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# RADIATIVE CORRECTIONS IN THE STANDARD MODEL AND THE RHO PARAMETER*

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Abstract

The problem of electroweak radiative corrections in the standard model is considered. Renormalization is discussed by fitting the bare parameters of the Lagrangian in terms of three data points. The $\rho-$parameter is considered and its importance for confrontation with experimental data is reviewed, expecially for the case where vector boson masses are input parameters.

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Recent years have shown a general consensus on the fact that electroweak interactions are entering in a new era of precision experiments. In a systematic investigation of the validity of the standard model we must therefore consider the radiative corrections as an indispensable part of the verification of the theory.

The central point in this issue is the comparison between theory and experiments, to be performed with the $e^+e^-$ colliders going into operation, which requires at least three measurements. The model indeed contains three free parameters, not counting fermion masses and mixing angles in the quark sector. The Higgs mass is also a free parameter but the measurements relevant to SLC/LEP are weakly dependent on $m_H$, the so called Veltman's screening theorem.

Most radiative corrections are less than 1% but as soon as the data will be better than 1% we must follow a well defined and unique procedure [1]. Given the data, subtract radiative corrections and determine the parameters $g, \theta_W$ and $M$. Once the parameters are fixed we are left with the problem of finding a gauge invariant quantity which represents the standard reference point. Indeed in gauge theories we are facing a constraint of internal consistency rather than a problem of parametrizing the data. If three measurements are obtained the rest must follow because of gauge invariance. A convenient way to realize this unambiguous procedure is to study the $\rho-$ parameter [2].

A well known definition of $\rho$ requires the knowledge of vector boson self-energies at low momentum. If $S_{ij}(p^2)$ represents the $i-j$ transition ($i = W, Z^0, \gamma$), we have

$$\rho = 1 - \frac{S_{WW}(0) - c_\theta^2 S_{ZZ}(0)}{M_W^2}$$

$c_\theta = \cos\theta_W$ being the cosine of the weak mixing angle. The deviation of $\rho$ from one signals the degree of isospin breaking as observed in the form of mass differences between the members of fermion doublets as well as a drastic departure from the minimal Higgs sector. In short $\rho$ is sensitive to the interaction of vector bosons with fermions and possibly other objects and the corresponding effects are visible even if the masses of these objects are larger than 100 GeV. Disregarding radiative corrections the result $\rho \geq 1/(2\,I)$ was proven in [2], where $I$ is the highest weak isospin found among Higgs particles. In the minimal version of the standard model $\rho = 1$ at lowest order. This expression for the $\rho-$ parameter follows from using low energy data as input. Since in this case masses are not measured directly we have to consider low energy cross sections as $\mu-$ decay and $\nu_\mu\, e^-$ scattering. However the original definition goes well beyond the low $q^2$ region. If $M, M_0$ and $c_\theta$ are the bare $W$ and $Z^0$ masses together with the bare $\cos\theta_W$ then with

$$\rho = \frac{M^2}{c_\theta^2 M_0^2}$$

the Higgs $\Delta I = 1/2$ rule implies $\rho = 1$. The bare parameters can be fixed by the knowledge of three experimental data, each of which is in principle defined at a different scale. It is evidently important to extend the $q^2 \sim 0$ considerations to $q^2 \sim M_W^2, M_Z^2$. From this point of view the precise values of $M_W$ and $M_Z$ will be a crucial test of the whole structure of the standard model, while until now the measurement of $\rho$ from low energy data has

represented our best window on unexplored energy scales.

$M_Z$ will be measured to within a fraction of a percent and we may expect that in the future precise measurements will be available from angular distributions in $W$ decay or total cross section in $W^+W^-$ production. For the three data points we may use $\alpha, G_F$ and $M_W$. The measure of $M_Z$ represents a constraint on the internal consistency of the theory reflected by the value of $\rho$. With the values of the masses at our disposal $\rho - 1$, computed from the previous definition, signals new physics and its importance must be reevaluate well beyond considering $\rho$ as a phenomenological parameter to enter $\nu e$ or $\nu$ hadron interactions.

For these reasons we advocate the reinstatement of the original interpretation as given in [1-3] instead of considering $\rho$ as an additional parameter to fit the data.

The $\rho-$parameter is given by the bare parameters of the theory which are fixed in terms of experimental quantities and in a near future we will be in a position where to test the region $q^2 \sim M_Z^2, M_W^2$. This is not the same as defining a $\rho(q^2)$ with $\rho(0)$ fixed by low energy data. First consider $q^2 = 0$. If we only include fermions into the radiative corrections then $S_{ZZ}(0)$ is actually equivalent to $S_{33}(0)$, namely we may put $s_\theta^2$ to zero in the corresponding self energy. This remains true when we include vector bosons as long as the renormalization is such that for the $Z^0 - \gamma$ transition we have $S_{Z\gamma}(0) = 0$. Now self energies appear in $\rho$ because by considering low energy cross sections the measured $M_Z^2$ is actually $M_Z^2 - S_{ZZ}(0)$ and $M_W^2$ is $M_W^2 - S_{WW}(0)$. Therefore

$$\begin{aligned}\rho &= \frac{M_W^2 - S_{WW}(0)}{c_\theta^2\,[M_Z^2 - S_{ZZ}(0)]} = 1 - \frac{S_{WW}(0) - c_\theta^2\, S_{ZZ}(0)}{M_W^2} \\ &= 1 - \frac{S_{WW}(0) - S_{33}(0)}{M_W^2}\end{aligned}$$

All of this corresponds to a physically well defined quantity. As soon as $q^2$ increases $c_\theta^2\, S_{ZZ}(q^2) \neq S_{33}(q^2)$ and the full expressions must be used. The correct definition requires for instance knowledge of $S_{ZZ}\left(-M_Z^2\right)$.

