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Jacques II Bernoulli and The Problem of the Vibrating Plate

Diarmuid Ó Mathúna
Dublin Institute for Advanced Studies

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Introduction.

In the history of the theory of Structural Mechanics the year 1820 stands as a natural dividing line. We are then on the threshold of the comprehensive work of Cauchy which would clarify once and for all the general concepts of stress and strain, give a full analysis of equilibrium and formulate the constitutive relations for a three-dimensional continuum. While the contributions of his contemporaries, notably Navier, Poisson and others, are not to be discounted, the work of Cauchy was to launch the Theory of Elasticity on its new and fruitful course, in which for a long time, problems related to the elastic plate would play a central role.

The subject of this talk definitely belongs to the earlier period and the fact that the equation proposed by Bernoulli is clearly incorrect has led historians, when they do mention it, to dismiss it - sometimes in a tone bordering on contempt. In this paper we suggest that the work is a not insignificant milestone on the road leading to the crisis that would be resolved only by the formulation of the general theory. Moreover, we shall see that Bernoulli missed the correct outcome only by an unlucky choice in timing for the application of his basic assumption.

We start by tracing the context in which the work appeared. The survey necessarily will be selective, but is intended to highlight the works that played a major influence on the approach of Jacques II Bernoulli.

2 Background of the Problem.

The earlier period may be considered to have begun with the enquiry of Galileo on the breaking of a beam put forward in his

> *Discorsi et Demonstrazioni matematiche intorno a due nuove scienze attinenti alla Mecanica e i movementi Locali, (Leiden, 1638).*

which appeared in 1638. Parallel with the subsequent formulation of the stress-strain law by Hooke, published in 1675, come the investigations of Mariotte, whose interesting work on beam rupture contained an elementary error that led to an incorrect conclusion regarding the position of the neutral fibre - namely, the erroneous belief that the position of the neutral fibre was irrelevant. Though subsequently corrected by Parent (1713) and later by Coulomb

(1773), investigators continued to place the neutral fibre on the concave face of the stressed beam, which, however, yielded fruitful results with the beam considered as a one dimensional problem. As has been pointed out by Truesdell, the question of the position of the neutral fibre is still with us, three hundred years on.

In the later years of the century, as noted by Speiser, the subject takes a significant turn when it attracted the attention of Leibniz who saw, in the analysis of deformation, the potential application for his newly discovered differential calculus. His interaction with Jakob I Bernoulli set the latter on a course of investigation which besides leading to the differential equation for the deflection curve of the elastica, also saw the first assertion of a variational principle as well as the introduction of concepts which would become mainstays in the emerging theory. Most significant for our purpose is the relation

$$M = f\left(\frac{1}{R}\right)$$

the general form of the constitutive moment-curvature relation of Jakob I for the beam under flexure. The simpler linear version of this functional relation was proposed by Daniel Bernoulli in 1732, in the form

$$M = D\frac{1}{R}$$

which inaugurates the next phase for our attention.

This is the period (1733-51) of the inter-related investigations of Euler and Daniel Bernoulli on the vibrations of elastic rods. The interesting story of the interaction between these two contrasting personalities throughout the 1730s and 40s has been diligently sorted out and discussed by Truesdell (L. Euler, Opera Omnia , Vol XI, Pt 2). The main points of Euler's work can be found in the Additamentum I, *De Curvis Elasticis*, in his book

> *Methodus Inveniendi lineas curvas maximi minimive proprietate gaudentes*,
> Lausanne, 1744,

published in 1744, while the results of Daniel Bernoulli appear in the Commentarii of St. Petersburg for the years 1741/3 published in 1751.

These studies include the introduction of the Elastic Energy as the integral of the square of the curvature for the originally straight rod by Daniel Bernoulli, the minimization of the energy through the application of the Calculus of Variations by Euler, and the identification of the vibrations of the rod with those of an isochronous simple pendulum of length l, permitting the formulation of the equation for rod vibration:

$$\frac{\partial^4 z}{\partial x^4} = \alpha^2 \frac{\partial^2 z}{\partial t^2}$$

in the form:

$$\frac{d^4 z}{dx^4} = \frac{z}{c^4}$$

where the linearized formula for the curvature has been used. The nodes are to be determined from the values of c (a constant having dimension of length), for which solutions exist for

the above equation satisfying the specified form of the end-conditions. Good agreement with experimental results is reported in the papers of Daniel Bernoulli.

