicted in (5.2). Suppose instead we were to calculate the quantity

$$\begin{aligned} & \langle M_s^*(t)|K^0\rangle\langle K^0|M_s(t)\rangle \\ = \; & \langle M_s^*|e^{iHt}|K^0\rangle\langle K^0|e^{-iHt}|M_s\rangle \\ = \; & e^{-i(M_s-M_s)t}\langle M_s^*|K^0\rangle\langle K^0|M_s\rangle \end{aligned} \tag{5.7}$$

which would be time independent apart form a phase factor. This does not correspond to what we are looking for.

5.3 Experimental interpretation of the "overlap function":

Consider a coherent superposition of K_ℓ and K_s beams. Again we are interested in the detection of K_0-signal downstream. The probability of detection is given by

$$\begin{aligned} & |\langle K^0|e^{-iHt}|M_s\rangle + C\langle K^0|e^{-iHt}|M_\ell\rangle|^2 \\ & = \text{“direct terms”} + \text{“cross terms”} \end{aligned} \tag{5.8}$$

where

$$\text{“cross terms”} = C\langle K^0|K_\ell(t)\rangle \cdot \langle K^0|K_s(t)\rangle^* + \text{c.c.} \tag{5.9}$$

One sees that the interference term is again given by the multiplication of two inner products belonging to two distinct spaces. The "overlap function" of (2.12) of the Lee-Oehme-Yang model is not a bonafide amplitude, *i.e.*it does not correspond to a well defined inner product. The amplitude which has a proper inner product should be $\langle M_\beta^*|M_\alpha\rangle$. As we see in (3.31), the latter has the expected orthogonality property. We have seen also that the neglect of the continuum contribution is not the essential ingredient which accounts for the

nonvanishing of the overlap function, since to the extent that one neglects the form factor variation, we still have $\langle M_\beta^*|V_i\rangle\langle V_i|M_\alpha\rangle \approx 0$. See (3.31).

6 Derivation of the Bell-Steinberger relation.

The Bell-Steinberger relation[18] is usually associated with the unitarity relation. It is instructive to see how the corresponding relation arises within the present framework. We recall (3.9). The equation of the discrete solution is given by

$$K_{ij}^{\alpha}\, a_j = 0\,, \tag{6.1}$$

where

$$K_{ij}(\lambda) = \lambda\delta_{ij} - E_{ij}(\lambda) - m_{ij}\,. \tag{6.2}$$

With analytic sontinuation one gets,

$$E_{ij}(\lambda) = \int_\Gamma \frac{g_{iq}^+(\omega^*)g_{qj}(\omega)}{\lambda - \omega + i\epsilon}\,. \tag{6.3}$$

We deform the unitarity cut running along the positive real axis to the contour Γ, such that it "exposes" the discrete state solution. See Fig. 2. In terms of the E-function, the discrete solution at $\lambda = M_\alpha$ is given by

$$[m_{ij} + E_{ij}(\lambda)]\, a_\lambda = M_\alpha a_\lambda\,. \tag{6.4}$$

Taking the hermitean conjugate for the discrete solution at $\lambda = M_\beta$ gives

$$a_\lambda^+ \left[m^+ + E^+(\lambda)\right] = M_\beta^* a_\lambda^+\,. \tag{6.5}$$

But

$$[E\,(M_\beta)]^+ \int_{\Gamma^*} dw' \, \frac{g^+(\omega'^*)g(\omega')}{M_\beta^* - \omega' - i\epsilon} \equiv E(M_\beta^*) \tag{6.6}$$

where

$$\left[g^+(\omega^*)g(\omega)\right]^+ = g^+(\omega)g(\omega^*) = g^+(\omega'^*)g(\omega') \tag{6.7}$$

and $\omega' = \omega^*$ were used.

We assume each Yukawa coupling function in the Hamiltonian can be characterized by a coupling constant g_{pi} and a cutoff L_p. To evaluate $E(z + i\epsilon)$, where there is say one discrete

solution in the lower half plane, we choose the contour Γ such that it barely misses the point z. The principal value part

$$\begin{aligned} P(E(z)) &= \sum_p g^+_{jp} g_{pi} \left[\int_0^{z-\delta} \frac{dw}{z-\omega+i\epsilon} + \int_{z+\delta}^{L_p} \frac{dw}{z-\omega+i\epsilon} \right] \\ &= \sum_p g^+_{jp} g_{pi} \left[\ell n \frac{z}{\delta} - \ell n \frac{L_p}{\delta} \right] \\ &= \sum_p g^+_{jp} g_{pi} \, \ell n \frac{z}{L_p} . \end{aligned} \tag{6.8}$$

