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BYZANTINE ASTRONOMY FROM A.D. 1300

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ABSTRACT

A Byzantine article from the 13th century contains advanced astronomical ideas and pre–Copernican diagrams. The models are geocentric but contain improvements on the trajectories of the Moon and Mercury. This talk presents several models and compares them briefly with the Astronomy of Ptolemy, Arabic Astronomies of that time and the heliocentric system.

1. Introduction

It is an important historical fact that Byzantium preserved the traditions and scientific knowledge of the ancient world. The Byzantines considered the traditions of ancient Greece and Rome to be their own heritage and preserved them for many centuries. Numerous studies have been written on the fields of literature, art, philopsophy, law, etc., but there are fewer studies on the scientific developments during the Byzantine period.

Among the valuable material delivered to us are scientific writings from ancient Greece. Historians of science state that "the majority of manuscripts on which our knowledge of Greek science is based are Byzantine codices, written between 500 and 1500 years after the lifetime of their authors".[1] Thus "while the Greek scientific heritage was [to a large extent] lost in Western Europe between the collapse of the Roman Empire in the fifth century and the translation movement of the twelfth and thirteenth centuries"[2], it remained intact in the Eastern Roman Empire (Byzantium) in manuscripts attributed to the ancient authors and at the same time it was modified in the articles and commentaries of Byzantine scholars. In contrast to Western Europeans "the Arabs had virtually full access to that [Greek] heritage from the eighth century onward. This occured because of a momentous translation effort whereby the great works of Greece and other cultures were translated in Arabic".[2] Later on (12th and 13th centuries) the classical knowledge was transmitted to Western Europe through Byzantine and Arabic sources and Irish monks who travelled across Europe founding monasteries and scriptoria.[3]

It is now interesting to ask, "As the Byzantines were copying the ancient texts for almost a thousand years, did they also study their contents?" We know that the texts were taught almost continuously at the University of Constantinople and the Patriarchal School, and, in addition, professors wrote commentaries and books (lecture notes) on these subjects. In addition recent studies of mathematical and astronomical texts[4] show that from the 11th to the 13th centuries Byzantine scholars began to question the ancient writings and started introducing their own improvements. Deviations from ancient theories have been established in astronomical texts, where the improvements in theorems and models are unambiguous.

Several studies of the past thirty years mention[5,6] that a short Byzantine article contains pre–Copernican figures and ideas. The article is of purely scientific nature and contains numerical parameters and 12 pages of diagrams which make possible the reconstruction of the models. For this reason it provides a unique opportunity for comparisons with the Astronomy of Ptolemy, Arabic Astronomies of this period and the heliocentric system developed later by Copernicus, Kepler and Galilei.

The article under discussion survives in three manuscripts. Two of them are in the Vatican Library and one in the Laurentiana Library of Florence.[5] It was written around A.D. 1280, and it is unsigned. David Pingree from Brown University made comparative studies and attributes the article to Gregory Chioniades,[7] who was born in Constantinople between 1240 and 1250 and died in Trabizond about 1320. Chioniades travelled extensively, first to Trabizond (Black Sea) and then to Tabriz (Iran) and became familiar with Persian and Arabic Astronomy. Since the article contains a complete astronomy of that time and deviates on several points from the classical tradition we prepared the edition and translation of the text together with an analysis of its contents. Our study appears in a book published together with Prof. P. Sotiroudis with the title *"The Schemata of the Stars"*.[8] I shall frequently refer to the Byzantine article as *"The Schemata of the Stars"*.

In this talk I will cover a few topics from the book trying to indicate the level of Astronomy at that time. Among them I will discuss:

1. Values for the obliquity and the precession of the equinoxes, which indicate the observational accuracy,

2. the shape of the earth, whether it is spherical or flat, and

3. models for the sun, the moon and the five planets. The models are very interesting because they apply a geometrical theorem of Arabic origin. For the comparison of the epicyclic with the Newtonian trajectories we need an analytic formalism for writing the epicyclic models. Several authors, including us, found it useful to visualize each circular motion as a rotating vector of the radius and then add up the rotating radii.[9] We discovered, in our studies, that the description of rotating vectors and the calculation of the resultant positions and velocities simplify tremendously when we write each vector as a complex function. The method will be used in the article and is briefly described in Appendix A.