Having outlined our strategy we now proceed in the implementation of a specific renormalization procedure, namely we construct a scheme for fitting the parameters. A major ingredient is given by the one-loop corrected propagators of the theory. Basically what we have to do amounts to the following [5]. Consider for a certain particle all self-energy diagrams that do not contain any further self-energy part in the internal lines. The sum of the contributions will be denoted by $S\left(\Delta_F, p^2\right)$, $\Delta_F$ being bare propagators ( for simplicity we consider scalar particles ). Next we define functions $\Delta_F^{(n)}$

$$i\,\Delta_F^{(n)}(p^2) = i\,\Delta_F(p^2)\left\{1 - i\,\Delta_F(p^2)\,S\left[\Delta_F^{(n-1)}, p^2\right]\right\}^{-1}$$

Here $\Delta_F^{(0)} = \Delta_F$. Assuming that the limit exists we have for the complete propagator

$$\overline{\Delta}_F(p^2) = \lim_{n\to\infty} \Delta_F^{(n)}(p^2)$$

In actual calculations the propagators must be calculated with a certain accuracy in the coupling constant. For our considerations the recipe is simple. We consider diagrams with one closed loop ( no self-energy loops ) and bare propagators, and tree diagrams with dressed propagators where $S$ is computed by considering one loop self-energy diagrams. Internal consistency requires that we neglect the two loop reducible diagrams, because two loop renormalization should be done according to the previous definition.

Before considering dressed propagators however, we discuss some technical details connected with renormalization. The way we perform renormalization avoids the explicit introduction of renormalization constants. Three quantities are computed, electric charge, Fermi coupling constant and position of the $Z^0$ peak. By comparison $g, s_\theta$ and $M$ are fixed.

$$\begin{aligned} g &= f_g\,(\alpha, G_F, M_Z) \\ s_\theta &= f_s\,(\alpha, G_F, M_Z) \\ M &= f_M\,(\alpha, G_F, M_Z) \end{aligned}$$

These quantities are by themselves infinite and scale dependent, but infinities and the arbitrary mass scale cancel out of all physical results, including $\rho$ as it should be. Consider the electric charge defined to be the residue of the pole at zero momentum transfer in $e\mu$ scattering. In a one loop renormalization what we get is a replacement of the type [3]

$$4\,\pi\,\alpha = g^2\,s_\theta^2\,(1+\delta\alpha)$$

$\delta\alpha$ denotes the contributions of wave-function factors, two vertex diagrams, $\gamma$ propagator insertions and $Z^0 - \gamma$ transition. It is a well known fact that purely e.m. contributions, exclusive of photon propagator diagrams, cancel. Vector bosons however modify this situation. First we compute infinities ( and finite parts as well ) containing $m_\mu$ and $m_e$. They cancel as expected and therefore we only consider the massless limit. Consider

$$\delta\alpha = \delta^W + \delta^V + \delta^\gamma + \delta^{Z\gamma}$$

$\delta^W$ is given in ref. [3] in terms of B form factors [6]. The two vertex diagrams give $\delta^V$ which can be cast into the form

$$\delta^V = \frac{g^2}{32\,\pi^2} \sum_{f=e,\mu} \left[ C_f(0) - \frac{v_\theta + 1}{4\,c_\theta^2}\, C_{ff}(0) \right]$$

where $v_\theta = 4\,s_\theta^2 - 1$ and using the C form factors [6]

$$C_{ff}(s) = -2\,C_{24} + (C_{11} + C_{23})\;s + 1$$
$$C_f(s) = 6\,C_{24} - (C_0 + C_{11} + C_{23})\;s - 1$$

The non e.m. contributions add up to a non zero result

$$\delta^W + \delta^V = \frac{g^2}{16\,\pi^2}\,(\Delta - \ln\,M^2)$$

where $\Delta = -2/(n-4) + \gamma - \ln\,\pi$. Also vector bosons give rise to a novel feature

$$\delta^{Z\gamma} = -v_\theta\,\frac{c_\theta}{s_\theta}\,\frac{S_{Z\gamma}(0)}{M^2} \neq 0$$

Indeed the $Z^0\;-\;\gamma$ transition at zero momentum is

$$S_{Z\gamma}(0) = -\frac{g^2}{8\,\pi^2}\,\frac{s_\theta}{c_\theta}\,M^2\;(\Delta - \ln\,M^2)$$

Being $\delta^W + \delta^V + \delta^{Z\gamma}$ different from zero we seem to loose a very appealing aspect of QED, namely the possibility of expressing electric charge renormalization via the summation of photon self- energy diagrams alone.

$$4\,\pi\,\alpha = \frac{g^2\,s_\theta^2}{1 - \frac{g^2 s_\theta^2}{16\,\pi^2}\,\Pi_{\gamma\gamma}(0)}, \qquad S_{\gamma\gamma} = \frac{g^2 s_\theta^2}{16\,\pi^2}\,p^2\,\Pi_{\gamma\gamma}$$