Using the identity

$$\frac{1}{z-\omega \pm i\epsilon} = P \frac{1}{z-\omega} \mp i\pi\delta(z-\omega) \tag{6.9}$$

$$E(z+i\epsilon) = \sum_p g^+_{jp} g_{pi} \, \ell n \left(\frac{z}{L_p} e^{-i\pi} \right) . \tag{6.10}$$

Assuming the bare mass matrix (m_{ij}) is hermitean, (6.4) and (6.5) lead to

$$\begin{aligned} & a^+_{\beta i} \left[E_{ij}(M^*_\beta) - E_{ij}(M_\alpha) \right] a_{j\alpha} \\ = \; & (M^*_\beta - M_\alpha) a^+_{\beta i} a_{i\alpha} = \sum_p \left(a^+_{\beta i} g^+_{jp} \right) \left(g_{pi} a_{i\alpha} \right) \left[2\pi i + \ell n \frac{M^*_\beta}{M_\alpha} \right] \end{aligned} \tag{6.11}$$

The last equality is a refined version of the Bell-Steinberger relation deduced based on the present theory.

For the kaon system, both the mass and the width differences between K_ℓ and K_s are small compared to the mean kaon mass, *i.e.*

$$\frac{M^*_\beta - M_\alpha}{M_\alpha} \ll 1 , \tag{6.12}$$

or

$$\ell n \left(\frac{M^*_\beta}{M_\alpha} e^{2\pi i} \right) \approx 2\pi i + \frac{M^*_\beta - M_\alpha}{M_\alpha} \approx 2\pi i . \tag{6.13}$$

Denote

$$\begin{aligned} \text{“}\langle\beta|\alpha\rangle\text{”} &= a^+_i(\beta) a_i(\alpha) , \\ \gamma^\alpha_p &= g_{pi} a_{i\alpha} , \\ \gamma^{\beta^+}_p &= a^+_{\beta i} g^+_{ip} \end{aligned} \tag{6.14}$$

(6.11) in the approximation (6.13) is reduced to the original form of the Bell-Steinberger relation:

$$\text{“}\langle\beta|\alpha\rangle\text{”}\left(M_\beta^* - M_\alpha\right) = 2\pi i \sum_p \gamma_p^{\beta+}\gamma_p^\alpha \,. \tag{6.15}$$

7 Summary

We have presented a theory for the neutral kaon system based on the extended Lee model. The spectrum of the theory consists of the discrete states on the second sheet, which are the K_ℓ and K_s states and the continuum states defined along a contour Γ. The spectrum spans the space $\mathcal{G}$. The bra states here are dual states of the ket states. For the discrete states, for both the bra states and the ket states are at $\lambda = M$. For the continuum states, if the ket state is defined at $\lambda + i\epsilon$ along the upper lip of the contour Γ, the bra state is at $\lambda - i\epsilon$, along the lower lip of Γ.

Our analysis indicates that the nonvanishing of the “$\langle K_\ell|K_s\rangle$” in LOY theory is related to the fact that the quantity does not correspond to a properly defined amplitude. It is the properly defined amplitude, which corresponds to the inner product in the $\mathcal{G}$ space, $\langle K_\ell^*|K_s\rangle$ is expected to vanish. As we see in 3.3, it does.

Lastly based on our present theory, we derived a refined version of the Bell-Steinberger relation. The refinement differs from the original relation in the order of $O\left(\frac{M_\ell^* - M_s}{M_s}\right)$. Although this difference is very insignificant for the neutral kaon, $D^0\bar{D}^0$, $B^0\bar{B}^0$ systems, it still remains a challenge to look for quantum systems in nature where such correction does lead to a detectable effect.

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Figure Captions

Fig. 1 The contour Γ and the exposed pole at M_α.

Fig. 2 Relationship between C_∞ and those counterclockwise contours enclosed by C_∞.

Appendix A: Orthogonality relations.

In this Appendix we verify the remainders of the orthonormality relations (4.8), (4.9) and (4.10), based on the wavefunctions defined in (4.1)–(4.4).