2. Observational Accuracy

I will begin this section with several definitions. As the earth moves around the sun, it defines a plane: the *ecliptic.* In addition, the earth rotates every 24 hours around its axis passing through the north pole. The extension of this axis intersects the celestial sphere at a point called the *celestial north pole.* The equator is a plane perpendicular to the earth's axis of rotation. The two planes of the ***equator*** and the *ecliptic* do not coincide but form an angle of 23° 27'. This angle is called the ***obliquity.***

In addition, the earth rotates like a "top" and its axis of rotation is not fixed but precesses in a conical motion; that is the north pole is not fixed but precesses on a circle and completes a revolution in 26,000 years. Consequently, the celestial north pole coincides now with the star Polaris, but in 12,000 years it will move very close to the bright star Vegas. Another way of describing the precession of the celestial north pole is in terms of the *equinoxes.* The extension of the earth's equator intersects the celestial sphere on a circle: the celestial equator. The two circles – the ecliptic and celestial equator – intersect at two points the *equinoxes.* The precession of the celestial north pole can be described as a precession of the equinoxial points.

In a geocentric system the definitions are similar. In this system, the ecliptic is defined as the plane of the apparent motion of the sun. The definition of the various quantities in the geocentric system is illustrated in figure 1. The obliquity and the

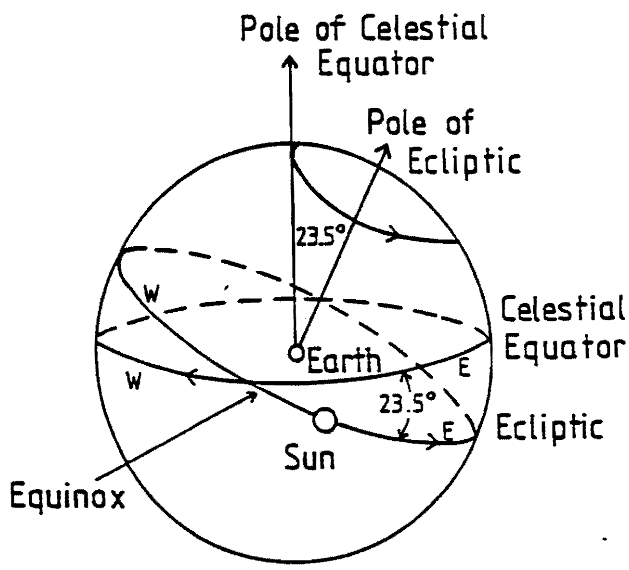

Fig. 1
Definitions of the celestial equator, the ecliptic and the obliquity.

precession of the equinoxes are responsible for two periodic events. The obliquity is responsible for the seasons and the precession of the equinoxes is important for the definition of the length of a year. In fact, the tropical year is defined as the time interval between two vernal equinoxes, being equal to 365.2422 days.

The Byzantine model is geocentric and has nine spheres. To explain the apparent motion of the fixed stars and the sun, Chioniades introduces three spheres: two for the fixed stars and one for the sun. The remaining six spheres are used for the moon and the five planets. The outer sphere of the universe is the ninth sphere, which rotates once every 24 hours. It carries with it the eight inner spheres with their stars and is responsible for day and night. All fixed stars, including the signs of the zodiac, are located on the eighth sphere which rotates very slowly and accounts for the precession of the equinoxes. Its axis of rotation is at an angle of approximately 23.5° relative to the axis of the ninth sphere. The angle between the two axes is the ***obliquity***. Chioniades reviews these values and we give a summary of them in Table 1.

	Chioniades	al–Tusi [10]
Astronomers preceding Ptolemy	24° 5'	
Ptolemy (127–151 A.D.)	23° 52'	23° 33' $< \varepsilon <$ 24°
Astronomers after Ptolemy	23° 35'	
At the time of Caliph al–Mamun (830 A.D.)	23° 33'	
al–Tusi (1260–1270 A.D.)	23° 30'	23° 35'

Table 1 : Values for the obliquity.