The same problem has been considered and solved in a different approach to radiative corrections by the authors of ref. [4]. Here we supply our solution. See however also the approach of ref [7]. Start with a triplet of vector bosons $B^i_\mu$ and a singlet $B^0_\mu$. The relevant terms in the interaction Lagrangian are

$$\mathcal{L} = \frac{i}{4}\,(\overline{\nu},\overline{e})\,\gamma^\mu \begin{pmatrix} g\,B^3_\mu + g''\,B^0_\mu & \sqrt{2}\,g\,W^+_\mu \\ \sqrt{2}\,g\,W^-_\mu & -g\,B^3_\mu + g''\,B^0_\mu \end{pmatrix} (1+\gamma^5) \begin{pmatrix} \nu \\ e \end{pmatrix}$$
$$+ \frac{i}{4} g'''\,B^0_\mu\,\overline{e}\gamma^\mu\,(1-\gamma^5)\,e - \tfrac{1}{2}\,g^2 F^2\,W^+_\mu W^-_\mu - \tfrac{1}{4}\,F^2\,\left(g\,B^3_\mu + g'\,B^0_\mu\right)^2$$

where for simplicity we considered only one fermion doublet. Moreover $F$ is the Higgs vacuum expectation value. Before diagonalizing the mass matrix we introduce a new coupling constant $\overline{g}$, related to $g$ by

$$\overline{g} = g\;(1 + g^2\,\Gamma)$$

Where $\Gamma$ is for the moment a free coefficient. Next we define fields $A_\mu$ and $Z^0_\mu$ by

$$Z^0_\mu = \frac{\bar{g}\, B^3_\mu + g'\, B^0_\mu}{\left(\bar{g}^2 + g'^2\right)^{1/2}}, \qquad A_\mu = \frac{-g'\, B^3_\mu + \bar{g}\, B^0_\mu}{\left(\bar{g}^2 + g'^2\right)^{1/2}}$$

The final result follows by using

$$g' = g'' = -\frac{s_\theta}{c_\theta}\,\bar{g}, \qquad g''' = g' + g'', \qquad F = \sqrt{2}\frac{M}{\bar{g}}$$

For the quadratic terms we get

$$\mathcal{L}_2 = -M^2\, W^+_\mu W^-_\mu - \tfrac{1}{2}\frac{M^2}{c^2_\theta}\, Z^0_\mu Z^0_\mu + \bar{g}^2 M^2 \Gamma \left( Z^0_\mu Z^0_\mu - \frac{s_\theta}{c_\theta}\, Z^0_\mu A_\mu + 2\, W^+_\mu W^-_\mu \right) + O\left(\bar{g}^4\right)$$

Charged and neutral currents receive extra contributions

$$\mathcal{L}^{extra}_{CC} = -\frac{i\,\bar{g}^3}{2\sqrt{2}}\Gamma\, \left[W^+_\mu\, \bar{\nu}\gamma^\mu \left(1+\gamma^5\right) e + W^-_\mu\, \bar{e}\gamma^\mu \left(1+\gamma^5\right) \nu\right]$$
$$\mathcal{L}^{extra}_{NC} = \frac{i}{4}\bar{g}^3\Gamma\, \left(s_\theta A_\mu + c_\theta Z^0_\mu\right) \left[\bar{e}\gamma^\mu \left(1+\gamma^5\right) e - \bar{\nu}\gamma^\mu \left(1+\gamma^5\right) \nu\right]$$

At this point we fix $\Gamma$ by requiring

$$\Gamma = \frac{1}{8\,\pi^2}\, B_0(0, M, M)$$

where $B_0$ is the scalar two-point function [6]. With this choice the vector boson self-energies are modified. Dropping from now on the bar we get

$$S_{\gamma\gamma} = \frac{g^2\, s^2_\theta}{16\,\pi^2}\, p^2\, \Pi_{\gamma\gamma}$$

$$S_{ZZ} = \frac{g^2}{16\,\pi^2\, c^2_\theta}\, \left[\left(S^0_{ZZ} - 32\,\pi^2\, M^2\, \Gamma\right) - 2\, s^2_\theta\, \left(S^0_{Z\gamma} + 16\,\pi^2\, M^2\, \Gamma\right) + s^4_\theta\, p^2\, \Pi_{\gamma\gamma}\right]$$

$$S_{Z\gamma} = \frac{g^2}{16\,\pi^2}\, \frac{s_\theta}{c_\theta}\, \left[\left(S^0_{Z\gamma} + 16\,\pi^2\, M^2\, \Gamma\right) - s^2_\theta\, p^2\, \Pi_{\gamma\gamma}\right]$$

$$S_{WW} = \frac{g^2}{16\,\pi^2}\, \left[\left(\Pi^0_{WW} + s^2_\theta\, \Pi^1_{WW}\right) p^2 + \left(\Sigma^0_{WW} - 32\,\pi^2\, M^2\, \Gamma\right) + s^2_\theta \Sigma^1_{WW}\right]$$

The explicit expressions will be given in appendix. The important fact to be noted here is the following

$$S_{Z\gamma}(0) = 0$$

which makes $\delta^{Z\gamma} = 0$ in the corrections to the electric charge. Moreover the extra $ee\gamma$ ($\mu\mu\gamma$) vertex gives rise to a contribution

$$\delta^{extra} = -\frac{g^2}{16\,\pi^2}\,(\Delta - \ln M^2)$$

Thus

$$\delta^{Z\gamma} = 0, \qquad \delta^W + \delta^V + \delta^{extra} = 0$$

and photon propagator diagrams alone contribute to the shift in $e^2$. At the same time we consider the $eeZ^0$ coupling. The infinite parts coming from wave-function factors and vertices add up to

$$\mathcal{L}_{eeZ} = -\,(2\,\pi)^4\; i\,\frac{i\,g^3}{32\,\pi^2}\,c_\theta \Delta\,\overline{v}\gamma^\mu\,(1+\gamma^5)\,u$$

which is exactly cancelled by the extra $eeZ^0$ vertex proportional to $\Gamma$. By means of the same mechanism all the required cancellations take place. There is no parity violation in low energy for the e.m. current, no ultraviolet divergence contained in the $\overline{\nu}\nu\gamma$ coupling and moreover the neutrino charge remains zero.