1. Verify $\langle\lambda^*, r|\alpha\rangle = 0$.

According to (4.8),

$$\langle\lambda^*, r|\alpha\rangle = \chi_{\lambda i}\eta_{\alpha i} + \int_\Gamma dw\, \zeta^r_{\lambda p}(\omega)\phi_{p\alpha}(\omega) \tag{A.1}$$

$$\int_\Gamma dw\, \zeta^r_{\lambda p}(\omega)\phi_{p\alpha}(\omega) = \int_\Gamma dw \left[\delta(\lambda-\omega) + \frac{\tilde{a}_{\lambda j}\tilde{g}_{ip}(\omega)}{\lambda-\omega-i\epsilon}\right]\frac{g_{pj}(\omega)c_{j\alpha}}{M_\alpha-\omega+i\epsilon}$$

$$= \frac{g_{pj}(\lambda)c_{j\alpha}}{M_\alpha-\lambda+i\epsilon} + \tilde{a}_{\lambda i}\int_\Gamma dw \frac{\tilde{g}_{ip}(\omega)g_{pj}(\omega)}{(\lambda-\omega-i\epsilon)(M_\alpha-\omega+i\epsilon)}\, c_{j\alpha}\,. \tag{A.2}$$

With the approach of (3.23), the second term

$$I_\Gamma = I_C \equiv \frac{\tilde{a}_{\lambda i}}{2\pi i}\int_C dw\, \frac{K_{ij}}{(\lambda-\omega-i\epsilon)(M_\alpha-\omega+i\epsilon)}\, c_{j\alpha}\,. \tag{A.3}$$

The contours are related by (analogous to Fig. 2)

$$C_\infty = C + C_\lambda,$$

or

$$I_C = I_\infty - I_\lambda\,, \tag{A.4}$$

$$I_\infty = -\frac{\tilde{a}_{\lambda i}}{2\pi i}\int_C dw\, \frac{\delta_{ij}\omega c_{j\alpha}}{\omega^2} = -\tilde{a}_{\lambda i}c_{\alpha i} = -\chi_{\lambda i}\eta_{i\alpha}, \tag{A.5}$$

$$I_\lambda = \frac{\tilde{a}_{\lambda i}K_{ij}(\lambda)c_{j\alpha}}{M_\alpha-\lambda+i\omega} = \frac{g_{pj}(\lambda)c_{j\alpha}}{M_\alpha-\lambda}\,. \tag{A.6}$$

Putting all these together, (A.1) is given by

$$\begin{aligned}\langle\lambda^* r|\alpha\rangle &= \chi_{\lambda i}\eta_{i\alpha} + \left[\frac{g_{ri}(\lambda)c_{j\alpha}}{M_\alpha-\lambda+i\epsilon} - \chi_{\lambda i}\eta_{i\alpha} + \frac{g_{rj}(\lambda)c_{j\alpha}}{M_\alpha-\lambda}\right] \\ &= 0\end{aligned} \tag{A.7}$$

2. Verify $\langle\beta^*|\lambda, r\rangle = 0$. The proof can be carried in similar fashion. We leave it as an exercise for the reader.

3. Verify $\langle\lambda^* r|\mu s\rangle = 0$.

$$
\begin{aligned}
\langle \lambda^* r | \mu s \rangle &= \left(\tilde{a}_\lambda,\, \delta(\lambda-\omega)\delta_{rp} + \frac{\tilde{a}_{\lambda j}\tilde{g}(\omega)_{jp}}{\lambda-\omega-i\epsilon} \right) \begin{pmatrix} a_\mu \\ \delta(\mu-\omega)\delta_{ps} + \dfrac{g_{pi}(\omega)a_{i\mu}}{\mu-\omega+i\epsilon} \end{pmatrix} \\
&= \tilde{a}_\lambda a_\mu + \delta(\lambda-\mu)\delta_{rs} + \frac{g_{ri}(\lambda)a_{i\mu}}{\mu-\lambda+i\epsilon} + \frac{\tilde{a}_{rj}\tilde{g}_{js}(\mu)}{\lambda-\mu-i\epsilon} \\
&\quad + \tilde{a}_\lambda \int_\Gamma dw\, \frac{\tilde{g}(\omega)g(\omega)\, a_\mu}{(\lambda-\omega-i\epsilon)(\mu-\omega+i\epsilon)} \qquad \text{(A.8)} \\
I_\Gamma &= \tilde{a}_\lambda \int_\Gamma \frac{\tilde{g}(\omega)g(\omega)a_\mu\, dw}{(\lambda-\omega-i\epsilon)(\mu-\omega+i\epsilon)} \\
&= -\frac{\tilde{a}_\lambda}{2\pi i} \int_C \frac{K(\omega)\, a_\mu\, dw}{(\lambda-\omega-i\epsilon)(\mu-\omega+i\epsilon)}. \qquad \text{(A.9)}
\end{aligned}
$$