For comparison, we included the values of al–Tusi from an Arabic article known as *al–Tadhkira.*[10] Chioniades gives precise values for three groups of ancient observers. The mention of al–Tusi is interesting in itself, because Chioniades refers to his precise measurement which was the best value at that time. The value attributed to al–Tusi is smaller than the one written in the *al–Tadhkira.* There is an explanation for this. The smaller value quoted here was obtained by al–Tusi after the writing of the *al–Tadhkira.*

For the precession of the equinoxes Chioniades states that "according to the ancients it is 1° in 100 years; according to later scholars 1° in 66 years and completes a revolution in 24,000 years". In Table 2 we give a summary of values quoted by various authors. One degree in 100 years is the value adopted by Ptolemy and one degree in 66 years was found by astronomers working for Caliph al–Mamoun. The contemporary value is 1° in 72 years. This shows an accuracy of 9%. A change in the value for the precession implies a modification for the length of the year. This necessitates the reform of the Julian calendar which Nikephoros Gregoras recognized and proposed in the 14th century, but was not adopted for fear of religious unrest. The new calendar was finally introduced by Pope Gregory XIII in 1582.

Astronomer	Remark	Value for Precession
Hipparchus (190–120 B.C.)	found not less than	1^o in 100 years
Ptolemy	$\sim$ 100–165 A.D	1^o in 100 years
Abi Mansur (A.D. 830)	Astronomer of al–Mamoun	1^o in 66 years
Simeon Seth	11th century A.D.	1^o in 66 years
G. Chioniades	$\sim$ A.D. 1280	1^o in 66 years
N. Gregoras ($\sim$ 1290–1360)	measurement	1^o in 66 years
Contemporary value		1^o in 72 years

Table 2 : Values for the Precession of the Equinoxes

In the Middle Ages, a great effort was invested in improving angular measurements. The best values were achieved by Arab astronomers and al–Tusi states that "a difference in position of less than 10' is undetectable."[11] Measurements of the distances to the planets were much worse and observers could measure accurately only the parallax of the moon. Other distances were much less accurate and were obtained by indirect methods.

3. The Shape of the Earth

It is generally believed that everybody who read Aristotle knew that the earth is spherical. The shape of the earth is not mentioned explicitly in "The Schemata of the Stars", but in five diagrams the earth is drawn as a spherical globe. In figure 2 I show two of the diagrams for solar and lunar eclipses where the earth is spherical. The article was written some 200 years before the voyage of Columbus to America and here we have one more evidence that the Byzantines considered the earth to be spherical. Arabic astronomy also considered the earth to be spherical.

An exception to this rule is an article by Cosmas Indicopleustes who wrote the "Christian Topography" around A.D. 550. There he describes the earth to be flat with an inclination relative to the sun, which explained the divisions of day and night. This simple view was not taken very seriously and it was not thought worthy of mention by medieval commentators.[12]

4. The Motion of the Sun

As mentioned already, the models for the sun and the planets are geocentric. For each celestial body we shall introduce a system of spheres whose axes and rates of rotation are at our disposal. They must be chosen appropriately, so that the resultant motion reproduces the correct longitudes, angular velocities as well as stations and retrogrations. These epicyclic models are an approximation to the elliptic motions through a superposition of uniform circular motions.

