Formation of Avalanches and Critical Exponents in Abelian Sandpile Model.
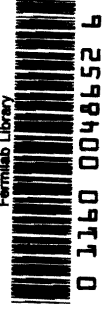
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Abstract:

The structure of avalanches in the Abelian sandpile model is analyzed. It is shown that an avalanche can be considered as a sequence of waves of decreasing sizes. Being more elementary events, waves admit the representation in terms of the q-component Potts model in the limit $q \to 0$. The decrement of waves follows the power law with the exponent α simply related with basic exponents of the sandpile model. Using known exponents of the Potts model, we derive α from scaling arguments.

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The sandpile model was introduced in the work [1] by Bak *et al.* to manifest the nature of "self-organized criticality" (SOC). The Abelian version of the model got most popular as it turned out analytically tractable [2]. Several characteristics of the Abelian sandpile were evaluated exactly: the total number of allowed configurations in the SOC state [2], the fractional number of sites having a given height [3,4], some height-height correlation functions [3,5], the expected number of topplings at a given site due to a particle added at another one [2].

Nevertheless, exact values of exponents characterizing avalanche processes remained unknown. The distribution of avalanches obeys the power law $P(S) \sim S^{-\tau}$ in which S is the number of distinct sites toppled during the relaxation. Exponents corresponding to the mass and linear extent of avalanches can be expressed in terms of τ [6,7]. Initial simulation studies of sandpiles [1] gave $\tau = 1$. The first theoretical predictions based on a continuous-energy model [8] and a Flory-like approximation [9] justified this result. Later on, Manna [10] undertook large-scale simulations and obtained the value $\tau = 1.22$. Meanwhile, the data of majority of numerical experiments were roughly consistent with $\tau = 7/6$ [6]. Simple mean-field arguments by Christiansen and Olami [7] led to a somewhat smaller value $\tau = 23/21$.

Recently, Pietronero, Vespignany and Zapperi [12] have presented a renormalization scheme of a new type that allowed them to estimate critical exponents of the sandpile model. They obtained $\tau = 1.253$.

Determination of τ needs a detailed analysis of the relaxation process. It would be desirable to represent the whole avalanche as a series of more elementary events and to express τ via auxiliary exponents. The first step in this direction has been made by Dhar and Manna who introduced the notion of inverse avalanches [13]. It was shown soon that there exists a direct procedure leading to the same representation of avalanches [14]. New objects representing basic elements of the avalanche were termed "waves of toppling".

In this Letter, we use the wave construction for finding the critical exponents of the 2D Abelian sandpile model. We will show that a typical avalanche can be considered as a

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sequence of waves of decreasing sizes. Each site involved into a wave topples only once. This permits us to define a spanning tree representation for waves and to find their distribution exactly. The decrement of size s of subsequent waves Δs follows also the power law $\Delta s \sim s^{\alpha}$ where the exponent α is simply related to τ. The problem of evaluation α can be formulated in terms of spanning trees or equivalently of the q-component Potts model in the limit $q \to 0$. Using known exponents of the latter model, we will derive the exponent α from scaling arguments. We estimate α from simulations and find good agreement between the measured and derived values.

The model we consider is a cellular automaton defined on a $N \times N$ square lattice $\mathcal{L}$. The sandpile is characterized by the number of particles or integer heights z_i at all sites i and is specified by two rules:

(i) Adding a particle at a random site: $z_i \to z_i + 1$;

(ii) Toppling of unstable sites: if any $z_i > 4$, then $z_j \to z_j - \Delta_{ij}$ for all $j \in \mathcal{L}$.

The toppling matrix Δ is the discrete Laplacian which has, in the case of a square lattice, non-zero elements $\Delta_{ii} = 4$ for all i and $\Delta_{ij} = -1$ for all pairs of adjacent sites i and j. It is convenient to introduce an additional site i_0 connected with all boundary sites to be a sink of toppled particles.

All stable configurations of heights which are allowed in the SOC state have the same probability [2]. To determine if a given configuration is allowed, Majumdar and Dhar [6] have introduced a "toppling from sink" together with a given order of preference for successive topplings of sites. By this procedure, one adds a particle to each site connected with i_0. All sites of $\mathcal{L}$ topple exactly once if and only if the configuration is allowed. Drawing all bonds connecting pairs of sites toppled at successive moments of time, one obtains a spanning tree covering a given lattice. The point i_0 is the root of the tree T_0. The collection of all possible rooted spanning trees $\{T_0\}$ is in one-to-one correspondence with the set of allowed configurations.