The twenty-year period 1746-66 might be justifiably called the era of the wave equation. Motivated by interest in the acoustic phenomena exhibited by the vibrating string, the equation was given its first formulation by d'Alembert and given its modern form and solution by Euler in sustained activity over this period.

Still motivated by the acoustical note, there appeared in 1767 two further papers by Euler that are relevant to our survey. The first deals with the vibrations of the drum-membrane modelled as a network of perfectly flexible strings for which the equation takes the form:

$$\frac{\partial^2 z}{\partial t^2} = \alpha^2 \frac{\partial^2 z}{\partial x^2} + \beta^2 \frac{\partial^2 z}{\partial y^2}$$

in which he then makes the argument that one can set $\alpha = \beta$, giving the equation its anticipated symmetric form. While the membrane has limited applicability since, by definition, it cannot resist a bending moment, this paper is significant in that it is the first formulation of a spatially two-dimensional problem.

The second paper addresses the vibrations of bells and concentrates on those of an annulus cut by two concentric cylinders parallel to the axis of the bell. To the annulus he applies the method previously derived for curved bars. Taking x as the distance measured along the bar, he arrives at the equation:

$$\frac{\partial^4 z}{\partial x^4} + \frac{1}{\alpha^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{\beta^2}\frac{\partial^2 z}{\partial t^2} = 0$$

However, there were many, including Euler himself who had misgivings about this theory and were keenly aware of the shaky assumptions on which it was based.

In the 1770s Euler returned once more to the problem of the elastic vibration of bars and after exhaustive analysis of solutions for various end-conditions, his final word on that topic appeared in the Acta of St. Petersburg for 1779, published in 1782. [In the return of Euler in the 1770s to such intense work on the vibration of bars, it is difficult to avoid the conjecture that he may have been backing away from any further attempt on the problem of the plate.] Also in 1782 in Verona appeared the work containing the extensive numerical studies by Giordano Riccati based on Euler's earlier (1740s) work.

Before finishing this background survey, cognizance should be taken of the study by Coulomb delivered to the Académie Française in 1773, and published in 1776, entitled:

> *Essai sur une application des Règles de Maximis et Minimis à Quelques Problèmes de Statique, relatifs à l'Architecture.*

In this attractively systematic analysis of the beam he re-directs attention to the long neglected thickness dimension and the question of the neutral fibre. Moreover, its very title should merit it a mention at this Symposium and, in fact, the work has already been cited by Mme. Patricia Radelet-de Grave in her lecture.

In 1787 there appeared the book by Chladni on the vibration characteristics of thin plates. This work, containing the first clear experimental results of the two-dimensional nodal formations, was the immediate stimulus for the work of Jacques II Bernoulli on the analysis of the elastic plate.

3 The Essai of Jacques II Bernoulli

3.1 Preliminaries

The work appears in Volume 5 of the Nova Acta Academiae Scientiarum Petropolitanae, St. Petersburg, 1789 under the title:

Essai Théorétique sur les Vibrations des Plaques Élastiques, Rectangulaires et Libres

Three authors are cited in the work besides Chladni, namely Daniel Bernoulli, Euler and Giordano Riccati.

Referring first to Euler's work on bells and recalling Euler's awareness of the precarious nature of the underlying hypothesis, he notes that comparison with the physical results of Chladni shows that this theory is not acceptable. Next he notes Euler's work on the drum - the two-dimensional membrane - and regrets that there are no physical results against which the latter theory can be tested. He, therefore, proposes to extend Euler's model of the membrane as a network of strings by treating the elastic plate as a network of elastic strips and thereby derive a theory, the results of which can then be tested against the physical results of Chladni. But first he gives a review of the prior work on elastic rods by Daniel Bernoulli and Euler.

Focusing on the the work of Euler as presented in the Additamentum of 1744, he gives a full review of the procedure for dealing with the elastic rod. The rod, whose weight is considered negligible, is considered to be subject at each point to a net transverse force denoted by ∂p: taking x as the abscissa along the rod and with z representing the transverse co-ordinate, he lets r denote the radius of curvature of the deformed rod: then with the constant D as a measure of the elasticity, the moment-curvature relation takes the form:

$$\int p\,dx = D\frac{1}{r},$$

and we recognise p as what we would now call the transverse shear. Implicit in this equation is the relation

$$M = \int p\,dx, \qquad \text{or} \qquad p = \frac{\partial M}{\partial x},$$

which appears to be the first cognizance of the moment-shear relation.