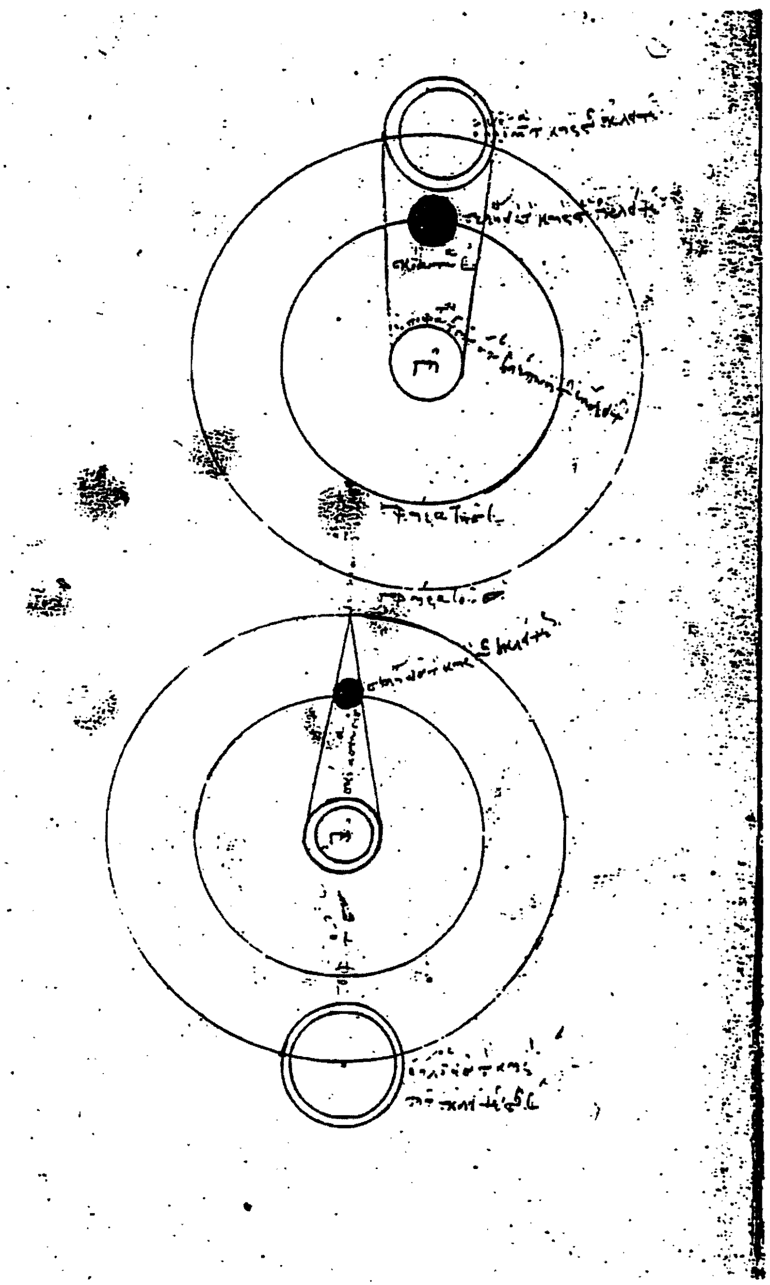

Fig. 2
Demonstration of solar (above) and lunar (below) eclipse with a spherical earth in the middle (from Vat. Gr. 211 f. 119 v).

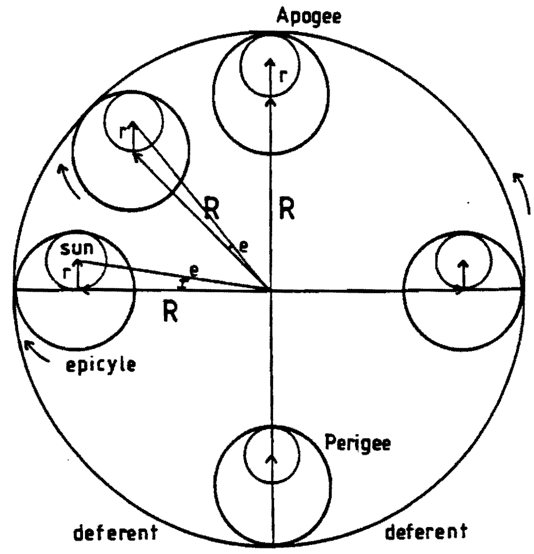

Fig. 3
Solar Model.

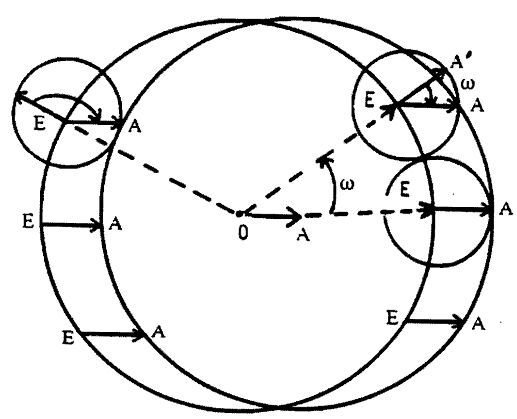

Fig. 4
Generation of an Eccentric Trajectory.

I begin with the model for the sun. First there is the fireball of the sun. This is the sphere where the sun is located. Next, the sphere of the sun is located inside the epicycle and the surface of the sun touches the concave surface of the epicyle. Finally the epicyle is tangent to the concave side of the major sphere, i.e. the deferent. The system for the circles of the sun is shown in figure 3. As the deferent rotates through the signs of the zodiac, it carries with it the epicycle. The epicycle rotates with the same angular velocity as the deferent but in opposite sense. The net effect is that the sun rotates on a circle which is eccentric relative to the earth. This model describes the apparent movement of the sun and it was adequate for the observational accuracy of the Middle Ages. It is an ancient model invented by Hipparchus, which was known to Ptolemy. It has been used by astronomers in Byzantium and the Arab World.

