Weyl-Gauging and Curved-Space Approach to Scale and Conformal Invariance*

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Abstract

Gauging with respect to scale symmetry, which was actually the first example of gauging, is revived as a tool for investigating the properties of conformal field theories. It turns out that for a certain class of diffeomorphic- and Weyl-invariant theories the gauge-potential may be replaced by the Ricci tensor or scalar and this replacement allows the improved energy momentum tensor and the Virasoro central charge to be derived in a simple intuitive manner. A criterion is derived to determine the class of theories for which the replacement is possible and is found to be the curved space analogue of the criterion for rigid scale invariance to imply conformal invariance.

1. Some Ancient History

We begin by explaining what is meant by Weyl-gauging. Following Einstein's gravitational theory, Weyl in 1918 attempted to incorporate electromagnetism into the theory by gauging the metric tensor [1] i.e. by letting

$$g_{\mu\nu} \to e^{\gamma \int W_\mu dx^\mu} g_{\mu\nu} \tag{1.1}$$

where γ was a constant and the vector-field W_μ was to be identified with the electromagnetic vector potential. It was pointed out by Einstein (in the same paper) that, although this idea was attractive, it was physically untenable because it would imply that the spacing of spectral lines would depend on the history of the emitting atoms, in manifest disagreement with experiment. However, after the advent of Wave Mechanics in 1926, the idea was resurrected by London [2] who pointed out that Weyl's original proposal was correct but had been applied to the wrong physical situations. What London noted was that the usual electromagnetic differential minimal principal $\partial_\mu \to \partial_\mu + \frac{i}{\hbar} A_\mu$ was equivalent to the integral minimal principle

$$\psi(x) \to e^{\frac{i}{\hbar}\int A_\mu . dx^\mu} \psi(x) \tag{1.2}$$

and that this was the correct version of Weyl's proposal, in which the constant γ was chosen pure imaginary and the electromagnetic factor was chosen to multiply the Schroedinger wave-function rather than the Einstein metric. The London observation was quite profound because it not only laid the foundations for modern gauge theory but brought electromagnetism into the realm of geometry. What we shall mean in this paper by Weyl-gauging is the extension of the original idea (1.1) to all fields, as will be explained in detail in section 7.

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2. Some Mediaeval History

To explain the problems which may be tackled by means of Weyl-gauging we consider some more recent developments. First, in 1971 there appeared a very interesting paper [3] in which it was pointed out that scale-symmetry, like γ_5-symmetry, had a quantum anomaly and that for a certain class of Lagrangian field theories scale invariance implied conformal invariance. A criterion for this class of Lagrangians was given, namely that a current j_μ, which they called the virial current, be a divergence. Formally

$$j_\mu = \partial^\nu J_{\mu\nu} \quad \text{where} \quad j_\mu = \pi^\nu (d\eta_{\mu\nu} + \Sigma_{\mu\nu})\phi \tag{2.1}$$

for some tensor $J_{\mu\nu}$, where π_μ is the canonical momentum, d is the scale dimension of the field, $\eta^{\mu\nu}$ is the Minkowski metric, $\Sigma_{\mu\nu}$ is the spin-generator and the Lagrangian is assumed to be a function only of the first derivatives of the fields. The Lagrangians which satisfy (2.1) have never been completely classified but the class includes at least the scale invariant Lagrangians which are of the usual renormalizable form. An interesting discussion of the quantum aspects of the problem is given in [4].

3. Some More Modern History

More recently, because of the interest in string theory, critical phenomena and statistical mechanics, attention has been focussed on 2-dimensional conformal field theories. One characteristic feature of these theories is that the conformal (Virasoro) algebra is infinite-dimensional, indeed the parameters are just the harmonic functions, and that it admits a central extension with parameter c. Recently, it has been shown that in the functional integral formulation of the Wess-Zumino-Witten model both the improved energy-momentum tensor and the value of the parameter c that characterizes the extension (and hence the physical model) can be obtained in a simple intuitive way by embedding the theory in curved space [5]. In that case the partition function

$$Z(g_{\mu\nu}) = \int d(\phi) e^{\int dx \sqrt{g} \mathcal{L}_{WZW}} \tag{3.1}$$