An avalanche is a perturbation of a stable state. It begins when a particle is dropped on a site of height 4 and stops when all sites become stable again. The Abelian property

admits an arbitrary order of topplings of non-stable sites during an avalanche. To introduce the waves of topplings, we carry out the process of relaxation in a specific way [14]. As usual, let us start with adding a particle to the site i of height 4 in an allowed configuration C. Topple it once and then topple all sites that become unstable keeping the site i out of the second toppling. We call the set of toppled sites "the first wave of topplings".

The site i loses 4 and receives m particles ($0 \leq m \leq 4$) besides the added one during the first wave. If the resulting height $z_i = 5$, we topple the site i the second time and continue the avalanche not permitting this site to topple the third time. The set of relaxed sites at this stage is "the second wave". The process continues producing intermediate configurations $C_1, C_2, ...C_n$ until the site i becomes stable and the avalanche stops.

All sites involved into the k-th wave ($k \geq 1$) topple only once during this wave. Indeed, to topple a site j twice, we have first to topple one of its neighbor sites j_1. The second toppling at j_1 is possible only after a second toppling at its neighbor $j_1 \neq j_2$. Continuing, we obtain the chain $j_1, j_2...$ which contains the initial site i for the finiteness of the wave. However, by definition, the site i topples once during the given wave, therefore other sites of the wave topple once, as well.

The construction of waves admits a spanning-tree interpretation. For this purpose, we introduce the sandpile model on an auxiliary lattice $\mathcal{L}'$, consisting of the original lattice $\mathcal{L}$, the site i_0, connected with boundary sites of $\mathcal{L}$ and an additional bond connecting the site i_0 and a given site i inside the lattice. If we consider the toppling from the sink for each allowed configuration on the new lattice $\mathcal{L}'$, we obtain, as a result, the set of spanning trees covering $\mathcal{L}'$ and having the root i_0. The trees obtained are of two classes. The first consists of trees without the bond $(i_0\ i)$ and therefore coincides with the set of one-rooted spanning trees $\{T_0\}$ defined above. The trees of the second class contain the bond $(i_0\ i)$. On removing the bond $(i_0\ i)$ a subtree of the whole tree gets disconnected. We obtain a two-rooted spanning tree on the original lattice $\mathcal{L}$ consisting of two components T_i' and T_0' having the roots at the sites i and i_0.

Now, we can select a particle dropped on i among all particles added to sites connected

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with i_0. This particle can be considered as a perturbation giving rise to an avalanche on $\mathcal{L}$. Since the site i on the lattice $\mathcal{L}'$ is connected with i_0, it topples only once and this avalanche is actually the wave. The corresponding subtree T_i' and its supplementary component T_0' are the graphic portrait of intermediate configurations C_k appearing after a k-th wave. To construct the subtree corresponding to the first wave, one can start with a configuration C which is allowed simultaneously on the lattices $\mathcal{L}$ and $\mathcal{L}'$. To select exactly the k-th wave for an arbitrary k, one can first add $k-1$ particles at i and then apply the toppling from the sink. An allowed configuration on $\mathcal{L}$ appears again after the last wave.

The graph representation of waves enables us to link the toppling process with the lattice Green function $G = \Delta^{-1}$ that is the solution of the Poisson equation with the boundary conditions $G_{i_0\,j} = 0$ for all $j \in \mathcal{L}$. In [14] the following proposition has been proven: For a lattice $\mathcal{L}$ with an additional vertex i_0

$$G_{ij} = \mathcal{N}^{(i,j)}/\mathcal{N} \qquad \cdot \; \cdot \tag{1}$$

where $\mathcal{N}^{(i,j)}$ is the number of two-rooted spanning trees having the roots i and j such that both the vertices i and j belong to the same subtree; $\mathcal{N}$ is the total number of spanning trees on $\mathcal{L}$.

The wave distribution follows immediately from Eq.(1) and the known asymptotics of the Green function $G(r) \sim \ln r$. Indeed, the probability $P(r_w \geq r_{ij})$ that the radius of the wave r_w is not less than the distance between i and j is

$$P(r_w \geq r_{ij}) \sim G_{ij} \tag{2}$$

Since the waves are compact, their sizes scale as $s \sim r^2$. Then, the asymptotic distribution of sizes $D(s)$ is

$$D(s) = D(r)\frac{dr}{ds} \sim \frac{1}{s} \tag{3}$$

where $D(r) = dP(r_w \geq r)/dr = 1/r$

The one-fold toppling of all sites in a wave is equivalent to a pass of particles over the boundary of the wave from sites inside the wave to neighboring sites outside. Typically, this

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leads to squeezing the next wave with respect to the previous one because a part of sites losing particles becomes unable to topple next time. So, the waves of increasing numbers $W_1, W_2, ..., W_n$ belonging to the same avalanche are generally of decreasing sizes $s_1, s_2, ...s_n$. An avalanche stops just at the moment when the boundary of the last wave reaches the initial point i.

