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# The Optical Theorem in $N$ Dimensions

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# THE OPTICAL THEOREM IN $N$ DIMENSIONS

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## Abstract

We obtain the formula for the general $N$-dimensional optical theorem in quantum mechanics. The result is given by the simple expression $\sigma_{tot} + 2(\frac{2\pi}{k})^{\nu}\mathrm{Re}(i^{\nu}f(0)) = 0$, where $\nu = (N-1)/2$. The formula holds for arbitrary (anisotropic, non-local) short-range potentials.

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**1.** In [1] Adawi wrote a formula for the optical theorem in $N$-dimensional quantum scattering; he assumed isotropic potentials and used partial wave analysis. In this note we shall deduce the same result for the case of a general, anisotropic, non-local hermitean and short-range potential.

The optical theorem stems from the continuity equation $\dot{\rho} + \mathrm{div}\boldsymbol{j} = 0$, which should hold for any (hermitean) potential, with an appropriate definition of the current $\boldsymbol{j}$. By integrating the continuity equation in a large volume $V$, the theorem expresses the fact that there is no net current flux through the boundary,

$$0 = \frac{d}{dt}Q + \int_V div\boldsymbol{j}\ dv = 0 + \int_{\partial V} \boldsymbol{j}\cdot d\boldsymbol{s}$$

$$\int_{\partial V} \boldsymbol{j}\cdot d\boldsymbol{S} = 0. \qquad (1)$$

where $Q \equiv \int |\psi(\boldsymbol{r})|^2\ dv =$ cons. and $\partial V \equiv$ boundary of $V$ and we are applying the general Stoke's theorem $\int_V dw = \int_{\partial V} w$.

**2.** For the general scattering situation in arbitrary dimension $N$ we have the separation

$$\psi(\boldsymbol{r}) = \psi_{inc} + \psi_{scatt} \xrightarrow{r\ large} e^{ikz} + \frac{1}{r^{\nu}}f(\Omega)e^{ikr} \qquad (2)$$

where $\nu = (N-1)/2$ and $f(\Omega)$ is the scattering amplitude in the solid (hyper-) angle $\Omega$. Far enough from the potential region the current $\boldsymbol{j} = (\hbar/2im)(\psi^*\boldsymbol{\nabla}\psi -$ *complex conjugate*$)$ has three terms: $\boldsymbol{j}_{inc}$, $\boldsymbol{j}_{scatt}$ and $\boldsymbol{j}_{intf}$; the flux of the first two in the far away boundary $\partial V$ is given by

$$\int \boldsymbol{j}_{inc}\cdot d\boldsymbol{S} = v\int \hat{\boldsymbol{z}}\cdot d\boldsymbol{S} = 0,$$
$$\int \boldsymbol{j}_{scatt}\cdot d\boldsymbol{S} = v\sigma_{tot} \qquad (3)$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the incoming direction, and the velocity is $v = \hbar k/m$; (3) indicates that the incoming flux enters and leaves the large volume, and for $\boldsymbol{j}_{scatt}$ that the scattered wave is purely outgoing.

The current due to interference between the incoming and the outgoing wave is

$$\boldsymbol{j}_{intf} = \frac{\hbar}{2im}\frac{ik}{r^{\nu}}\left\{(e^{-ikz}e^{ikr}f(\Omega)\hat{\boldsymbol{r}} + e^{-ikr}e^{ikz}f^*(\Omega)\hat{\boldsymbol{z}} - c.c.)\right\} \qquad (4)$$

which oscillates wildly for large $r$ except in the forward direction $\Omega = \theta = 0$. Its contribution to the flux will be computed by taking as boundary $\partial V$ e.g. a sphere of very large radius

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$R$:

$$\int \boldsymbol{j}_{intf} \cdot d\boldsymbol{S} = \left(\frac{\hbar k}{m}\right) R^{\nu} \Re \int_{S_R^{2\nu}} d\Omega f(\Omega) \cdot \cdot e^{ikR(1-\cos\theta)}(1+\cos\theta). \quad (5)$$

No assumptions on the potential are made, besides hermiticity and short range character; for the usual three-dimensional case this is a well-known formula, see e.g. Gottfried [2].

As $R \to \infty$, the contribution is dominated by those angles which are close to $\theta = 0$; assuming that $f(\Omega)$ behaves smoothly in this limit, $f(\Omega) \simeq f(0)$; what remains is a purely geometric integral

$$\int_{S^{2\nu}} d\Omega e^{ikR(1-\cos\theta)}(1+\cos\theta) = (vol(S^{2\nu-1})) \int_0^{\pi} \sin^{2\nu-1}\theta \cdot \cdot e^{ikR(1-\cos\theta)}(1+\cos\theta) d\theta \quad (6)$$

where

$$\text{vol}\,(S^{n-1}) = \int d^n r e^{-r^2} / \int_o^{\infty} r^{n-1} e^{-r^2} dr = 2\pi^{\frac{n}{2}} / \Gamma(\frac{n}{2}). \quad (7)$$