may be regarded as a Schwinger function with external currents $g_{\mu\nu}$ and its variation with respect to $g^{\mu\nu}$ produces the expectation value of the improved energy momentum tensor according to

$$\frac{\delta Z}{\delta g^{\mu\nu}} = < T_{\mu\nu} > \tag{3.2}$$

Furthermore

$$< T^\mu_\mu > = cR \tag{3.3}$$

where R is the Riemann scalar and c is the central charge. Although these results were obtained originally only for the free WZW theory, they are actually valid for a much wider class of theories, including interacting theories such as abelian and non-abelian Toda theories, provided that the fields are coupled to the Ricci tensor $R_{\mu\nu}$ and/or scalar R in an appropriate way . For example, for the Liouville theory with one scalar field ϕ and potential e^ϕ the coupling takes the form $eR\phi$, where e is the constant $1 + (\hbar/4\pi)$. In general it is the coupling to R that produces the improvement term for the energy momentum tensor, a result that can be seen immediately for the Liouville model, since, in the flat-space limit,

$$\frac{\delta \int R\phi}{\delta g^{\mu\nu}(x)} \quad \rightarrow \quad (\nabla_\mu \nabla_\nu - g_{\mu\nu}\nabla)\phi(x) \tag{3.4}$$

and the right-hand side of (3.4) will be recognized as the standard improvement term [6] [7] for the energy-momentum tensor of that theory.

4. Present Paper

What is the relevance of the ancient and mediaeval history to the more recent procedure of embedding the theory in curved space in order to obtain the improved energy-momentum tensor and the Virasoro centre? The answer is that the recent procedure does not always work and thus raises the question as to why and when it does work It turns out that the procedure works for those theories in which W_μ may be converted by partial integration to its derivatives, which, in turn, may be converted to the Ricci tensor or scalar. At the infinitesimal level the criterion for this turns out to be just the criterion (2.1) except that the ordinary derivative is replaced by the covariant one. What is interesting, however, is that the present derivation of (2.1) is based on gauge-theory whereas the former derivation was based on the Lie algebra of the conformal group. The alternative derivation thus affords a completely new insight into the meaning of the criterion.

For example, in curved space the scale-invariance $\rightarrow$ conformal-invariance result is seen to be a simple consequence of the fact that the conformal group is contained in those subgroups of the local Weyl group for which R and $R_{\mu\nu}$ scale homogeneously and for which, therefore, the Lagrangian need not be gauged.

In this paper we shall consider for simplicity only the case of scalar fields. In the scalar case the criterion (2.1) reduces to the condition that the virial current be a gradient i.e.

$$j_\mu = \partial_\mu S \tag{4.1}$$

for some scalar S. The scalar-field Lagrangians for which (4.1) holds are essentially the standard Lagrangians which are bilinear in the first derivatives of the fields with coefficients which are for the most part numerical but could possibly depend on the field in question. A counter-example for which the coefficients depend on other fields will be given as illustration.

5. Rigid Scale Invariance

Let us consider Poincare-invariant Lagrangians that are quadratic in the first derivatives of a set of scalar fields ϕ^a denoted collectively by ϕ,

$$L = \frac{1}{2}\eta^{\mu\nu} h_{ab}(\phi)\partial_\mu \phi^a \partial_\nu \phi^b + V(\phi) \tag{5.1}$$

The condition that a general Lagrangian be rigid scale-invariant is the condition that it be invariant with respect to $x_\mu \rightarrow \lambda x_\mu$ and $\phi^a \rightarrow \lambda^{d_a}\phi^a$ where λ is constant and d_a is the dimension of the field ϕ^a. One sees that for $n \geq 3$ the condition that the

Lagrangian (5.1) be scale-invariant reduces the Lagrangian and field transformation to

$$L = \frac{1}{2}\eta^{\mu\nu}h_{ab}\partial_\mu\phi^a\partial_\nu\phi^b + V(\phi) \qquad \text{and} \qquad \phi^a \to \lambda^d\phi^a \qquad d = \frac{2-n}{2} \tag{5.2}$$

respectively, where the h_{ab} are constants and V transforms homogeneously with degree $-n$

$$V(\lambda^d\phi^a) = \lambda^{-n}V(\phi^a) \quad \text{e.g} \quad V(\phi) = f\phi^{\frac{2n}{(n-2)}} \tag{5.3}$$

where ϕ is a single field and f is a constant. Note that because the kinetic term is bilinear in all the fields they have a universal scaling index $d_a = d$.