Relationship between various contours is (analogous to Fig. 2):

$$C_\infty = C + C_\lambda + C_\mu\,,$$

and the corresponding intergrals are:

$$
\begin{aligned}
I_\Gamma &\equiv I_C = I_\infty - I_\lambda - I_\mu\,, \qquad \text{(A.10)} \\
I_\infty &= -\frac{\tilde{a}_\lambda}{2\pi i} \int_{C_\infty} \frac{\delta_{ij}\omega a_{\mu j}}{(-\omega)^2}\, dw = -\tilde{a}_\lambda\, a_\mu\,, \qquad \text{(A.11)} \\
I_\lambda &= +\frac{\tilde{a}_\lambda\, K(\lambda)a_\mu}{\mu-\lambda+i\epsilon} = \frac{g(\lambda)a_\mu}{\mu-\lambda+i\epsilon}\,, \qquad \text{(A.12)} \\
I_\mu &= +\frac{\tilde{a}_\lambda\, K(\mu)a_\mu}{\lambda-\mu-i\epsilon} = \frac{\tilde{a}_\lambda\tilde{g}(\mu)}{\lambda-\mu-i\epsilon}\,. \qquad \text{(A.13)}
\end{aligned}
$$

Putting all these together, (A.8) gives

$$
\begin{aligned}
\langle \lambda^* r | \mu s \rangle &= \tilde{a}_\lambda a_\mu + \delta(\lambda-\mu)\delta_{rs} + \frac{g(\lambda)a_\mu}{\mu-\lambda+i\epsilon} + \frac{\tilde{a}_\lambda\tilde{g}(\mu)}{\lambda-\mu-i\epsilon} \qquad \text{(A.14)} \\
&\quad + \left[-\tilde{a}_\lambda a_\mu - \frac{g(\lambda)a_\mu}{\mu-\lambda+i\epsilon} - \frac{\tilde{a}_\lambda\tilde{g}(\mu)}{\lambda-\mu-i\epsilon} \right] = \delta(\lambda-\mu)\delta_{rs} \qquad \text{(A.15)}
\end{aligned}
$$

Appendix B: Completeness relations.

In this appendix we verify the completeness relations of (4.12)-(4.15) based on the wave functions of (4.1)-(4.4).

1. Verify $\langle V_i|I_{\mathcal{G}}|V_j\rangle = \delta_{ij}$.

Based on (4.12),

$$\langle V_i|I_{\mathcal{G}}|V_j\rangle = \eta_{i\alpha}\chi_{\alpha j} + \int_\Gamma d\lambda\, \eta_{i\lambda}\chi_{\lambda j}\,, \tag{B.1}$$

$$\int_\Gamma d\lambda\, \eta_{i\lambda}\chi_{\lambda j} = \int_\Gamma d\lambda\, a_{i\lambda}\tilde{a}_{\lambda j}\,. \tag{B.2}$$

Since from (2.8), $Ka = \tilde{g}$

$$\begin{aligned} a\tilde{a} &= K^{-1}\tilde{g}\, g\widetilde{K^{-1}} \\ &= \frac{1}{2\pi i}\left[K^{-1}(K-\widetilde{K})\widetilde{K^{-1}}\right] \\ &= \frac{1}{2\pi i}\left[\widetilde{K^{-1}} - K^{-1}\right], \end{aligned} \tag{B.3}$$

where $K \equiv K(\lambda + i\epsilon)$, $\widetilde{K} \equiv K(\lambda - i\epsilon)$ with $\pm i\epsilon$ defined with respect to the contour Γ. Substituting (B.3) into (B.2),

$$I_\Gamma \equiv \int_\Gamma d\lambda a\tilde{a} = \frac{1}{2\pi i}\int_C K^{-1}d\lambda. \tag{B.4}$$

Cauchy theorem implies (analogous to Fig. 2): $C_\infty = \sum_\alpha C_\alpha + C$, or

$$I_C = -\sum_\alpha I_\alpha + I_\infty \tag{B.5}$$

$$K^{-1}(\lambda)_{ij} = \frac{\mathrm{Cof}\, K_{ij}}{\det K} \xrightarrow{\lambda\to\infty} \frac{\delta_{ij}}{\lambda} \tag{B.6}$$

so

$$I_\infty = \frac{1}{2\pi i}\int_{C_\infty} K^{-1}_{ij}d\lambda = \delta_{ij}\,. \tag{B.7}$$