I describe the model with complex functions because it is easy to generalize it for models with many epicycles. All the motions of the planets lie on planes. Planar motions are easily described in polar coordinates or with complex functions. The location of the point E of the major circle is given by

$$Z_0(t) = Re^{i\omega t}. \tag{1}$$

This complex function defines the vector $\overrightarrow{OE}$, in figure 4, which rotates with angular velocity $\omega_1 = \omega$. The epicycle defines a second vector $\vec{r}_1 = \overrightarrow{EA}$ which rotates with angular velocity $\omega_2 = -\omega$. Because the center of the epicycle is carried around by the deferent, the angular velocity of the vector $\overrightarrow{EA}$ relative to the axes ReZ and ImZ, after the combined rotations ω_1 and ω_2, is

$$\tilde{\omega}_2 = \omega_1 + \omega_2 = \omega - \omega = 0. \tag{2}$$

This is shown explicitly in figure 4 where after the first rotation the radius of the epicycle is in direction of EA'. After the second rotation ω_2 the radius of the epicycle is restored to the direction $\overrightarrow{EA}$. The position of the point A is given by

$$Z(t) = Re^{i\omega_1 t} + r_1 e^{i(\omega_1+\omega_2)t} = Re^{i\omega t} + r_1 \tag{3}$$

for $\omega_1 = \omega$ and $\omega_2 = -\omega$, as it is illustrated in figure 4. In Cartesian coordinates one obtains

$$\begin{aligned} x &= R\cos\omega t + r_1 & (4)\\ y &= R\sin\omega t, & (5) \end{aligned}$$

which is the equation of an eccentric circle.

5. Comparison with the Newtonian Trajectory

The eccentric trajectory is an approximation to the elliptic orbit. It is worthwhile to ask how accurate is the eccentric trajectory. According to Newtonian mechanics the elliptic trajectory is determined as

$$r(\theta) = \left(\frac{L^2/m\alpha}{1-\varepsilon\cos\theta}\right) \tag{6}$$

where L is the angular momentum, m is the mass of the planet, M is the mass of the sun, G is Newton's gravitational constant

$$\alpha = GM\,m \quad \text{and } \varepsilon \text{ the eccentricity.} \tag{7}$$

The earth rotates around the sun following equation (6), but when we consider the earth at rest then the kinematic equations describe the position and velocity of the sun as is observed from the earth.

The angular velocity of the sun, as seen from the earth, is derived from Kepler's second law. The law states that "the radius from earth to the sun sweeps equal surfaces at equal times" and gives

$$\frac{d\theta}{dt} = \frac{L}{mr^2}. \tag{8}$$

We can substitute r in terms of θ using equation (6), to obtain

$$\frac{d\theta}{dt} = \frac{m\alpha^2}{L^3}\left\{1 - 2\varepsilon\cos\theta + \varepsilon^2\cos^2\theta\right\}. \tag{9}$$

The angular velocity depends on the longitude of the sun. For the sun–earth system the eccentricity is

$$\varepsilon = 0.017\,, \tag{10}$$

and the other two terms in eq. (9) are

$$2\varepsilon\cos\theta \;=\; 0.034\cos\theta \tag{11}$$

$$\varepsilon^2\cos^2\theta \;=\; 2.9\times10^{-4}\cos^2\theta \tag{12}$$

which demonstrates that the corrections to the constant angular velocity are small.

Alternatively, the angular velocity of the sun can be calculated for the eccentric trajectory to be

$$\frac{d\theta}{dt} = \omega\left\{1 - \delta\cos\theta + \delta^2\cos2\theta + O(\delta^3)\right\} \tag{13}$$

with $\delta = r/R$ and $\theta = \omega t$. The derivation of this equation follows the method of Appendix A. After we identify $\delta = 2\varepsilon$ and $\omega = m\alpha^2/L^3$, the terms linear in $\cos\theta$ are identical and the difference is

$$\frac{d\theta}{dt}\Big|_{\text{Eccentric}} - \frac{d\theta}{dt}\Big|_{\text{Newton}} = (7\cos^2\theta - 4)\varepsilon^2\omega \leq 0.05' \;/\; \text{day}. \tag{14}$$

We see that the two trajectories and angular velocities are periodic, repeating themselves every time θ reaches 2π. The maximum difference in the angular velocities at any value of θ is $0.05'$ / day. One should note that such a small difference was undetectable with the experimental accuracy of that time (1300 A.D.).