The case $n = 2$ is special because the fields ϕ are then dimensionless and this allows the two possibilities

$$(a) \qquad L = \frac{1}{2}\eta^{\mu\nu}h_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b \qquad\qquad \phi^a \to \phi^a \tag{5.4}$$

which is the sigma-model, and

$$(b) \qquad L = \frac{1}{2}\eta^{\mu\nu}h_{ab}(\hat{\phi})\partial_\mu\phi^a\partial_\nu\phi^b + e^\theta V(\hat{\phi}) \qquad\qquad e^\theta \to \lambda^{-2}e^\theta \quad \hat{\phi} \to \hat{\phi} \tag{5.5}$$

where θ is a linear combination of the fields ϕ and the $\hat{\phi}$ are the remaining fields. The special case of (5.5) in which the Lagrangian takes the form

$$L = \frac{1}{2}\eta^{\mu\nu}\left[\partial_\mu\theta\partial_\nu\theta + \hat{h}_{\alpha\beta}(\hat{\phi})\partial_\mu\phi^\alpha\partial_\nu\phi^\beta\right] + e^\theta V(\hat{\phi}) \tag{5.6}$$

is just the Liouville-model [6] [7] supplemented by some scalar fields $\hat{\phi}$ whose potential $V(\hat{\phi})$ is arbitrary.

6. Rigid Scale Invariance $\leftrightarrow$ Rigid Weyl Invariance

For Poincare-invariant Lagrangians the coordinates x_μ enter the Lagrangian only through the derivatives ∂_μ and the measure d^nx. Since the derivatives enter only in the bilinear form $\eta^{\mu\nu}..\partial_\mu..\partial_\nu$ and since

$$\eta_{\mu\nu} \to \lambda^2\eta_{\mu\nu} \quad \Rightarrow \quad \sqrt{g} \to \lambda^n\sqrt{g} \tag{6.1}$$

one sees that the scale transformation of the coordinates may be converted to the transformation (6.1) of the metric and conversely. It follows that any rigid scale (and Poincare) invariant Lagrangian is invariant with respect to the transformations

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \quad \text{and} \quad \phi \to \lambda^{-\frac{(n-2)}{2}}\phi \quad (n \geq 3) \qquad \phi \to \phi \quad \text{or} \quad \phi - 2\ln\lambda \quad (n=2) \tag{6.2}$$

of the metric and field, and conversely. The transformations (6.2), *which do not involve any change of coordinates*, are called rigid Weyl transformations. Thus for Poincare-invariant Lagrangians rigid scale symmetry and rigid Weyl symmetry are equivalent. The importance of this fact is that, as will be seen in the next section, rigid Weyl symmetry can be converted to local Weyl symmetry by gauging it in the usual manner.

7. Diffeomorphic Invariance

In order to convert rigid Weyl invariance into a local Weyl invariance we have to let $\lambda \to \lambda(x)$. But as this means letting the metric become x-dependent we see that the proper context for local Weyl invariance is the curvilinear version of the theory. Strictly speaking, since the metric only scales under the local Weyl group, it would be sufficient to consider only conformally flat curved spaces. But it will be more useful to consider generally curved spaces, at least for the present. As is well-known, any standard Poincare-invariant theory can be converted to its curvilinear version by gauging with respect to the diffeomorphic group

$$A = \int d^n x L(\eta^{\mu\nu}, \partial_\sigma, \Phi) \quad \to \quad \mathcal{A} = \int d^n x \sqrt{g} \mathcal{L}(g^{\mu\nu}, \nabla_\sigma, \Phi) \tag{7.1}$$

where ∇_μ denotes the covariant derivative, Φ is a generic notation for the fields and $g^{\mu\nu}$ is a metric that may describe either curvilinear coordinates or a genuine curvature of space. In the case of scalar fields ϕ we have, of course, $\nabla_\mu \phi = \partial_\mu \phi$.