The second experimental quantity we need is given by $G_F$ or rather by the muon lifetime

$$\frac{1}{\tau_\mu} = \frac{m_\mu^5}{192\,\pi^3}\,\frac{g^4}{32\,M^4}\,\left(1 + \delta_\mu^{e.m.} + \delta_\mu^{weak}\right)$$

where

$$\delta_\mu^{e.m.} = \frac{\alpha}{2\,\pi}\left(\frac{25}{4} - \pi^2\right)$$

If we compute $\delta_\mu^{weak}$ from wave-function factors, four vertex diagrams and five box diagrams [3], we obtain

$$\delta^W + \delta^V + \delta^B = \frac{g^2}{\pi^2}\left[\tfrac{3}{4} + \tfrac{1}{2}\,(\Delta - \ln M^2) + \tfrac{1}{4}\,\frac{7 - 4\,s_\theta^2}{4\,s_\theta^2}\,\ln c_\theta^2\right]$$

The extra contribution from $\mathcal{L}_{CC}^{extra}$ gives

$$\delta_\mu^{extra} = -4\,g^2\Gamma = -\frac{g^2}{2\,\pi^2}\,(\Delta - \ln M^2)$$

Therefore we have an infrared and ultraviolet finite correction beside $W$ self-energy diagrams

$$\delta_\mu = \frac{g^2}{4\,\pi^2}\left[3 + \frac{7-4s_\theta^2}{4\,s_\theta^2}\,\ln\,c_\theta^2\right] + \delta_\mu^{e.m.}$$

We are now in a position to write down the one loop dressed propagators.

$$\Delta_{\mu\nu}^{\gamma\gamma} = \frac{1}{(2\,\pi)^4\,i}\,\frac{\delta_{\mu\nu}}{p^2}\left[1 - \frac{g^2 s_\theta^2}{16\,\pi^2}\,\Pi_{\gamma\gamma}\right]^{-1}$$

$$\Delta_{\mu\nu}^{ZZ} = \frac{1}{(2\,\pi)^4\,i}\,\delta_{\mu\nu}\left\{p^2 + \frac{M^2}{c_\theta^2} - \frac{g^2}{16\,\pi^2\,c_\theta^2}\left[\overline{S}_{ZZ}^0 - 2\,s_\theta^2\,\overline{S}_{Z\gamma}^0 + s_\theta^4\,p^2\,\Pi_{\gamma\gamma}\right]\right\}^{-1}$$

$$\begin{aligned}\Delta_{\mu\nu}^{Z\gamma} =& \frac{1}{(2\,\pi)^4\,i}\,\delta_{\mu\nu}\frac{g^2}{16\,\pi^2}\,\frac{s_\theta}{c_\theta}\left(\frac{\overline{S}_{Z\gamma}^0}{p^2} - s_\theta^2\,\Pi_{\gamma\gamma}\right)\\ &\times\left\{p^2 + \frac{M^2}{c_\theta^2} - \frac{g^2}{16\,\pi^2\,c_\theta^2}\left[\overline{S}_{ZZ}^0 - 2\,s_\theta^2\,\overline{S}_{Z\gamma}^0 + s_\theta^4\,p^2\,\Pi_{\gamma\gamma}\right]\right\}^{-1}\end{aligned}$$

where with $\overline{S}$ we denote the corresponding $S$ opportunely subtracted. Before constructing $\rho$ we discuss in some details how renormalization actually works for $e^+e^- \to \mu^+\mu^-$. With the outlined procedure we only need to worry about self-energy diagrams. In fitting the parameters the combination $g s_\theta = e$ is easily fixed in terms of $\alpha$ from $e\mu$ scattering

$$\frac{1}{g^2 s_\theta^2} = \frac{1}{4\pi\alpha} + \frac{1}{16\pi^2}\,\Pi_{\gamma\gamma}(0)$$

Thus photon exchange in $e^+e^- \to \mu^+\mu^-$ gives an amplitude proportional to

$$-(2\,\pi)^4 i\,\frac{4\pi\alpha}{p^2}\left\{1 - \frac{\alpha}{4\pi}\left[\Pi_{\gamma\gamma}(p^2) - \Pi_{\gamma\gamma}(0)\right]\right\}^{-1}$$

where the combination of self-energies is ultraviolet finite. For the $Z^0$ propagator the situation is slightly more complicated. Given

$$\Delta^{-1} = p^2 + \frac{M^2}{c_\theta^2} - S_{ZZ}(p^2)$$

we first obtain

$$\Delta = \frac{c_\theta^2}{g^2}\left[\frac{c_\theta^2}{g^2}p^2 + \frac{M^2}{g^2} - \frac{c_\theta^2}{g^2}S_{ZZ}(p^2)\right]^{-1}$$