Observing that in the deformed position the rod element ∂s is subject to the reactive restoring force $\frac{\partial p}{\partial s}$, and noting further that for small deflections we have:

$$\partial s \approx \partial x \qquad \text{and} \qquad \frac{1}{r} \approx \frac{\partial^2 z}{\partial x^2}$$

it follows that:

$$\left[\frac{\partial p}{\partial s} =\right] \qquad \frac{\partial p}{\partial x} = D\frac{\partial^4 z}{\partial x^4} \qquad \left[= D\frac{\partial^2}{\partial x \partial s}\frac{1}{r}\right].$$

Then applying the technique of Daniel Bernoulli for identifying the accelerative force with the restoring accelerative force of the isochronous simple pendulum of length l, and setting [1]:

$$c^4 = Dl,$$

[1] For a discussion of the dimensions of c, see Appendix

we retrieve the basic equation for rods in the form:

$$\frac{d^4z}{dx^4} = \frac{z}{c^4},$$

with solutions given by:

$$e^{\pm\frac{x}{c}}, \qquad e^{\pm i\frac{x}{c}}.$$

This is then applied to the case of the rod of length $2a$ with free ends, for which there results the following characteristic equation for the determination of c:

$$\frac{a}{c} = (2n+1)\frac{\pi}{2} \pm \frac{2}{e^{\frac{\pi}{2}(2n+1)} + e^{-\frac{\pi}{2}(2n+1)}}$$

with the plus sign holding for n odd and the minus sign holding for n even.

The solutions represent an infinity of sounds, rising in pitch with increasing n, whose strength is reflected in the amplitude.

3.2 Modelling of the Plate

Now consider two strips - AB of length a, and CD of length b, crossing at right angles at F having infinitesimal width η, thickness θ, density k, and specific elasticity E, so that the elastic constant for each strip is $E\eta\theta^3$.

The plane of the strips being horizontal, let us take the elementary transverse (vertical) forces to which the strips are subjected as ∂p for strip AB and ∂q for the strip CD. Consider the x-abscissa along AB and the y-abscissa along CD and let z denote the (assumed small) transverse displacement: further let r_x denote the radius of curvature of the deformed strip AB and r_y the radius of curvature of the deformed strip CD. Then following the previous results for rods, Jacques Bernoulli writes the moment-curvature relations for the two strips in the form:

$$\int p\,dx = E\eta\theta^3\frac{1}{r_x}, \qquad \int q\,dy = E\eta\theta^3\frac{1}{r_y}$$

From these and the approximations valid for small deflections:

$$\frac{1}{r_x} \approx \frac{\partial^2 z}{\partial x^2}, \qquad \frac{1}{r_y} \approx \frac{\partial^2 z}{\partial y^2}$$

it follows that:

$$\frac{\partial p}{\partial x} = E\eta\theta^3\frac{\partial^4 z}{\partial x^4}, \qquad \frac{\partial q}{\partial y} = E\eta\theta^3\frac{\partial^4 z}{\partial y^4}$$

Since η is the width of each strip, it follows that, for the element of intersection common to the two strips of dimension $\eta \times \eta = \eta^2$, the motive forces acting are given respectively by the above factors multiplied by η: noting further that as the mass of the element is given by $k\eta^2\theta$ it is necessary to divide the motive forces by this mass to get the accelerative forces in the form:

$$\frac{E}{k}\theta^2\frac{\partial^4 z}{\partial x^4} \qquad \text{and} \qquad \frac{E}{k}\theta^2\frac{\partial^4 z}{\partial y^4}$$

respectively. Here Jacques Bernoulli makes explicit his binding assumption namely:

(***) the common area of the two strips, being as glued together must move together and share equally between them the motive forces acting on them,

from which he deduces that for each element the accelerative force is given by:

$$\frac{1}{2}\frac{E}{k}\theta^2\left[\frac{\partial^4 z}{\partial x^4}+\frac{\partial^4 z}{\partial y^4}\right]$$

so that, following the procedure for identifying the vibration with that of the isochronous pendulum of length l, and setting:

$$c^4=\frac{1}{2}\frac{E}{k}\theta^2 l$$

the governing equation takes the form:

$$\frac{\partial^4 z}{\partial x^4}+\frac{\partial^4 z}{\partial y^4}=\frac{z}{c^4} \qquad \text{(F)}$$

the *"fundamental equation of the entire theory."*

He observes that as the tone for the simple pendulum is proportional to $\sqrt{1/l}$ it follows that for the plate it is proportional to $\theta\sqrt{E/k}$ consistent with experimental results for rods.