8. Standard Weyl Gauging

Suppose now that we try to extend the rigid Weyl symmetry to a local Weyl symmetry by letting $\lambda \to \lambda(x)$ so that the Weyl scaling becomes $g_{\mu\nu} = \lambda^2(x) g_{\mu\nu}$. For the sigma-model this is automatic because ϕ does not change. For the other two cases, $n \geq 3$ and the Liouville model for $n = 2$, one has to gauge in the usual manner. For $n \geq 3$ the Weyl gauging consists of letting

$$\partial_\mu \phi \to (\partial_\mu + W_\mu)\phi \quad \text{where} \quad W_\mu \to W_\mu + \frac{n-2}{2} \partial_\mu \sigma \tag{8.1}$$

and for the Liouville model it consists of letting

$$\partial_\mu e^\theta \to (\partial_\mu + W_\mu) e^\theta \quad \text{where} \quad W_\mu \to W_\mu + 2\partial_\mu \sigma \tag{8.2}$$

where in both cases $\sigma(x) = \ln \lambda(x)$. Note that for the fields θ and $\hat{\phi}$ the gauging is inhomogeneous and trivial respectively

$$\partial_\mu \theta \to \partial_\mu \theta + W_\mu \qquad \partial_\mu \hat{\phi} \to \partial_\mu \hat{\phi} \tag{8.3}$$

Thus the diffeomorphic and Weyl invariant version of the action (5.2) is

$$\mathcal{A} = \int \sqrt{g} g^{\mu\nu} \frac{1}{2} (\partial_\mu \phi + W_\mu \phi).(\partial_\nu \phi + W_\nu \phi) + \sqrt{g} V(\phi) \tag{8.4}$$

where $\phi.\phi$ is shorthand for $h_{ab}\phi^a \phi^b$ and the diffeomorphic and Weyl invariant version of the Liouville action is

$$\mathcal{A} = \int \frac{1}{2} \sqrt{g} g^{\mu\nu} \Big[(\partial_\mu \theta + W_\mu)(\partial_\nu \theta + W_\nu) + (\partial_\mu \hat{\phi}.\partial_\nu \hat{\phi}) \Big] + \sqrt{g} e^\theta V(\hat{\phi}) \tag{8.5}$$

The important point is that by partial integration these actions can be written as

$$\int \sqrt{g}\Big\{\frac{1}{2}[\partial^\mu\phi.\partial_\mu\phi] + V(\phi) - \frac{1}{2}\Xi(\phi.\phi)\Big\} \quad \text{where} \quad \Xi = \nabla^\mu W_\mu - W^\mu W_\mu \tag{8.6}$$

and

$$\int \sqrt{g}\Big\{\frac{1}{2}g^{\mu\nu}\left[(\partial_\mu\theta)(\partial_\nu\theta) + (\partial_\mu\hat{\phi}.\partial_\nu\hat{\phi})\right] + e^\theta V(\hat{\phi}) - \Xi\theta + W^\mu W_\mu\Big\} \qquad \Xi = \nabla^\mu W_\mu \tag{8.7}$$

respectively. Thus for these potentials the vector W_μ manifests itself only through the diffeomorphic scalar Ξ and the gauging consists simply of adding terms of the form $\Xi(\phi.\phi)$ and $\Xi\theta$ to the respective Lagrangians.

9. Scaling Properties of Ξ

Let $\hat{X}$ denote any quantity X after a finite local Weyl scaling $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \lambda(x)^2 g_{\mu\nu}$ Since

$$\Xi = \nabla^\mu W_\mu - \epsilon W^\mu W_\mu \quad \text{where} \quad \nabla^\mu = \partial^\mu + \Gamma^\mu \qquad \Gamma^\mu = \frac{1}{\sqrt{g}}\partial_\nu\left(g^{\mu\nu}\sqrt{g}\right) \tag{9.1}$$

(and $\epsilon = 1, 0$ for $n \geq 3, n = 2$) the quantities we are most interested in are W_μ and ∇^μ. The scaling property of W_μ is given in (8.1) and (8.2) and from the definition of the Christoffel symbols or from [8] we find that the scaling properties of Γ^μ and ∇^μ are

$$\hat{\Gamma}^\mu = \lambda^{-2}(\Gamma^\mu + (n-2)\partial^\mu\sigma) \qquad \rightarrow \qquad \hat{\nabla}^\mu = \lambda^{-2}(\nabla^\mu + (n-2)\partial^\mu\sigma) \tag{9.2}$$