Using the second fitting equation, obtained from $\mu-$ decay, the combination $M^2/g^2$ is fixed

$$8\,\frac{M^2}{g^2} = \frac{1+\delta_G}{G_F} + \frac{1}{2\pi^2}\,\overline{\Sigma}_{WW}(0)$$

where $\delta_G = 1/2\,\delta_\mu$. Therefore we get

$$\begin{aligned}\Delta =& 8\,G_F\frac{c_\theta^2}{g^2}\left\{8\,G_F\frac{c_\theta^2}{g^2}p^2 + 1 + \delta_G + \frac{G_F}{2\pi^2}\,\overline{\Sigma}_{WW}(0)\right.\\ &\left.-\frac{G_F}{2\pi^2}\left[\overline{S}^0_{ZZ}(p^2) - 2s_\theta^2\overline{S}^0_{Z\gamma}(p^2) + s_\theta^4 p^2\Pi_{\gamma\gamma}(p^2)\right]\right\}^{-1}\end{aligned}$$

The third fitting equation, namely the position of the $Z^0$ pole, gives

$$\begin{aligned}\frac{M^2}{c_\theta^2} &= M_Z^2 + \frac{g^2}{16\,\pi^2 c_\theta^2}\,Re\left[\overline{S}^0_{ZZ}\left(-M_Z^2\right) - 2s_\theta^2\overline{S}^0_{Z\gamma}\left(-M_Z^2\right) - s_\theta^4 M_Z^2\Pi_{\gamma\gamma}\left(-M_Z^2\right)\right]\\ &\equiv M_Z^2 + \frac{g^2}{16\,\pi^2 c_\theta^2}\,Re\,f\left(-M_Z^2\right)\end{aligned}$$

The previous equation defines $f(p^2)$. By combining the two equations we easily obtain a solution for $c_\theta^2/g^2$

$$8\,G_F M_Z^2\frac{c_\theta^2}{g^2} = 1 + \delta_G + \frac{G_F}{2\,\pi^2}\,Re\,\left[\overline{\Sigma}_{WW}(0) - f\left(-M_Z^2\right)\right]$$

Finally $\Delta$ may be cast into the following form

$$\begin{aligned}\Delta =& 8\,G_F M_Z^2\frac{c_\theta^2}{g^2}\left\{\left[1 + \delta_G + \frac{G_F}{2\,\pi^2}\,\overline{\Sigma}_{WW}(0)\right]\,(p^2 + M_Z^2)\right.\\ &\left.-\frac{G_F}{2\,\pi^2}Re\,\left[M_Z^2 f(p^2) + p^2 f\left(-M_Z^2\right)\right] - i\frac{G_F M_Z^2}{2\,\pi^2}Im\,f(p^2)\right\}^{-1}\\ &= 8\,G_F M_Z^2\frac{c_\theta^2}{g^2}\,P_Z^{-1}\end{aligned}$$

The quantity in braket is ultraviolet finite. Inside the function $f$ we used the zero'th order value for $s_\theta^2$, which follows from the fitting equations

$$\bar{s}_\theta^2 = \tfrac{1}{2}\left[1 - \sqrt{1 - \frac{2\,\pi\alpha}{G_F M_Z^2}}\right]$$

Up to first order the solution for $s_\theta^2$ is found to be

$$s_\theta^2 = \bar{s}_\theta^2 \left(1 + \frac{\bar{c}_\theta^2}{\bar{c}_\theta^2 - \bar{s}_\theta^2}\,\kappa\right)$$
$$\kappa = \delta_G - \frac{\alpha}{4\,\pi}\Pi_{\gamma\gamma}(0) + \frac{G_F}{2\,\pi^2}\left[\overline{\Sigma}_{WW}(0) - Re\,f\left(-M_Z^2\right)\right]$$

Once more the corrections to $s_\theta^2$ are not finite. However the $Z^0 - Z^0$ propagator diagram together with the two $Z^0 - \gamma$ transition diagrams give a finite answer

$$-\left(2\,\pi^2\right) i\,\frac{G_F M_Z^2}{2\,P_Z}\left\{\gamma^\mu\left(\bar{v}_\theta - \gamma^5\right)\otimes\gamma^\mu\left(\bar{v}_\theta - \gamma^5\right) - \left[4\,\frac{\bar{s}_\theta^2\bar{c}_\theta^2}{\bar{c}_\theta^2 - \bar{s}_\theta^2}\,\kappa - \frac{\alpha}{\pi}\left(\frac{\overline{S}^0_{Z\gamma}}{p^2} - \bar{s}_\theta^2\,\Pi_{\gamma\gamma}\right)\right]\right.$$
$$\left.\times\left[\gamma^\mu\gamma^5\otimes\gamma^\mu + \gamma^\mu\otimes\gamma^\mu\gamma^5 - 2\,\bar{v}_\theta\gamma^\mu\otimes\gamma^\mu\right]\right\}$$