(6) can be evaluated in terms of Bessel functions, as the $\theta$-integral is proportional to $J_{\nu-\frac{1}{2}}(kR)$. This is not necessary, however, as we can substitute (6) by an integral in the euclidean space $\mathbb{R}^{2\nu}$ tangent to the boundary sphere at the forward direction $\theta = 0$, the result being identical in our limit $R \to \infty$. So we have

$$\lim_{R\to\infty}(6) = \lim_{R\to\infty} \int_{S^{2\nu}} (2) e^{ikR\theta^2/2} d\Omega = \lim_{R\to\infty} 2 \int d^{2\nu} x e^{ikRx^2/2} = \lim_{R\to\infty} 2 \left[\int_{-\infty}^{+\infty} e^{ikRx^2/2} dx\right]^{2\nu} = \lim_{R\to\infty} 2 \left(\frac{2\pi i}{kR}\right)^{\nu}. \quad (8)$$

Therefore from (1), (3), (5), and (8) we obtain the *general optical theorem* [3]

$$\sigma_{tot} + 2(2\pi/k)^{(N-1)/2} \Re \left\{ i^{(N-1)/2} f(0) \right\} = 0. \quad (9)$$

**3.** Formula (9) agrees with [1], where it is expressed in terms of gamma functions, it is deduced through a much longer analysis, and holds only for local, isotropic potentials; our procedure is a simplification of that of Gottfried [2] (see also [4]), and applied to arbitrary dimensions. For $N = 2$ the integral in (6) is evaluated in [5].

In low dimensions we have the well known results:

$$\sigma_{tot} + 2\Re f(0) = 0 \quad (1\text{-}D) \quad (10)$$

$$\sigma_{tot} + 2\sqrt{\frac{2\pi}{k}} \{ \Re f(0) - \Im f(0) \} / \sqrt{2} = 0 \quad (2\text{-}D) \quad (11)$$

$$\sigma_{tot} = (4\pi/k) \Im f(0) \quad (3\text{-}D) \quad (12)$$

obtained respectively in [6], [5], and [3].

Some final comments are in order: 1) The theorem is eminently plausible on physical grounds: it says that in order to have scattering you have to deplete the incoming wave, and this is achieved by generating a forward scattering wave to interfere destructively with the incoming wave; this is exactly what formula (9) says; the factor $k^{-\nu}$ is obviously necessary for dimensional reasons; and the formula is non-linear,

characteristic of a unitary interference process, as $\sigma_{tot}$ and $f$ scale differently, $\dim[\sigma_{tot}] = [length]^{2\nu=N-1}$, $\dim[f(\Omega)] = [length]^{\nu}$.

2) The phase factor $\phi(i^{\nu}) = (N-1)(\frac{\pi}{4})$ is more subtle; it implies a phase increase by $\pi/4 = 45°$ as the spatial dimension $N$ increases by one; therefore, it shows *periodicity eight*. This is reminiscent of the periodicity theorem for the homotopy of the orthogonal groups of Raoul Bott; the periodicity is also present at the level of Clifford algebras [7]. In fact, the wave function $\psi(\boldsymbol{r})$ is a kind of spinor, or "square root" of the orthogonal tensorial object $\rho(\boldsymbol{r}) = |\psi(\boldsymbol{r})|^2$; this rough argument can probably be refined.

3) Our choice of the phase of the scattering amplitude $f(\Omega)$ in eq.(2) is not the only possible, so the final expression of the theorem might differ from ours, usually by factors of $\sqrt{i}$. This is true of Ref. [1]; also in [8] Adhikari argues that a different phase for $f$ might result in better analytical properties.

4) The optical theorem is a direct consequence of the continuity equation, that is, of the conservation of the probability; this, in turn, comes from the hermitean character of the hamiltonian, and consequently, of the *isometric* property of the $S$-matrix, $S^{\dagger}S = \mathbb{1}$. In fact, one can look at the relation (9) as an elaborate consequence of the normalization condition $\langle\psi \mid \psi\rangle = 1$, where $|\psi\rangle = |\psi_{inc}\rangle + |\psi_{sc}\rangle$, carried out in coordinate space, $\langle \boldsymbol{r} \mid \psi\rangle = \psi(\boldsymbol{r})$.

5) Complementary to the optical theorem, its cousin the *Levinson theorem* ( [9]; see also [10]) is also an elaboration of the completeness relation $\int |\psi\rangle\langle\psi| = \mathbb{1}$, which comes from the *unitary* character of the $S$-matrix, $SS^{\dagger} = \mathbb{1}$, this time in momentum space. We have now the split $\Omega_0 = \Lambda + \Omega = \mathbb{1}$, where $\Lambda$ projects to the bound states, and $\Omega(\Omega_0)$ to the continuum states of the actual (free) system. We hope to delve in this analogy in a future publication.

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