Self-similarity of avalanches implies self-similarity of their components. Therefore, one can expect that the difference between successive waves $\Delta s = s_k - s_{k+1}$ obeys also a power law

$$\Delta s \sim s^\alpha \tag{4}$$

The exponent α, if exists, can be related with τ by a scaling relation. Let n denote the number of waves in an avalanche which coincides with the number of topplings at the site i. Equation (4) can be rewritten in the differential form $ds/dn \sim s^\alpha$ or

$$dn \sim \frac{1}{s^\alpha} ds \tag{5}$$

The wave of size s belongs to an avalanche of size $S \geq s$ which has the probability $P(S \geq s) \sim s^{1-\tau}$. Then, the distribution of waves belonging to diverse avalanches is

$$D(s) \sim \frac{1}{s^{\alpha+\tau-1}} \tag{6}$$

Comparing (6) with (3), we obtain the scaling relation

$$\alpha + \tau = 2 \tag{7}$$

Majumdar and Dhar [6] introduced an exponent y assuming that n scales with the size of an avalanche as $n \sim s^{y/2}$. To be consistent, the exponents α and y must be related as

$$2\alpha + y = 2 \tag{8}$$

We have studied the statistics of waves numerically generating 10^6 avalanches on the lattices of sizes up to $N = 500$. In Fig.1, we have plotted Δs versus the wave size s on a $\log - \log$ scale, which displays a clear power-law behavior.

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In [15], Grassberger and Manna have introduced clusters of sites A_n which toppled $\geq$ n times, $n \geq 1$, during an avalanche. If waves of a given avalanche obey the relations $W_1 \supset W_2 \supset ... \supset W_n$ strictly, the structure of waves coincides completely with that of clusters $\{A\}$. At the same time, Dhar and Manna who investigated inverse avalanches registrated situations when the wave W_k overlaps the preceding one W_{k-1}. They argued that these events are nevertheless relatively rare and on the average the last waves scale as the clusters of maximal topplings. Our simulations show generally that the distributions of waves $\{W\}$ and clusters $\{A\}$ follow the same asymptotical law (4). Taking into account these observations, we neglect the overlapping of waves and deal only with the decreasing of waves.

The above construction allows us to determine α from scaling arguments. To this end, we have to link the decrease in the size of waves Δs with the spanning tree characteristics. Given a rooted tree T_i and two sites $j_1, j_2 \in T_i$, we shall say that the site j_1 is a predecessor of j_2 if the unique path connecting j_2 and the root i passes via j_1. Let T_i' be the subtree corresponding to the wave W_k. As all sites involved into W_k topple exactly once, all internal sites of W_k remain unchanged. The wave W_{k+1} following W_k will repeat its order of topplings until the relaxation process reaches the boundary of W_k. Accordingly, the subtree T_i'' that represents W_{k+1} will coincide with T_i' until its sites have no predecessors among the boundary sites of W_k. Denote by B_j a set of sites of T_i' having a boundary site j as a predecessor. If the site j gets stable with respect to the next wave W_{k+1}, all sites of B_j get stable too as the toppling process penetrates into B_j via the point j. As a result, the sites of B_j as well as the site j itself contribute to Δs. Generally, Δs consists of all boundary sites $j_1, j_2...$ of the wave W_k getting stable with respect to W_{k+1} and of sites of all sets $B_{j_1}, B_{j_2}, ...$ having $j_1, j_2, ...$ as predecessors.

Instead of the boundary sites, it is convenient to consider a closed path Γ separating subtrees T_i' and T_0' on the dual lattice $\mathcal{L}_D$. If one cuts the adjacent bonds of the boundary site j, the set B_j will also be surrounded by the closed path γ_j on $\mathcal{L}_D$ as B_j is a branch of the subtree T_i'. The loop γ_j is attached to Γ and they have at least one common dual bond.