In conclusion we turn to the $\rho-$parameter. As discussed at length in the first part we assume precise knowledge of four quantities, $\alpha, G_F, M_W$ and $M_Z$. Suppose to fix the parameters with $\alpha, G_F$ and $M_W$ and to consider the measure of $M_Z$ as a constraint on the internal consistency of the theory. Solving for $g^2, s_\theta^2$ and $M^2$ we get

$$g^2 = \frac{8\,G_F\,M_W^2}{1 + \frac{\alpha}{4\,\pi}\,Re\,\delta_g}, \qquad s_\theta^2 = \frac{\bar{s}_\theta^2}{1 + \frac{\alpha}{4\,\pi}\,Re\,\delta_s}$$

where

$$\bar{s}_\theta^2 = \frac{\pi\alpha}{2\,G_F M_W^2}$$

$$S^i_{WW}\left(-M_W^2\right) = -M_W^2\,\Pi^i_{WW}\left(-M_W^2\right) + \Sigma^i_{WW}\left(-M_W^2\right)$$

and

$$\delta_g = \frac{4\,\pi}{\alpha}\,\delta_G - \frac{1}{M_W^2}\left\{\frac{1}{\bar{s}_\theta^2}\left[\overline{S}^0_{WW}\left(-M_W^2\right) - \overline{\Sigma}^0_{WW}(0)\right] + \overline{S}^1_{WW}\left(-M_W^2\right) - \overline{\Sigma}^1_{WW}(0)\right\}$$

$$\delta_s = \Pi_{\gamma\gamma}(0) - \delta_g$$

Also

$$M^2 = \frac{M_W^2}{1 + \frac{\alpha}{4\pi} Re\, \delta_M}$$

$$\delta_M = -\frac{1}{M_W^2} \left[ \frac{1}{\bar{s}_\theta^2} \overline{S}_{WW}^0 \left(-M_W^2\right) + \overline{S}_{WW}^1 \left(-M_W^2\right) \right]$$

Now let $M_0$ be the $Z^0$ bare mass, we get

$$c_\theta^2 M_0^2 = \frac{\bar{c}_\theta^2 M_Z^2}{1 + \frac{\alpha}{4\pi} Re\, \delta'_M}$$

$$\delta'_M = \frac{1}{\bar{s}_\theta^2 \bar{c}_\theta^2} \left\{ -\frac{1}{M_Z^2} \left[ \overline{S}_{ZZ}^0 \left(-M_Z^2\right) - 2\bar{s}_\theta^2 \overline{S}_{Z\gamma}^0 \left(-M_Z^2\right) \right] - \frac{\bar{s}_\theta^2}{M_W^2} \left[ \overline{S}_{WW}^0 \left(-M_W^2\right) - \overline{\Sigma}_{WW}^0(0) \right] \right.$$
$$\left. + \bar{s}_\theta^4 \left[ \frac{4\pi}{\alpha} \delta_\mu + \Pi_{\gamma\gamma}\left(-M_Z^2\right) - \Pi_{\gamma\gamma}(0) - \frac{\overline{S}_{WW}^1 \left(-M_W^2\right) - \overline{\Sigma}_{WW}^1(0)}{M_W^2} \right] \right\}$$

From the previous results it follows

$$\begin{aligned} \rho &= \frac{M_W^2}{M_Z^2 \bar{c}_\theta^2} \left[ 1 + \frac{\alpha}{4\pi} Re\, \left(\delta'_M - \delta_M\right) \right] \\ &= \frac{M_W^2}{M_Z^2 \bar{c}_\theta^2} \left[ 1 + \delta\rho \right] \end{aligned}$$

In $\delta'_M - \delta_M$ we use $M_0 = M_W/\bar{c}_\theta$. Although the various quantities appearing in $\delta'_M$ and $\delta_M$ are by themselves infinite, the combination occurring for the $\rho-$parameter is finite, as it should be. In conclusion we consider the dependence of the $\rho-$parameter on a new as yet undiscovered quark doublet ( or top-bottom as well ). From low energy data we obtain the well known result

$$\rho_{ud} = 1 + \frac{3}{8} \frac{G_F}{\pi^2} \left[ m_u^2 + m_d^2 + 2 \frac{m_u^2 m_d^2}{m_u^2 - m_d^2} \ln \frac{m_u^2}{m_d^2} \right]$$

If masses are input parameters then

$$
\begin{aligned}
\frac{\pi^2\,\bar{c}_\theta^2}{G_F}\,\delta\rho_{ud} = {} & \left[\tfrac{3}{4}\left(m_u^2+m_d^2\right)\bar{c}_\theta^2 - M_W^2\left(\tfrac{1}{2}-\bar{s}_\theta^2+\tfrac{10}{9}\,\bar{s}_\theta^4\right)\right]\ln\bar{c}_\theta^2 + \tfrac{3}{4}\left(m_u^2+m_d^2\right)\bar{s}_\theta^2\ln\frac{m_d^2}{M_W^2} \\
& + \left[M_W^2\left(\tfrac{3}{4}-2\,\bar{s}_\theta^2+\tfrac{8}{3}\bar{s}_\theta^4\right) - \tfrac{3}{4}\,m_u^2\,\bar{c}_\theta^2\right]\sum_i Re\,G_1\left(x^i_{Zu}\right) \\
& + \left[M_W^2\left(\tfrac{3}{4}-\bar{s}_\theta^2+\tfrac{2}{3}\bar{s}_\theta^4\right) - \tfrac{3}{4}\,m_d^2\,\bar{c}_\theta^2\right]\sum_i Re\,G_1\left(x^i_{Zd}\right) \\
& + 2\left[\tfrac{3}{4}\left(m_d^2-m_u^2\right) - \tfrac{3}{2}\,M_W^2\right]\left(\bar{c}_\theta^2-\bar{s}_\theta^2\right)\sum_i Re\,G_2\left(x^i_W\right) \\
& - 3\,M_W^2\left(\tfrac{1}{2}-\tfrac{4}{3}\,\bar{s}_\theta^2+\tfrac{16}{9}\,\bar{s}_\theta^4\right)\sum_i Re\,G_3\left(x^i_{Zu}\right) \\
& - 3\,M_W^2\left(\tfrac{1}{2}-\tfrac{2}{3}\,\bar{s}_\theta^2+\tfrac{4}{9}\,\bar{s}_\theta^4\right)\sum_i Re\,G_3\left(x^i_{Zd}\right) \\
& + 3\,M_W^2\left(\bar{c}_\theta^2-\bar{s}_\theta^2\right)\sum_i Re\,G_3\left(x^i_W\right) + 3\,m_u^2\left(\tfrac{1}{2}-\bar{s}_\theta^2\right)\sum_i Re\,G_1\left(x^i_W\right) \\
& + \tfrac{2}{9}\,M_W^2\,\bar{s}_\theta^4\left(\ln\frac{m_u^2}{m_d^2} - 5\ln\frac{m_u^2}{M_W^2}\right) + \tfrac{3}{2}\,m_u^2\,\bar{s}_\theta^2\,F_1(x) + \tfrac{3}{4}\left(m_d^2-m_u^2\right)\bar{s}_\theta^2\,F_2(x)
\end{aligned}
$$