From (9.2) it follows that the scaling property of Ξ is

$$\hat{\Xi} - \lambda^{-2}\Xi = \frac{(n-2)}{2}\lambda^{-2}\left[\nabla^2\sigma + \frac{(n-2)}{2}\partial^\mu\sigma\partial_\mu\sigma\right] \tag{9.3}$$

Note that the terms in W_μ have cancelled on the right-hand side, leading to the interesting result that although Ξ depends quadratically on W_μ the quantity $\hat{\Xi} - \lambda^{-2}\Xi$ is independent of W_μ.

For $n = 2$ the argument is a little different because $\Xi = \nabla^\mu W_\mu$. However in that case Γ^μ scales homogeneously i.e. $\hat{\Gamma}^\mu = \lambda^{-2}\Gamma^\mu$ and hence

$$\hat{\Xi} = \hat{\nabla}^\mu \hat{W}_\mu = \lambda^{-2}(\nabla^\mu W_\mu - 2\nabla^2\sigma) \quad \rightarrow \quad \hat{\Xi} - \lambda^{-2}\Xi = -2\lambda^{-2}\nabla^2\sigma \tag{9.4}$$

which is the analogue of (9.3) for the case $n = 2$.

10. Metrical Weyl Gauging

We now come to a remarkable feature of these theories, namely that the scaling properties of the quantity Ξ are (up to a numerical factor) identical to the scaling properties of the Riemann scalar R! Indeed with the conventions of [8] the scaling property of R is

$$\hat{R} - \lambda^{-2}R = 2(n-1)\lambda^{-2}\left[\nabla^2\sigma + \frac{(n-2)}{2}\partial^\mu\sigma\partial_\mu\sigma\right] \tag{10.1}$$

which is identical to (9.3) if R is replaced by Ξ and $2(n-1)$ by $(n-2)/2$. Note that is true only because the there is no W_μ dependence in (9.3). From (10.1) and (9.3) we have

$$\hat{\Xi} - \lambda^{-2}\Xi = m\left(\hat{R} - \lambda^{-2}R\right) \quad \text{where} \quad m \equiv \frac{(n-2)}{4(n-1)} \tag{10.2}$$

or equivalently

$$\left(\hat{\Xi} - m\hat{R}\right) = \lambda^{-2}\left(\Xi - mR\right) \tag{10.3}$$

In other words the combination $\Xi - mR$ scales homogeneously. This shows that if $\Xi = mR$ at any scale it is true at all scales.

But since W_μ was introduced only to compensate for the change in the field under scaling Ξ can be chosen at will at a given scale and hence in particular we can choose $\Xi = mR$ at a particular scale. With this choice we have

$$\Xi \equiv \nabla^\mu W_\mu - W^\mu W_\mu = mR \tag{10.4}$$

at all scales. Note that from the point of view of the Weyl field W_μ, for which $F_{\mu\nu}$ is not necessarily zero (10.4) is just a scale-invariant choice of gauge-fixing.

Once (10.4) holds the $n \geq 3$ Weyl-gauged action (8.6) becomes the purely metrical Lagrangian

$$\mathcal{A} = \frac{1}{2}\int \sqrt{g}\left\{g^{\mu\nu}\left[\partial_\mu\phi.\partial_\nu\phi\right] + V(\phi) - \frac{m}{2}R(\phi.\phi)\right\} \tag{10.5}$$

For $n = 2$ we may in the same way choose

$$\Xi \equiv \nabla^\mu W_\mu = -R \tag{10.6}$$

at any scale and it will be true at all scales. In that case the Weyl-gauged action (8.7) becomes

$$\mathcal{A} = \int \frac{1}{2}\sqrt{g}g^{\mu\nu}\left[(\partial_\mu\theta)(\partial_\nu\theta) + (\partial_\mu\hat{\phi}\partial_\nu\hat{\phi})\right] + \sqrt{g}e^\theta V(\hat{\phi}) + \sqrt{g}R\theta \tag{10.7}$$

where we have dropped the term $W^\mu W_\mu$ since it does not couple to the other fields. The Actions (10.5) and (10.7) are manifestly diffeomorphic invariant and one can verify directly that they are local Weyl-invariant.