The discrete state contribution is evaluated in Appendix C giving

$$I_\alpha = \frac{1}{2\pi i}\int_{C_\alpha} K^{-1}\, d\lambda = \eta_{i\alpha}\chi_{\alpha j} \tag{B.8}$$

Putting all these together, (B.1) gives

$$\langle V_i|I_{\mathcal{G}}|V_j\rangle = \eta_{i\alpha}\chi_{\alpha j} + [-\eta_{i\alpha}\chi_{\alpha j} + \delta_{ij}] = \delta_{ij}\,. \tag{B.9}$$

2. Verify $\langle N\theta_p|I_{\mathcal{G}}|V_i\rangle = 0$.

From (4.13),

$$\langle N\theta_p|I_{\mathcal{G}}|V_i\rangle = \phi_{p\alpha}\chi_{\alpha i} + \int_\Gamma d\lambda \phi^r_{p\lambda}\chi_{\lambda i} \tag{B.10}$$

$$\begin{aligned}\int d\lambda \phi^r_{p\lambda}\chi_{\lambda i} &= \int d\lambda \left[\delta(\lambda-\omega)\delta_{rp} + \frac{g_{pj}(\omega)a_{j\lambda}}{\lambda-\omega+i\epsilon}\right]\tilde{a}_{\lambda i} \\ &= \tilde{a}_i(\omega) + g_{pj}\int_\Gamma d\lambda \frac{a_{j\lambda}\tilde{a}_{\lambda i}}{\lambda-\omega+i\epsilon}\end{aligned}$$

$$I_\Gamma = g_p \int_\Gamma d\lambda \frac{a_\lambda \tilde{a}_\lambda}{\lambda-\omega+i\epsilon} = \frac{g_p}{2\pi i}\int_C \frac{d\lambda\, K^{-1}}{\lambda-\omega+i\epsilon} \tag{B.11}$$

where in the last step (B.4) was used. For the contour convention analogous to Fig. 2,

$$C_\infty = C + \sum_\alpha C_\alpha + C_\omega$$

or

$$I_\Gamma \equiv I_C = I_\infty - \sum_\alpha I_\alpha - I_\omega \tag{B.12}$$

$$(I_\infty)_{pi} = \frac{g_{pj}}{2\pi i}\int_{C_\infty} \frac{\delta_{ji}\, d\lambda}{\lambda^2} = 0 \tag{B.13}$$

$$\begin{aligned}(I_\alpha)_{pi} &= \frac{g_{pj}}{2\pi i}\int_{C_\alpha} \frac{K^{-1}_{ji} d\lambda}{\lambda-\omega+i\epsilon} = \frac{g_{pj}\eta_{j\alpha}\chi_{\alpha i}}{M_\alpha-\omega+i\epsilon} = \phi_{p\alpha}\chi_{\alpha i} \\ (I_\omega)_{pi} &= g_{pj}K^{-1}_{ji}(\omega) = \tilde{a}_i\,.\end{aligned} \tag{B.14}$$

Putting all together, (B.10) becomes

$$\langle N\theta_p|I_{\mathcal{G}}|V_i\rangle = \phi_{p\alpha}\chi_{\alpha i} + [\tilde{a}_i(\omega) - \phi_{p\alpha}\chi_{\alpha i} - \tilde{a}_i(\omega)] = 0. \tag{B.15}$$

3. Verify $\langle V_i|I_{\mathcal{G}}|N\theta_p\rangle = 0$.

From (4.14),

$$\langle V_i|I_{\mathcal{G}}|N\theta_p\rangle = \eta_{i\alpha}\zeta_{\alpha p} + \int_\Gamma d\lambda\, \eta_{i\lambda}\zeta_{\lambda p} \tag{B.16}$$

$$\int_\Gamma d\lambda \eta_\lambda \left[\delta(\lambda-\omega+\frac{\tilde{a}_\lambda\tilde{g}}{\lambda-\omega-i\epsilon}\right] = \eta(\omega) + \int_\Gamma d\lambda \frac{a_\lambda\tilde{a}_\lambda}{\lambda-\omega-i\epsilon}\tilde{g}(\omega) \tag{B.17}$$

$$I_\Gamma = \frac{1}{2\pi i}\int_C d\lambda \frac{K^{-1}_\lambda}{\lambda-\omega-i\epsilon}\tilde{g}(\omega)\,. \tag{B.18}$$