For the accumulating effect over a year we must integrate eq. (13) over time. What is the corresponding equation for the Newtonian trajectory? Most books give the radius r as a function of the angle, but we need the angle as a function of time. The analytic solution does not exist but it is given by a power series expansion in the eccentricity. To the order that we need it[13]

$$\theta(t) = \left\{\omega t + 2\varepsilon \sin \omega t + \frac{5}{4}\varepsilon^2 \sin 2\omega t\right\} + O(\varepsilon^3). \tag{15}$$

It will be useful if textbooks warn the students that such equations and the corresponding one for $r(t)$ exist. Now comparing eq. (15) with the integrated eq. (13), we see that the terms $O(\varepsilon)$ are the same and the difference is $\frac{1}{4}\varepsilon^2 \sin 2\omega t$, which was again unobservable in the 13th century.

We demonstrated in this section that the epicyclic/eccentric model is very accurate. In fact the structure of eq. (15) indicates that the solution of the Kepler problem is approximated well by a trigonometric series and the epicyle models lead to such a series. All this means that the ancient and medieval astronomers were working with a very good approximation. However, they were missing a dynamic foundation for their models.

6. The Lunar Model

The model for the moon is very interesting because in all models the earth is located inside the moon's trajectory. In addition, the moon is relatively close to the earth so that irregularities on its trajectory and the parallax are large and were observable[14] at that time. The irregularities motivated astronomers to introduce models with many epicycles.

To keep track of the corrections they applied a theorem which classifies the perturbations. The theorem, or device, to be described below, was introduced by al–Tusi[11], and it is also used by Chioniades and Copernicus without giving credit to the previous authors. Nowadays it is known as the Tusi–device and its purpose was to produce a rectilinear motion out of two uniform circular motions. The theorem has several corollaries, which were not stated by the medieval astronomers but were incorporated in their models.

<u>Theorem</u>: Consider a circle of radius $R/2$ which is located inside and is tangent to the concave surface of a larger circle of radius R. Let the large circle rotate with angular velocity ω carrying with it the small circle, which is pivoted at its center and rotates in the opposite direction with angular velocity 2ω. Then any point on the circumference of the small circle moves along a diameter of the large circle.

Corollary 1: The motion of any point on the circumference of the small circle is a simple harmonic oscillation (see eq. (17)). This property is not stated in the Arabic or Byzantine texts.

Proof: Consider the point A which is the common point of the two circles and at time $t = 0$ lies on the real axis. As the large circle rotates, the center of the small circle, denoted as C, describes again a circle given by

$$C = \frac{R}{2} e^{i\omega t}. \tag{16}$$

Since the small circle rotates around its center with the angular velocity 2ω and in the opposite sense, the radius CA rotates with angular velocity $-\omega$, relative to the fixed axes, as demonstrated in figure 5. The trajectory of the point A is

$$Z(t) = \frac{R}{2} e^{i\omega t} + \frac{R}{2} e^{-i\omega t} = \frac{R}{2} \left[e^{i\omega t} + e^{-i\omega t} \right] = R \cos \omega t. \tag{17}$$

The first term, in this equation, corresponds to the motion of the center of the small circle. The second term gives the rotation of the radius CA of the small circle. We gave a proof with complex functions, but the original proof was geometric.

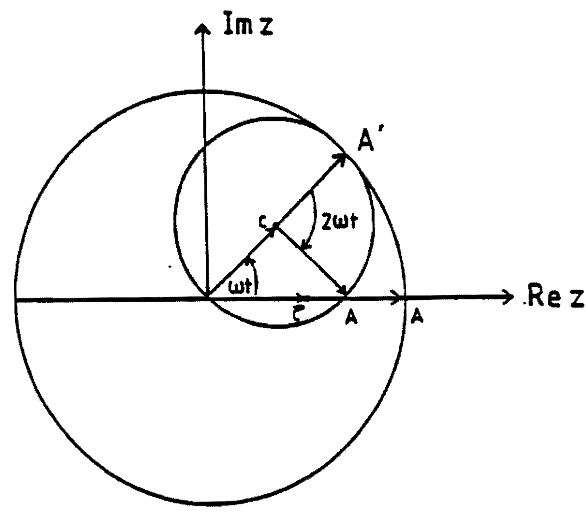

Fig. 5
Demonstration of the Theorem.

Corollary 2: When the ratio of the radii is not $1:2$, then every point on the circumference of the small circle describes an ellipse.