The equation is then solved by separation of variables and the procedure followed for the case of rods is applied. The comparison of the results with the physical results of Chladni does not suggest any confirmation except perhaps for the particular cases where the nodal lines are parallel to the plate-edges - as apparently was anticipated by Chladni for such a model. Beyond recognising the ingenuity displayed, we shall not discuss further these later sections of Jacques II's Essai.

3.3 Discussion of Approach

Rather we wish to focus on the fundamental equation (F) and in particular on the operator on the left hand side which we now know should be the biharmonic operator, namely:

$$\frac{\partial^4 z}{\partial x^4}+\underline{2\frac{\partial^4 z}{\partial x^2\partial y^2}}+\frac{\partial^4 z}{\partial y^4}\ ,$$

The equation (F) is incorrect in that the operator on the left is deficient in not including the mixed term underlined above. We can ask where is the source of this deficiency and would it have been possible for Jacques II to correct it?

There are two factors, either of which - or both combined - could be considered the reason for the absence of the mixed term in the final equation (F), namely:

1. The model does not admit the possibility of a twisting moment;

2. There is a lack of symmetry in the manner in which the restoring force is distributed.

It is difficult to see how the problem with the twisting moment could have been remedied. Even apart from the limitations of the model, the concept of the twisting moment would seem to require a clearer concept of shear stress and a fuller equilibrium analysis than was then available.

However, even staying within the framework of the model there does appear to be potential for remedy in the second point. The exploration of this latter possibility followed from a (private) communication from Professor David Speiser, prompting a review of the basic assumption in relation to symmetry considerations. Clearly the basic assumption not only does not violate, but rather asserts the symmetry expectations. As the order of the final equation is correct (fourth order) and its deficiency lies in the absence of the mixed term whose inclusion would restore its symmetry, we are forced to the conclusion that the error must lie in a misapplication of the basic assumption. This led to the following reconsideration.

As already mentioned, *at the end of his analysis* Jacques Bernoulli makes explicit the assumption that is meant to reflect the binding of the overlapping strips namely:

(***) the common area of the two strips, being as glued together must move together and share equally between them the motive forces acting on them,

on which is based the final addition leading to equation (F). However, it would appear that as this is a reflection of the binding of the "glued" elements, this assumption should have been introduced earlier and applied strictly.

If instead of taking over without modification the previous results for independent rods, one admits at the outset that the combined elastic restoring force is proportional to the sum of the curvatures of the constituent strips, it (i.e. the combined restoring force) takes the form:

$$E\eta\theta^3\left[\frac{1}{r_x}+\frac{1}{r_y}\right] \approx E\eta\theta^3\left[\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right] = E\eta\theta^3 \triangle z$$

where we have written $\triangle$ to represent the two-dimensional Laplacian operator, i.e.

$$\triangle \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Then assuming in accordance with (***) that this restoring force is shared equally by the reactive shear forces one has the moment-curvature relations in the symmetric form:

$$\int p\,dx = \frac{1}{2}E\eta\theta^3 \triangle z \quad , \int q\,dy = \frac{1}{2}E\eta\theta^3 \triangle z$$

so that there follows for the components of the transverse motive forces:

$$\frac{\partial p}{\partial x} = \frac{1}{2}E\eta\theta^3\frac{\partial^2}{\partial x^2}\triangle z \quad , \qquad \frac{\partial q}{\partial y} = \frac{1}{2}E\eta\theta^3\frac{\partial^2}{\partial y^2}\triangle z$$

These, on multiplication by η can be added to give the resultant transverse motive force, which, on division by the combined mass, $2k\eta^2\theta$, gives the resultant accelerative force in the form:

$$\frac{1}{4}\frac{E}{k}\theta^2 \triangle \triangle z$$

which, on following the procedure for the isochronous pendulum, and setting,

$$c^4 = \frac{1}{4}\frac{E}{k}\theta^2 l$$

yields the fundamental equation:

$$\triangle \triangle z = \frac{z}{c^4}$$

or, explicitly,

$$\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 x}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} = \frac{z}{c^4} \qquad \text{(B)}$$

replacing (F). It would appear that only the delayed application of his basic assumption to the last stage of his derivation, rather than inserting it together with its implications at the outset of his analysis, deflected Jacques II from obtaining the correct equation in his "premier essai".