Analogous to Fig. 2,

$$C_\infty = C + \sum_\alpha C_\alpha + C - C_\omega,$$

or

$$I_\Gamma = I_\infty - \sum_\alpha I_\alpha - I_\omega \,, \tag{B.19}$$

$$I_\infty = 0 \,, \tag{B.20}$$

$$I_\alpha = \frac{1}{2\pi i}\int_{C_\alpha} \frac{d\lambda K_\lambda^{-1}}{\lambda-\omega-i\epsilon}\tilde{g}(\omega) = \frac{\eta_\alpha \chi_\alpha \tilde{g}}{M_\alpha-\omega-i\epsilon} = \eta_{i\alpha}\zeta_{\alpha p} \,, \tag{B.21}$$

$$I_\omega = \frac{1}{2\pi i}\int_{C_\omega} d\lambda \frac{K_\lambda^{-1}}{\lambda-\omega-i\epsilon}\tilde{g}(\omega) = K^{-1}(\omega)\tilde{g}(\omega) = \eta(\omega) \,. \tag{B.22}$$

Putting all these together, (B.16) becomes

$$\langle V_i|I_{\mathcal{G}}|N\theta_p\rangle = \eta_\alpha\zeta_{\alpha p} + (\eta - \eta_\alpha\zeta_{\alpha p} - \eta) = 0 \tag{B.23}$$

4. Verify $\langle N\theta_p(\omega)|I_{\mathcal{G}}|N\theta_q(\omega')\rangle = \delta(\omega-\omega')\delta_{pq}$.

From (4.15),

$$\langle N\theta_p(\omega)|I_{\mathcal{G}}|N\theta_q(\omega')\rangle = \phi_{p\alpha}\zeta_{\alpha q} + \int_\Gamma d\lambda \phi^r_{p\lambda}\zeta^r_{\lambda q} \tag{B.24}$$

$$\int_\Gamma d\lambda \left[\delta(\lambda-\omega)\delta_{rp} + \frac{g_p(\omega)a_\lambda}{\lambda-\omega+i\epsilon}\right]\left[\delta(\lambda-\omega')\delta_{rq} + \frac{\tilde{a}_\lambda\tilde{g}_q(\omega')}{\lambda-\omega'-i\epsilon}\right]$$

$$= \delta(\omega-\omega')\delta_{pq} + \frac{\tilde{a}_\omega\tilde{g}_q(\omega')}{\lambda-\omega'-i\epsilon} + \frac{g_p(\omega)a_{\omega'}}{\omega'-\omega+i\epsilon} + I_\Gamma \tag{B.25}$$

where

$$I_\Gamma = \int_\Gamma d\lambda \frac{g(\omega)a_\lambda\tilde{a}_\lambda\tilde{g}(\omega')}{(\lambda-\omega+i\epsilon)(\lambda-\omega'-i\epsilon)}$$

$$= \frac{g(\omega)}{2\pi i}\int_C \frac{d\lambda\, K^{-1}}{(\lambda-\omega+i\epsilon)(\lambda-\omega'-i\epsilon)}\tilde{g}(\omega') \equiv I_C \,. \tag{B.26}$$

Using the contour notation analogous to Fig. 2, here

$$C_\infty = C + \sum_\alpha C_\alpha + C_\omega + C_{\omega'} \tag{B.27}$$

or

$$I_C = I_\infty - \sum_\alpha I_\alpha - I_\omega - I_{\omega'} \,, \tag{B.28}$$

$$I_\infty = 0, \tag{B.29}$$

$$I_\alpha = \frac{g(\omega)a\tilde{a}\tilde{g}(\omega')}{(M_\alpha-\omega)(M_\alpha-\omega')} = \phi_\alpha\zeta_\alpha \,, \tag{B.30}$$

$$I_\omega = \frac{g(\omega)K^{-1}(\omega)\tilde{g}(\omega')}{\omega-\omega'-i\epsilon} = \frac{\tilde{a}(\omega)\tilde{g}(\omega')}{\omega-\omega'-i\epsilon} \,, \tag{B.31}$$

$$I_{\omega'} = \frac{g(\omega)K^{-1}(\omega')\tilde{g}(\omega')}{\omega'-\omega+i\epsilon} = \frac{g(\omega)\tilde{a}(\omega')}{\omega'-\omega+i\epsilon} \,. \tag{B.32}$$

Putting all these together, (B.24) becomes

$$\langle N\theta_p(\omega)|I_{\mathcal{G}}|N\theta_q(\omega)'\rangle = \delta(\omega - \omega')\delta_{pq} \tag{B.33}$$

Appendix C: Integral representation of discrete contributions.