Thus for the above Actions in curved space we have the remarkable result that, in contrast to the usual internal symmetry groups, the Weyl group can be gauged without introducing a new field in the form of a vector potential. Indeed (10.5) and (10.7) could have been chosen as the definition of Weyl-gauged actions. However we have proceeded via the gauge-potential W_μ so as to bring the argument into line with the usual procedure of gauging a rigid symmetry.

One might ask why the same procedure could not be adopted for the gauge-potential W^μ itself. For $n \geq 3$ there is an equation corresponding to (10.3) for W_μ, namely

$$\hat{W}^\mu - \frac{1}{2}\hat{\Gamma}^\mu = \lambda^{-2}(W^\mu - \frac{1}{2}\Gamma^\mu) \tag{10.8}$$

and thus it is tempting to set $W^\mu = \frac{1}{2}\Gamma^\mu$. But this would violate the diffeomorphic invariance because from the diffeomorphic point of view W^μ is a vector and Γ^μ is not (it is a connection). For $n = 2$ the situation is even worse because Γ^μ scales homogeneously and thus there is no equation analogous to (10.8).

11. Harmonic Weyl Group

Let us now consider the set of Weyl transformations with respect to which the Riemann scalar R (and hence Ξ) scales homogeneously. It is clear that this set will form a subgroup and from the scaling property of the Riemann scalar given in (10.1) we see that the elements $\lambda_h = e^{\sigma_h}$ are defined by

$$\nabla^2\sigma_h + \frac{(n-2)}{2}(\partial\sigma_h)^2 = 0 \tag{11.1}$$

Here it is understood that the transformations defined by σ_h are finite, not infinitesimal. Note that, in spite of being non-linear in σ_h, the property (11.1) is compatible with the abelian nature of the group because the metric also scales. Since the condition (11.1) reduces to $\nabla^2\sigma_h = 0$ for any σ_h for $n = 2$ and infinitesimal σ_h for any n we shall call the subgroup defined by (11.1) the harmonic Weyl group $\mathcal{W}_h$.

The importance of the harmonic Weyl group from our point of view is the following: For general Weyl transformations the local Weyl invariance of the Weyl-gauged actions (10.5) and (10.7) is due to the fact that the inhomogeneous parts of the Weyl variations of the ungauged part and R-part cancel. Indeed it was to effect this cancellation that the Weyl group was gauged in the first place! But for the harmonic Weyl transformations the R-part has no inhomogeneous variation. Hence the ungauged Lagrangian must also have no inhomogeneous variation. But since it is rigid Weyl invariant, it has also no homogeneous variation. Thus it is scale-invariant. It follows that, for the Lagrangians that we have considered we can omit the R-term if we gauge only with respect to the harmonic Weyl group. Thus, for these Lagrangians, we have the free-lunch
Theorem I: For Lagrangians that permit metrical Weyl gauging Rigid Weyl Invariance automatically implies Harmonic Weyl Invariance

Here by automatic we mean without any kind of gauging.

12. The Conformal Group

The conformal group is defined to be the set of coordinate transformations for which the change in the metric can be interpreted as a Weyl transformation i.e. for which

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}g_{\alpha\beta}(y) \equiv \hat{g}_{\mu\nu}(x) = e^{2\sigma_c(x)}g_{\mu\nu}(x) \tag{12.1}$$

The infinitesimal form of this condition, namely

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = 2 g_{\mu\nu} \sigma_c \quad \text{where} \quad \epsilon^\mu = \delta x^\mu \tag{12.2}$$

is called the conformal Killing equation. It is clear that the Weyl transformations $\sigma_c(x)$ induced by conformal transformations form a subgroup of the Weyl group and we shall call this subgroup the conformal subgroup $\mathcal{W}_c$. It is natural to ask what is the relationship between the conformal subgroup and the harmonic subgroup of the Weyl group. To answer this question we take the flat space limit of (12.2), from which it is easy to obtain the relations $\nabla.\epsilon = n\sigma_c$ and hence