Proof: Let us denote by r the radius of the small circle. The trajectory is given by

$$Z(t) = (R - r)e^{i\omega t} + re^{-i\omega t}. \tag{18}$$

In Cartesian coordinates

$$x = R\cos\omega t \tag{19}$$
$$y = (R-2r)\sin\omega t \tag{20}$$

which is an ellipse. I return now to the problem of the moon.

Astronomers in ancient and medieval times could observe the position of the moon, its angular velocity and parallax. They noticed that the position of the moon and its angular velocity have different periods. This means when the moon comes back to the same position in the sky it does not always have the same angular velocity. They tried to account for this new phenomenon by introducing one more epicycle with a new angular velocity, called the *anomalistic velocity.* Putting all facts together, Ptolemy devised a complicated model, which accounted for the position and the angular velocity but predicted that the earth–moon distance should change by a factor of two. This disagreed with observations of the parallax.

The models of al–Tusi and Chioniades are based on the theorem and the corollaries discussed in this section. The models consist of

(i) a deferent (concentric or eccentric),

(ii) two epicyles which form a Tusi–couple, and

(iii) an additional epicycle which accounts for the *anomalistic velocity* of the moon.

The Tusi–couple introduces a small oscillation around the deferent which is easy to visualize.

Comparing al–Tusi with Chioniades, they seem to have the same model except for the last epicycle. In both cases the last epicycle rotates with the anomalistic velocity relative to a frame of reference. The two authors seem to select different frames of reference. Comparing *The Schemata of the Stars* with *al–Tadhkira*, we note that Chioniades exchanged the roles of the third and fourth spheres, but this does not have a physical consequence, because planar rotations commute. This was known to al–Tusi who also exchanged the order of the third and the fourth sphere in some of his articles.

Copernicus introduces[15] a simpler model with three spheres. He knows Tusi's theorem which he demonstrates with a diagram and gives a proof. For the lunar model, his first two spheres satisfy the conditions of corollary 2 and the third sphere introduces the anomalistic velocity. Then he selects the radii at the approximate ratios of 60 : 6 : 1, which produces a rapidly converging trigonometric series. This clever choice improves the value for the parallax. It has been discovered by Roberts,[16] however, that the lunar model and the numerical values for the radii in *De Revolutionibus* are identical to several decimals with those of Ibn ash–Shatir (1304–1375/76) – an astronomer from Damascus.

It is evident from this section that astronomical theories were brought to Byzantium from Persian and Arabic schools. The theories were studied by the Byzantine astronomers, who adopted the new methods to their knowledge and tradition. Later on the *Schemata of the Stars* arrived to Italy and became part of their scientific knowledge. When Copernicus was studying in Italy he must have been exposed to this scientific knowledge, and when he developed his astronomy he arrived at the same conclusions. Finally, it is a remarkable coincidence that the lunar models of Copernicus and Ibn ash-Shatir are so similar.

7. The Planets

The theories for the outer and inner planets also introduce new properties. For lack of space I will discuss only the model for Mercury and two of its properties.

Mercury and Venus are inner planets and, as seen from the earth, appear to follow the sun; being sometimes ahead and at other times trailing the sun. The maximum elongation for Venus is 47° and for Mercury 27°. To account for the motion of Mercury, Chioniades uses Corollary 2 to create an elliptic trajectory which follows the position of the mean sun. In the text he wrote, "The second sphere rotates in opposite sense to the signs of the zodiac at the rate 0° 59' per day. The third sphere rotates in the sense of the zodiac with angular velocity 1° 59' ... The mean distance between the centers of the two spheres is 6° 0'..." We note that the angular velocities are at the ratio of $\sim 1 : 2$ (within a minute) and the radii are at the ratio $10 : 1$. According to corollary 2 this arrangement produces an ellipse whose major axis precesses slowly. Thus the center of the epicycle moves on an ellipse which follows the sun. To this motion Chioniades adds one more epicylic which now complicates the trajectory. More details and figures for the trajectory can be found in our book[8]. After completing our book, I noticed the figures for the trajectories of Mercury, Venus and Mars appearing in the book are very similar[17] to trajectories surviving in copper etchings from A.D. 1609 and woodcuts from A.D. 1742, which means that the same computational methods were used for a long time.