From (B.4), we saw

$$
\begin{aligned}
(I_\Gamma)_{ij} &\equiv \int_\Gamma d\lambda \langle V_i|\lambda\rangle\langle\lambda^*|V_j\rangle = \int_\Gamma d\lambda \eta_{i\lambda}\chi_{\lambda j} \\
&= \frac{1}{2\pi i}\int_C d\lambda \left(K_\lambda^{-1}\right)_{ij} d\lambda \equiv (I_C)_{ij}\,. \qquad \text{(C.1)}
\end{aligned}
$$

Through the deformation of the contour the corresponding discrete state contribution at $\lambda = M_\alpha$ is given by

$$
I_{C_\alpha} = \frac{1}{2\pi i}\int_{C_\alpha} K_\lambda^{-1} d\lambda\,. \qquad \text{(C.2)}
$$

In this appendix we want to verify

$$
I_{C_\alpha} = \frac{1}{2\pi i}\int_{C_\alpha} K_\lambda^{-1} d\lambda = \langle V_i|\alpha\rangle\langle\alpha^*|V_j\rangle \qquad \text{(C.3)}
$$

This is used a number of times in Appendix B in proving the completeness relations.

Denote the 2×2 matrix

$$
K_\lambda = \begin{pmatrix} L_\lambda & I_\lambda \\ J_\lambda & L_\lambda \end{pmatrix}. \qquad \text{(C.4)}
$$

The subscript indicates that all the elements I, J, L as well as the matrix K depend on λ. From the residue theorem,

$$
I_{C_\alpha} = \frac{\text{Cof}\, K_\alpha}{(\det K_\alpha)'} \qquad \text{(C.5)}
$$

where

$$
\text{Cof}\, K_\alpha = \begin{pmatrix} L_\alpha & -I_\alpha \\ -J_\alpha & L_\alpha \end{pmatrix} \qquad \text{(C.6)}
$$

and

$$
(\det K_\alpha)' = 2L_\alpha L'_\alpha - I'_\alpha J_\alpha - I_\alpha J'_\alpha\,. \qquad \text{(C.7)}
$$

The positions of the discrete states are at $\lambda = M_\alpha$, which satisfy the constraints of $\det K_\alpha = 0$, or

$$
L_\alpha = \pm\sqrt{I_\alpha J_\alpha}\,. \qquad \text{(C.8)}
$$

Consider the "+" case. This corresponds to the K_s state, where

$$
\frac{I_s}{L_s} = \sqrt{\frac{I_s}{J_s}} \equiv \frac{p_s}{q_s}\,. \qquad \text{(C.9)}
$$

From (C.5) we get

$$
\begin{aligned}
I_{C_s} &= \frac{\mathrm{Cof}\, K_s}{\mathrm{Det}\, K_s} = \frac{\begin{pmatrix} 1 & -\frac{I_s}{L_s} \\ -\frac{J_s}{L_s} & 1 \end{pmatrix}}{2L'_s - \frac{J_s}{L_s} I'_s - \frac{I_s}{L_s} J_s} \\
&= \frac{\begin{pmatrix} 1 & -\frac{p_s}{q_s} \\ -\frac{q_s}{p_s} & 1 \end{pmatrix}}{2L'_s - \frac{q_s}{p_s} I'_s - \frac{p_s}{q_s} J'_s} . \qquad \text{(C.10)}
\end{aligned}
$$

To evaluate the right hand side of (C.3), we use the explicit forms (see (3.16) and (3.20)):

$$
c_s = \begin{pmatrix} p_s \\ -q_s \end{pmatrix} \text{ and } d_s = (q_s,\ -p_s) \qquad \text{(C.11)}
$$

$$
\begin{aligned}
\langle V_i|\alpha\rangle\langle\alpha^*|V_j\rangle &\equiv \eta_{i\alpha}\chi_{\alpha j} = N_s^2 c_{i\alpha} d_{\alpha j} \\
&= N_s^2 \begin{pmatrix} p_s \\ -q_s \end{pmatrix} (q_s,\ -p_s) \\
&= N_s^2 \begin{pmatrix} p_s q_s & -p_s^2 \\ -q_s^2 & p_s q_s \end{pmatrix} \qquad \text{(C.12)}
\end{aligned}
$$

$$
\begin{aligned}
N_s^{-2} &= d_s\, K' c_s = (q_s,\ -p_s) \begin{pmatrix} L'_s & -I'_s \\ -J'_s & L'_s \end{pmatrix} \begin{pmatrix} p_s \\ -q_s \end{pmatrix} \\
&= 2p_s q_s L'_s - q_s^2 I'_s - p_s^2 J'_s . \qquad \text{(C.13)}
\end{aligned}
$$