$$(n-1)\nabla^2 \sigma_c = 0 \quad \text{and} \quad (n-2)\partial_\mu \partial_\nu \sigma_c = 0 \tag{12.3}$$

from which we see that σ_c must be linear in x for $n \geq 3$ but need be only harmonic for $n = 2$. Thus in the flat space limit the conformal subgroup is a subgroup of the harmonic subgroup for $n \geq 3$ and coincides with it for $n = 2$. This result agrees with the well-known fact that in flat space the infinitesimal conformal transformations are given by

$$\epsilon^\mu = 2(c.x)x^\mu - c^\mu x^2 \quad \sigma_c = 2c.x \quad (n \geq 3) \quad \text{and} \quad \epsilon_\pm = f_\pm(x_\pm) \quad 2\sigma_c = \partial.f \quad (n = 2) \tag{12.4}$$

where c^μ is a constant vector and $f_\pm$ are arbitrary differentiable functions.

13. The Flat-Space Limit

From the previous section we see that in the Minkowski (or Euclidean) limit the harmonic Weyl group is equal to the conformal group for $n = 2$ and contains it for $n \geq 3$. If we combine this observation with the equivalence of rigid Weyl-invariance and rigid scale-invariance discussed in section 2 we see that in the Minkowski limit theorem I becomes

Theorem II: For Actions that admit metrical Weyl gauging

$$Rigid-Scale-Invariance \;\leftrightarrow\; Rigid-Weyl-Invariance$$
$$\rightarrow Harmonic-Weyl-Invariance \;\rightarrow\; Conformal-Invariance$$

But this is just the well-known result that for such actions rigid scale-invariance implies conformal invariance. However, it shows that the class of Actions for which rigid scale-invariance implies conformal invariance are just those for which the Weyl gauging can be made metrical. Thus it provides a new criterion for this phenomenon to happen, namely that W_μ may be converted to Ξ by partial integration.

One may obtain an intuitive feeling as to what kind of Lagrangians might be expected to satisfy this criterion as follows: First, since W_μ occurs in the action only in the combination $\partial_\mu + W_\mu$ one would not expect Actions containing higher powers than quadratic of the first derivatives, or higher derivatives, such as $(\partial\phi)^4$ or $(\partial^2\phi)^2$, to qualify. Thus, in general, one would expect the criterion to be satisfied for at most those Actions which are quadratic in the first derivatives of the fields. Furthermore, since even for such Actions the partial integration would be obstructed if the coefficient of the kinetic term for a given field depended on any other field, we should expect the coefficients to be extremely restricted. Indeed the Actions would be essentially the traditional Actions with kinetic terms that are independent of the fields and quadratic in their derivatives.

14. Counter-Example for Quadratic-Derivative Lagrangians

To illustrate the argument in the last paragraph of section 9 we construct a class of counter-examples for which the Action is bilinear in the first derivatives of the fields but the coefficients are field-dependent. This class is obtained by considering the version of the $n = 2$ Lagrangian (5.5) in which

$$L = \frac{1}{2}\eta^{\mu\nu}\left[h(\hat{\phi})\partial_\mu\theta\partial_\nu\theta + \partial_\mu\hat{\phi}_\alpha\partial_\nu\hat{\phi}^\alpha\right] + e^\theta V(\hat{\phi}) \tag{14.1}$$

The Lagrangians (14.1) are in a certain sense the opposite of the Liouville Lagrangians for which the coefficient $h(\hat{\phi})$ was attached to the conformal-scalar fields $\hat{\phi}$ rather than θ. The Weyl-gauged version of (14.1) is evidently

$$L = \frac{1}{2}\eta^{\mu\nu}\left[h(\hat{\phi})(\partial_\mu\theta + W_\mu)(\partial_\nu\theta + W_\nu) + \partial_\mu\hat{\phi}_\alpha\partial_\nu\hat{\phi}^\alpha\right] + e^\theta V(\hat{\phi}) \tag{14.2}$$

and it is easy to see that unless the coefficient $h(\hat{\phi})$ is a constant the W_μ cannot be converted to Ξ plus terms which decouple. According to our analysis, therefore, the action (14.1) should not be conformally invariant and this can be verified directly.

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