At the end of the section on Mercury, Chioniades makes an interesting remark concerning latitude. He states that among the five planets four of them have their apogees in the northern hemisphere of the stellar globe, except for Mercury whose apogee is in the southern hemisphere. Here Chioniades may be stating the result of observations or he may be echoing an ancient tradition. In the Roman times Marcianus Capella, quoting Heraklides of Pontus, made the same remark. This description of the latitudes survived after the introduction of the heliocentric system with Rheticus and Copernicus making a similar statement.

Finally, a mixed model with Venus and Mercury rotating around the Sun, and all of them together with the outer planets rotating around the earth was introduced by Heraklides of Pontus. It was developed again by

Pierre D'Abano of Padua and is known to us as the Tychonic model. Such a model is also hinted in a manuscript of Mount Athos.[18] Here we have another example where the old scientific heritage was not forgotten but survived in articles of the Middle Ages.

8. Concluding Remarks

In this talk I tried to demonstrate that in the Middle Ages, as in our time, new theories did not appear overnight. New theories come about as the result of gradual improvements, which open the debate for new and radical postulates. In the case of astronomy the classical heritage played a crucial role. The classical knowledge survived in manuscripts of Byzantine codices, where there are also notable improvements.

The Byzantine era was also a time of scientific exchanges with the Arab world. Persian and Arabic articles were brought to Byzantium, translated into Greek and incorporated into their scientific knowledge. We described briefly improvements which were introduced for the trajectories of the Moon and Mercury.

At the same time we emphasized that the observational accuracy was not good enough to distinguish the geocentric from the heliocentric systems. Still the movement of the Sun, the Moon and the five planets remained one of the most outstanding and challenging problems in medieval scholarship. As the Byzantine articles came to the West, they must have also called the attention of other astronomers. Several discoveries of Byzantine and/or Arab origin appear in the same form in western articles – which means that they were taught or discussed actively in western scientific circles. This gradual improvement of methods and ideas culminated with the introduction of the heliocentric system.

Appendix A

Planar motion and its description in terms of complex functions

The latitudes of planets and the moon are relatively small and it is a good approximation to consider the trajectories as planar. The positions of the planets are determined by the radii of the deferent and epicycles which rotate. The description of these motions in terms of cartesian coordinates becomes complicated. We found that they are easily described with complex functions and explain the method in detail. The location of a point on the deferent is given by the function

$$z(t) = R\, e^{i\omega t}.$$

The quantity $\hat{e}_r = e^{i\omega t}$ is a unit vector in the radial direction, and $\hat{e}_\theta = i\, e^{i\omega t}$ is a unit vector in the angular direction. Describing the position of a point in terms of unit vectors and the calculus of complex functions we can calculate all relevant quantities. Consider the general motion of a point on the complex plane. The radius $r(t)$ and the angle $\theta(t)$ are both functions of time. The general position is

$$z(t) = r(t)e^{i\theta(t)} \tag{A.1}$$

and the velocity is given by the time derivative

$$v(t) = \dot{z}(t) = \dot{r}(t)e^{i\theta(t)} + r\dot{\theta}(t)(i\,e^{i\theta}) \tag{A.2}$$

with the dot indicating the derivative relative to time. Similarly the acceleration is given by the second derivative

$$a(t) = \ddot{z}(t) = (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (2\dot{r}\dot{\theta} + r\ddot{\theta})(i\,e^{i\theta}). \tag{A.3}$$

In summary:

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta}, \tag{A.4}$$

$$a_r = \ddot{r} - r\ddot{\theta} \quad \text{and} \quad a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} \tag{A.5}$$

are the well–known formulae for velocity and acceleration in polar coordinates. We can solve the above equation for the components. We note

$$\dot{z}z^* = r\dot{r} + i\,r^2\dot{\theta} \quad \text{with} \quad r = |z| = \sqrt{zz^*}$$

and therefore

$$\dot{r} = Re\,\frac{\dot{z}z^*}{|z|} \qquad \text{and} \tag{A.6}$$

$$\dot{\theta} = Im\,\frac{\dot{z}z^*}{|z|^2}. \tag{A.7}$$

With equations (A.6) and (A.7) we can calculate analytically several quantities occuring in the planetary motions. Applying the same steps to equation (A.3), we derive the components of the acceleration

$$a_r = Re\,\frac{\ddot{z}z^*}{|z|} \qquad \text{and} \tag{A.8}$$

$$a_\theta = Im\,\frac{\ddot{z}z^*}{|z|}. \tag{A.9}$$

We shall not have the occasion to use the last two equations, because in the late medieval times it was not recognized yet that acceleration is an important concept in physics and astronomy.

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