Substituting (C.13) into (C.12) and comparing with (C.10), we see (C.3) is satisfied. The "$-$" case corresponds to the K_ℓ–state where $c_\ell = \binom{p}{q}$ and $d_\ell = (q\ \ p)$.

Here

$$
\begin{aligned}
I_{C_\ell} &= \langle V_i|\ell\rangle\langle\ell^*|V_j\rangle \\
&= \frac{\begin{pmatrix} 1 & +\frac{p_\ell}{q_\ell} \\ +\frac{p_\ell}{q_\ell} & 1 \end{pmatrix}}{2L'_\ell + \frac{q_\ell}{p_\ell} I'_\ell + \frac{p_\ell}{q_\ell} J'_\ell} . \qquad \text{(C.14)}
\end{aligned}
$$

Finally as an extension to the identity (C.3), for any function $W(\lambda)$ which is regular in the neighborhood of $\lambda = M_\alpha$,

$$
\frac{1}{2\pi i} \int_{C_\alpha} W(\lambda) K_\lambda^{-1} d\lambda = W(M_\alpha)\langle V_i|M_\alpha\rangle\langle M_\alpha^*|V_j\rangle . \qquad \text{(C.15)}
$$

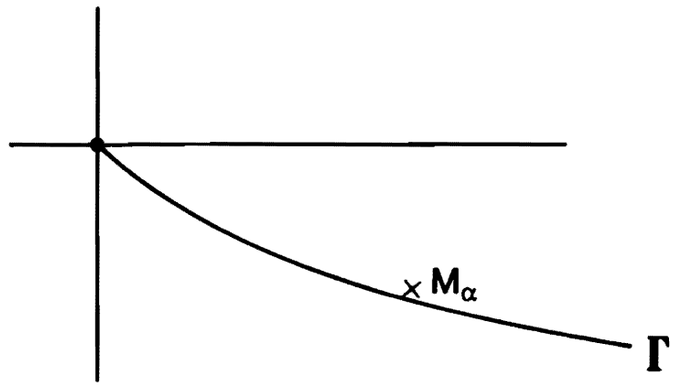
M_α
Γ

Fig. 1

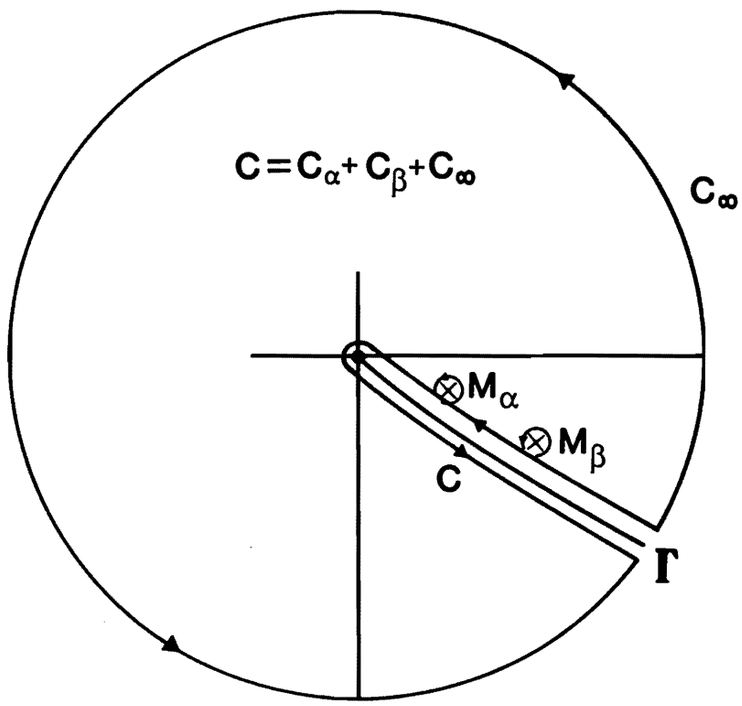
$C = C_\alpha + C_\beta + C_\infty$
C_∞
M_α
M_β
C
Γ

Fig. 2