Where we have defined the following quantities

$$
\begin{aligned}
x_W^{1,2} \quad & \text{are the roots of} \quad M_W^2 x^2 + \left(m_u^2 - m_d^2 - M_W^2\right)x + m_u^2 - i\epsilon = 0 \\
x_{Zu(d)}^{1,2} \quad & \text{are the roots of} \quad M_0^2 x^2 - M_0^2 x + m_{u(d)}^2 - i\epsilon = 0
\end{aligned}
$$

and

$$
x = \frac{m_u^2}{m_u^2 - m_d^2}
$$

The previous result follows when we express the $B$ form factors in terms of $F$ and $G$ functions [6]

$$
\begin{aligned}
B_0\left(p^2, m_1, m_2\right) &= i\,\pi^2\left[\Delta - \ln(-p^2 - i\epsilon) - \sum_i G_1(x_i)\right] \\
B_1\left(p^2, m_1, m_2\right) &= -\tfrac{1}{2}i\,\pi^2\left[\Delta - \ln(-p^2 - i\epsilon) - 2\sum_i G_2(x_i)\right] \\
B_{21}\left(p^2, m_1, m_2\right) &= \tfrac{1}{3}i\,\pi^2\left[\Delta - \ln(-p^2 - i\epsilon) - 3\sum_i G_1(x_i)\right]
\end{aligned}
$$

$\Delta$ is the ultraviolet factor, namely $\Delta = -2/(n-4) + \gamma - \ln \pi$, $x_i$ are the roots of $-p^2x^2 + (p^2 + m_2^2 - m_1^2)x + m_1^2 - i\epsilon = 0$ and

$$G_n(z) = \int_0^1 dx\, x^{n-1} \ln(x-z), \qquad F_n(z) = -\int_0^1 dx \frac{x^n}{x-z}$$

For $m_u \gg M_W, m_d$ we use the asymptotic behavior of the $F, G-$ functions [2]

$$G_n\left(x_W^1\right) \sim \frac{1}{n} \ln\left(-\frac{m_u^2}{M_W^2}\right), \quad G_1\left(x_W^2\right) \sim -1 + i\pi, \quad G_2\left(x_W^2\right) \sim -\tfrac{3}{4} + \tfrac{i}{2}\pi$$
$$F_1(x) \sim \ln\left(\frac{m_u^2}{m_d^2}\right) - 1, \qquad F_2(x) \sim \ln\left(\frac{m_u^2}{m_d^2}\right) - \tfrac{3}{2}$$
$$G_1\left(x_{Zu}^{1,2}\right) \sim \tfrac{1}{2}\left(\ln \frac{m_u^2}{M_W^2} + \ln \bar{c}_\theta^2\right) \mp \tfrac{i}{2}\pi$$

As a consequence we find a cancellation of the logarithms in the asymptotic behavior of $\delta\rho_{ud}$ and the expected result emerges

$$\delta\rho_u \sim -\tfrac{3}{8}\frac{G_F m_u^2}{\pi^2}$$

For a very heavy quark the same result is obtained, independently from the input parameters, i.e. low energy data or vector boson masses. However for $m_{top} \approx 100 - 200\,\mathrm{GeV}$ we have to use the full expression for $\rho_{ud}$ and not its asymptotic form. For degenerate fermions ($m_u = m_d = m$) we easily find that terms proportional to $G_F m^2 \ln(m^2/M_W^2)$ and to $G_F m^2$ cancel out in the final answer when $m \gg M_W$, as expected. Actually we get $\delta\rho_{ud} = O(\mid m_u^2 - m_d^2 \mid)$ for $m_u \approx m_d \gg M_W$. For light quarks, $m_u, m_d \ll M_W$ we only need the behavior of $G_n(x)$ near 0 and 1

$$x \to 1, \qquad G_1(x) \sim -1 + i\pi, \quad G_2(x) \sim -\tfrac{3}{4} + \tfrac{i}{2}\pi$$
$$x \to 0, \qquad G_n(x) \sim -\frac{1}{n^2}$$