If several boundary sites $j_1, j_2, ...$ are in turn predecessors of j, then the cutting of bonds adjacent to j creates a dual loop γ_j around a combined cluster consisting of sets $B_{j_1}, B_{j_2}, ...$. In Fig 2, we show a typical cluster Γ together with the set of loops $\{\gamma\}$. By construction, two main quantities contribute to Δs: the length of the contour Γ and the area of loops $\{\gamma\}$.

Denoting by R a linear extent of the wave W_k, we can estimate the length of the contour Γ as $R^{5/4}$ since Γ is a chemical path on the dual spanning tree [16]. Then, the contribution from Γ gives

$$\Delta s \sim R^{5/4} \sim s^{5/8} \tag{9}$$

which implies 5/8 for the exponent α. We shall see, however, that the leading contribution comes from the second quantity determined by the interior of loops $\{\gamma\}$.

Consider a single loop γ. It is characterized by the distance l between points x and y where it is attached to the contour Γ and the linear extent r (see Fig 2). The cluster surrounded by γ is an unrooted subtree of the rooted tree. Accordingly to (3), the rooted trees are distributed as $D(r) \sim 1/r$. The root can occupy any of r^2 positions inside γ. Therefore, unrooted subtrees are distributed as $1/r^3$. Let us consider a circle C of radius l having the center at point x. The average number of intersections between C and Γ is of order $l^{1/4}$ due to fractal dimensions of the chemical path. The point y can occupy any of l points of C with equal probability. Thus, we obtain the asymptotical joint distribution of loops γ

$$D_\gamma(l, r) \sim \frac{l^{1/4}}{r^3 l} \tag{10}$$

The maximal extent of both r and l is of order R. The minimal extent of r is of order l whereas l is bounded from below by the lattice spacing. Integrating over r and l, we obtain the contribution to Δs from the single loop γ

$$\Delta_\gamma s \sim \int_1^R \int_l^R r^2 D_\gamma(l, r) dr dl \sim R^{1/4} \tag{11}$$

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The number of loops is proportional to the length of Γ, that is $R^{5/4}$. Then, the total Δs is

$$\Delta s \sim R^{3/2} \sim s^{3/4} \tag{12}$$

Comparing (12) with (4) and using (7) we finally get $\alpha = 3/4$ and $\tau = 5/4$.

Our numerical estimation of $\alpha = 0.73$ extrapolated to infinite N is quite consistent with the obtained value.

The distribution (10) is based on scaling arguments. To verify its validity, we have used an exact result coming from the analogy between a Coulomb gas and spanning trees. Saleur and Duplantier [17] evaluated the probability that vicinities of two points x and y separated by the distance l are connected by two paths on the tree. They found for large l

$$D_2(l) \sim \frac{1}{l^{3/2}} \tag{13}$$

To derive (13) from (10), we consider two paths as a loop and compare conditions leading to (10) and (13). The distribution (10) is restricted by the presence of the external contour Γ that fixes the position of the initial point x. In the latter case, the point x can occupy any site of the perimeter proportional to $r^{5/4}$. The linear extent r of the loop varies from l to infinity, so the integration over r gives

$$D_2(l) \sim \int_l^\infty r^{5/4} D_\gamma(l,r) dr \sim \frac{1}{l^{3/2}} \tag{14}$$

in accordance with (13).

If α is known, other exponents of the sandpile model can be readily found. For instance, using the identity [6]

$$\tau_s - 1 = 2(\tau - 1)/(2 + y) \tag{15}$$

we find from (7) and (8) the exponent of the total number of topplings $\tau_s = 6/5$.

The numerical result by Manna for $\tau_s = 1.2008$ [11] is in excellent agreement with our theoretical prediction.

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Figure captions

Fig. 1. Double logarithmic plot of averaged decrement Δs against cluster size s for the statistics of 10^6 avalanches on the square lattice of size $L = 500$.