Sequel

Unfortunately there was little in the way of a sequel. The equation proposed by Jacques Bernoulli was recognised as inadequate even by its author - yet nobody thought of exploring a more symmetric application of his assumptions. We can speculate what difference the absence of the great and experienced Euler might have made at this point. As it was proposed as "un premier essai" it is fair to assume that Jacques Bernoulli intended to return to it. Sadly he was to die tragically in a drowning accident later that year of 1789, when he was barely thirty years old. We can only conjecture what he might have done.

Nothing much happened until after the turn of the century when Chladni on a lecture tour was making his results known in France. In 1809 the appearance of the French translation of his work aroused an excitement that infected many including Napoleon. This led to the announcement by the French Institute of a Special Prize for a satisfactory analysis of the Plate Problem, which on its third offering was awarded to Sophie Germain in 1816. That she was the sole applicant on all three occasions was at least partly due to the fact that Lagrange had let it be known that he considered it a problem of extreme difficultly - though it was he also who played the crucial role in rectifying the incorrect initial essay of Sophie Germain. But now we are nearing the end of the early period and the new controversies surrounding the Plate Problem had forced its attention on Cauchy to whom we owe the formulation of the general theory. The role of the Plate Problem in the new era is an even longer story.

Jacques II grew up in the shadow of his Uncle Daniel and of that of his wife's grandfather, Leonhard Euler, neither of whom made a frontal attack on this problem. It seems his approach to the problem was closer to getting the right equation than either he or anyone else has recognised. But if we admit that he should have - or at least could have - got it right, it is still hazardous to speculate on the consequences. It has been convincingly argued by Truesdell that it was the very success with special problems in the eighteenth century that may have postponed the formulation of the general theory. Whichever way one speculates, it appears that this work merits more sympathetic consideration than it has received heretofore.

Appendix: A Note on Dimensions.

In the analysis of Section 3.2, when identifying the vibration with that of the isochronous pendulum, it appears that Bernoulli has set the gravity acceleration factor $g = 1$. Strictly we would have:

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = \frac{z}{c^4} \quad , \qquad \text{with} \qquad c^4 = \frac{1}{2}\frac{E}{k}\theta^2\frac{l}{g}.$$

Noting that

$$E \text{ is } ML^{-1}T^{-2}, k \text{ is } ML^{-3}, g \text{ is } LT^{-2},$$

so that c has the dimension of length. Similarily in Section 3.3 where:

$$\triangle\triangle z = \frac{z}{c^4}, \qquad \text{with} \qquad c^4 = \frac{1}{4}\frac{E}{k}\theta^2\frac{l}{g}$$

it is clear that the c-factor again has dimension of length.

In Section 3.1, when recapitulating prior work, in order to make the identification with the isochronous pendulum, we should introduce a density κ for the rod (mass per unit length) and the identification takes the form

$$D\frac{\partial^4 z}{\partial x^4} = \frac{\partial p}{\partial x} = \kappa\frac{\partial^2 z}{\partial t^2} = \kappa\frac{g}{l}z$$

so that when we write

$$\frac{d^4 z}{dx^4} = \frac{z}{c^4}, \qquad \text{we should have} \qquad c^4 = \frac{Dl}{g\kappa}.$$

Noting the dimensions of g and κ it follows that

$$g\kappa \text{ is } ML^{-2}$$

and within the two-dimensional framework, the stiffness factor D is given by

$$D = \left(\frac{\text{Force}}{\text{Unit length}}\right) \times (\text{Moment of Inertia of cross section}) \quad \text{is} \quad MT^{-2}L^3.$$

it follows that

$$c^4 \text{ is } L^4$$

showing that here also c has the dimension of length.

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