This gives

$$\delta\rho_{ud} = \frac{G_F M_W^2}{\pi^2 \bar{c}_\theta^2}\left[\left(-\tfrac{8}{9}\ln\frac{m_u^2}{M_W^2} - \tfrac{2}{9}\ln\frac{m_d^2}{M_W^2} - \tfrac{50}{27}\right)\bar{s}_\theta^4 + \left(-\tfrac{1}{2} + \bar{s}_\theta^2 - \tfrac{10}{9}\bar{s}_\theta^4\right)\ln \bar{c}_\theta^2\right]$$

For completeness we give the explicit expressions for the functions $\Pi_{\gamma\gamma}, \ldots, \Sigma_{WW}^1$ in terms of $B-$ form factors.

$$\Pi_{\gamma\gamma} = \tfrac{2}{3} - 12\,B_{21}(M,M) + 7\,B_0(M,M) + 4\sum_f \left[B_f(m_e,m_e) + \tfrac{4}{3}\,B_f(m_u,m_u) + \tfrac{1}{3}\,B_f(m_d,m_d)\right]$$

$$S^0_{Z\gamma} = p^2\,\{\tfrac{2}{3} - 10\,B_{21}(M,M) + \tfrac{13}{2}\,B_0(M,M) + \sum_f \left[B_f(m_e,m_2) + 2\,B_f(m_u,m_u) + B_f(m_d,m_d)\right]\} - 2\,M^2 B_0(M,M)$$

$$S^0_{ZZ} = p^2\,\Pi^0_{ZZ} + \Sigma^0_{ZZ}$$

$$\Pi^0_{ZZ} = \tfrac{2}{3} - 9\,B_{21}(M,M) + \tfrac{25}{4}\,B_0(M,M) - B_{21}(M_0,m_H) - B_1(M_0,m_H) - \tfrac{1}{4}\,B_0(M_0,m_H) + \tfrac{1}{2}\sum_f \left[B_f(m_e,m_e) + B_f(m_\nu,m_\nu) + 3\,B_f(m_u,m_u) + 3\,B_f(m_d,m_d)\right]$$

$$\Sigma^0_{ZZ} = -\,2\,M^2 B_0(M,M) + \tfrac{1}{2}\,M_0^2 B_1(M_0,m_H) + \tfrac{5}{4}\,M_0^2 B_0(M_0,m_H) - \tfrac{1}{2}\,m_H^2 B_1(M_0,m_H) - \tfrac{1}{4}\,m_H^2 B_0(M_0,m_H) - \tfrac{1}{2}\sum_f \left[m_\nu^2 B_0(m_\nu,m_\nu) + m_e^2 B_0(m_e,m_e) + 3\,m_u^2 B_0(m_u,m_u) + 3\,m_d^2 B_0(m_d,m_d)\right]$$

$$\Sigma^0_{WW} = \tfrac{9}{2}\,(M_0^2 - M^2) B_1(M_0,M) + \tfrac{1}{4}\,(13M_0^2 - 21M^2) B_0(M_0,M) + \tfrac{1}{2}\,(M^2 - m_H^2) B_1(M,m_H) + \tfrac{1}{4}\,(5M^2 - m_H^2) B_0(M,m_H) + \sum_f \left[(m_e^2 - m_\nu^2) B_1(m_\nu,m_e) - m_\nu^2 B_0(m_\nu,m_e) + 3\,(m_d^2 - m_u^2) B_1(m_u,m_d) - 3\,m_u^2 B_0(m_u,m_d)\right]$$

$$\Sigma^1_{WW} = 2\,(M^2 - M_0^2)\,[2B_1(M_0,M) + B_0(M_0,M)] - 2\,M^2\,[2B_1(\lambda,M) + B_0(\lambda,M)]$$

$$\Pi^0_{WW} = \tfrac{2}{3} - 9B_{21}(M_0,M) - 9B_1(M_0,M) + \tfrac{7}{4}\,B_0(M_0,M) - B_{21}(M,m_H) - B_1(M,m_H) - \tfrac{1}{4}\,B_0(M,m_H) + 2\sum_f \left[B_{21}(m_\nu,m_e) + B_1(m_\nu,m_e) + 3\,B_{21}(m_u,m_d) + 3\,B_1(m_u,m_d)\right]$$

$$\Pi^1_{WW} = 8\,B_{21}(M_0,M) - 2\,B_0(M_0,M) + 8\,B_1(M_0,M) - 8\,B_{21}(\lambda,M) - 8\,B_1(\lambda,M) + 2\,B_0(\lambda,M)$$

where $B_f = 2\,B_{21} - B_0$ and $\lambda$ is the photon mass. Ultraviolet finiteness for physical quantities can be verified very easily by using the following infinite parts

$$P.P.\left(\overline{S}^0_{ZZ}\right) = \left(r_z p^2 + \mu^2\right)\,\Delta \qquad P.P.\left(\overline{S}^0_{Z\gamma}\right) = r_z p^2\,\Delta$$

$$P.P.\left(\overline{S}^0_{WW}\right) = \left(r_z p^2 + \mu^2\right)\,\Delta \qquad P.P.\left(\Pi_{\gamma\gamma}\right) = r_\gamma\,\Delta$$

where $P.P.$ stands for pole part and

$$r_\gamma = 3 - \tfrac{32}{9}\,N_f, \qquad r_z = \tfrac{19}{6} - \tfrac{4}{3}\,N_f$$

$$\mu^2 = M_0^2 - 6\,M^2 - \tfrac{1}{2}\sum_f \left(m_\nu^2 + m_e^2 + 3\,m_u^2 + 3\,m_d^2\right)$$

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