Fig. 2. A typical contour Γ with the set of loops $\{\gamma\}$. The loop γ_2 is attached to Γ in points x and y separated by the distance l. The linear extent of γ_2 is r.
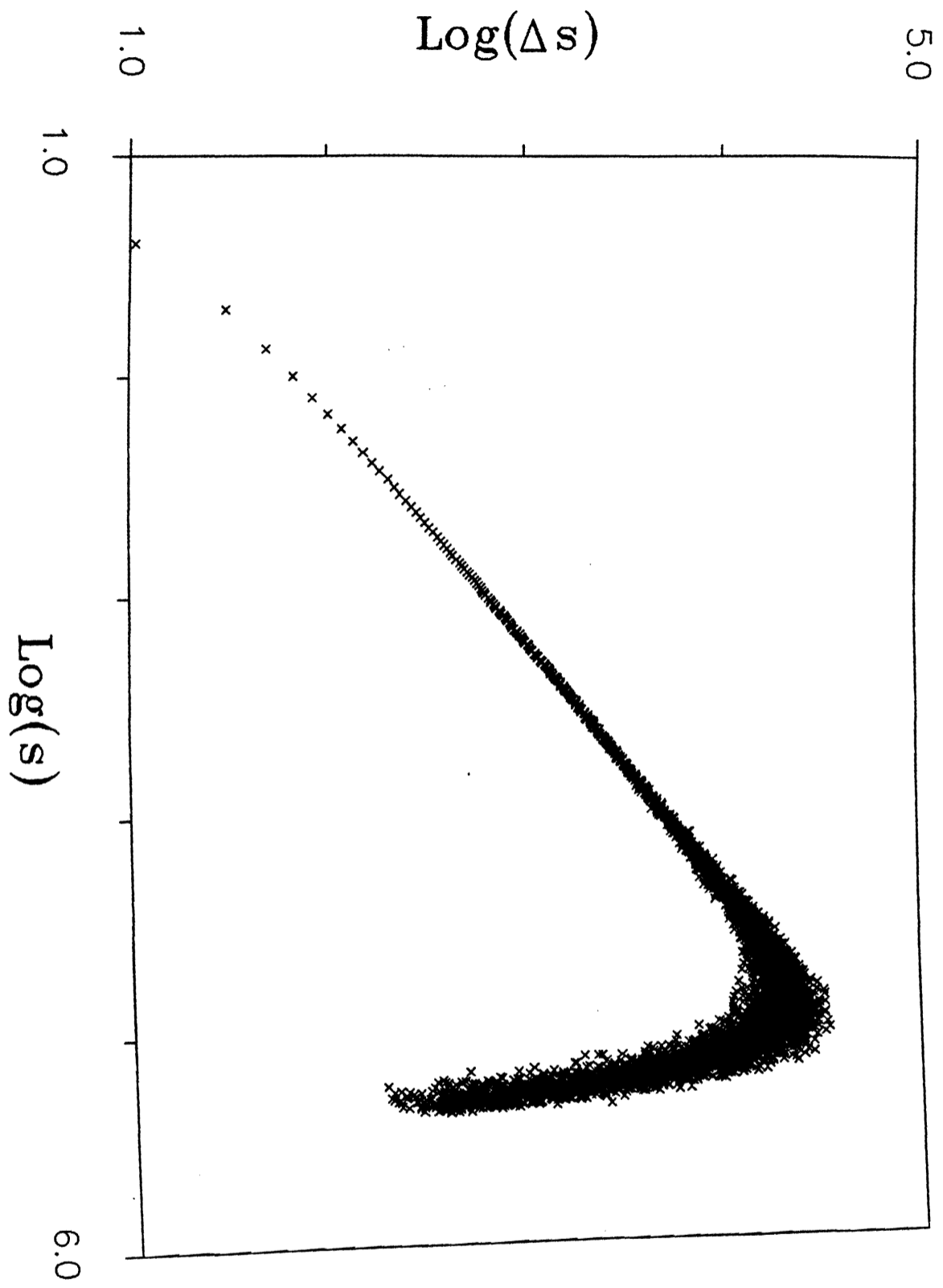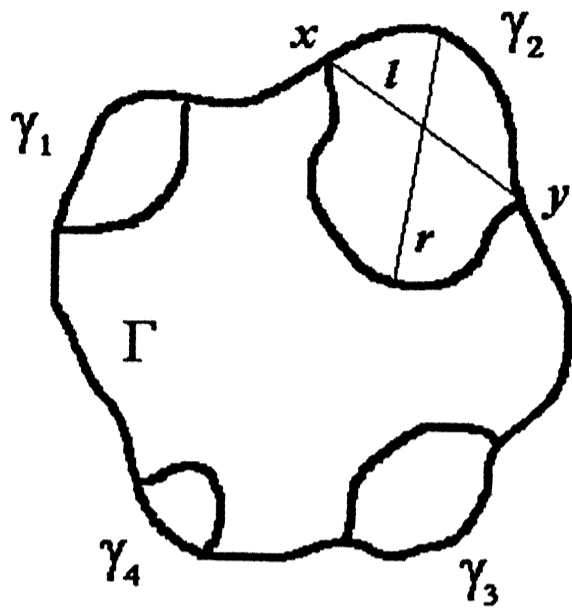
Fig. 1

Fig. 2