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TWO-DIMENSIONAL BLACK HOLES AND GENERAL RELATIVITY

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ABSTRACT

The Einstein-Hilbert action with a cosmological term is used to derive a new action in 1+1 spacetime dimensions. It is shown that the two-dimensional theory is equivalent to planar symmetry in General Relativity. The two-dimensional theory admits black holes and free dilatons, and has a structure similar to two-dimensional string theories. Since by construction these solutions also solve Einstein's equations, such a theory can be the key to bring two-dimensional results into the four-dimensional real world. In particular the two-dimensional black hole is also a black hole in General Realtivity.

General Relativity is thought to be the correct theory from the largest conceivable scales up to the Planck length, 10^{-33}cm. A place to test these tinyest radii can be found in the late stages of black hole evaporation. This is a non-trivial issue and one is still probing regions where a semi-classical approximation is valid. The two-dimensional (2D) black holes found in the context of string theories [1,2] are being used to compute and analyze the back-reaction of the radiation on the geometry [3]. Such 2D theories are, in principle, toy models and it seems essential to build a bridge into the four-dimensional (4D) world. With this purpose, we consider the following action,

$$S = \frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\phi}\left(R + 2\left(\nabla\phi\right)^2 + 4\lambda^2\right). \tag{1}$$

where g is the determinant of the 2D metric, R is the scalar curvature, ϕ is a scalar field and λ is a constant. Equation (1) is very similar to the simplest 2D action derived by imposing the vanish of the β functions in string theory, and given by,

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left(R + 4 (\nabla\phi)^2 + 4\lambda^2 \right). \tag{2}$$

As we will show, action (1) comes from 4D Einstein gravity and also admits black hole solutions. It is tempting to generalize both these actions into,

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left(R - 4\omega (\nabla\phi)^2 + 4\lambda^2 \right), \tag{3}$$

where ω is a parameter. Equations (1) and (2) have $\omega = -\frac{1}{2}$ and $\omega = -1$, respectively. For $\omega = 0$ one has the Jackiw-Teitelboim theory [4], where the scalar curvature is a constant. Equation (3) is a 2D Brans-Dicke theory [5]. It seems plausible to have black hole solutions for several different values of ω.

How can we obtain the action (1) from 4D General Relativity? The idea is to reduce from $3+1$ to $1+1$ spacetime dimensions, loosing the less possible information. This is the case if, (i) the 3-space manifold splits into a direct product of a 2D manifold Σ_2 with the real line, $M_3 = \Sigma_2 \mathrm{x} R$ and, (ii) the physics and geometry (metric) on Σ_2 are invariant under the action of the 3-parameter group, G_3, of motions, i.e., the action is an isometry. Conditions (i) and (ii) imply planar symmetry. Thus the infinitesimal generators of the group G_3 are two orthogonal translations and one rotation about the axis. By a known theorem [6] if a group G_3 of motions has spatial orbits of dimension 2, these orbits admit orthogonal spacetime 2-surfaces. We will label the spatial planar 2-surfaces by (y, z).

Now, Einstein-Hilbert action is,

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g^{(4)}} \left(R^{(4)} - 2\Lambda \right), \tag{4}$$

where the superscript $^{(4)}$ denotes 4D quantities and Λ is the 4D cosmological constant. Also, the most general plane-symmetric metric can be written as,

$$ds^2 = g_{ab}dx^a dx^b + e^{-2\phi}\left(dy^2 + dz^2\right), \tag{5}$$

where $a, b = 0, 1$, and g_{ab} and ϕ are functions on the spacetime 2-surfaces. The scalar function ϕ is called for obvious reasons the dilaton. From standard dimensional reduction techniques [7] on (4) and (5), we obtain equation (1). Thus the 2D graviton-dilaton theory given by equation (1) can be related in a very direct manner with General Relativity. To obtain (1) we have integrated over a 'spurious' 2D planar compact manifold, say a 2D planar torus, on which we have imposed the normalization $\int dydz = 8$, and set $\Lambda = -2\lambda^2$. Adding a pure 2D cosmological term to the action (1), $\frac{1}{2\pi}\int d^2x\sqrt{-g}l$, l a constant, is equivalent to spherical symmetry in General Relativity [8]. With a different motivation, but also using dimensional reduction arguments, equation (2) has been derived out of a Chern-Simons action [9].

Variation of (1) with respect to g^{ab} and ϕ yields the gravitational and dilaton field equations, respectively,

$$\frac{1}{2}G_{ab} + D_a D_b\phi - (D_a\phi)(D_b\phi) - g_{ab}D_c D^c\phi +$$

$$+\frac{3}{2}g_{ab}(D_c\phi)(D^c\phi) - g_{ab}\lambda^2 = T_{ab}e^{2\phi}, \tag{6}$$

$$2D_c D^c\phi - 2(D_c\phi)(D^c\phi) + R + 4\lambda^2 = 0, \tag{7}$$

where D represents the covariant derivative. We have add a matter term S_m to (1) such that $\frac{\delta S_m}{\delta g^{ab}} \equiv -\frac{1}{\pi}\sqrt{-g}T_{ab}$. In 2D the Einstein tensor is $G_{ab} \equiv 0$.

To find solutions of this theory we have to exhibit explicitly the 2D metric g_{ab}. By performing a coordinate transformation we can put g_{ab} into diagonal form,

$ds^2 = -e^{2\nu}dt^2 + e^{2\mu}dx^2$, where ν and μ, (as well as ϕ), are functions of the spacetime coordinates (t, x). We still have the freedom to choose a gauge, any gauge will do. We choose the unitary gauge, $\mu = 0$. Then the metric is,

$$ds^2 = -e^{2\nu}dt^2 + dx^2. \tag{8}$$

If we now look for static (exists a Killing vector $\frac{\partial}{\partial t}$), vacuum ($T_{ab} = 0$) spacetimes we obtain from (6), (7) and (8) the following three equations, (only two of them are independent),

$$\phi_{,xx} - \frac{3}{2}\phi_{,x}^2 + \lambda^2 = 0, \tag{9}$$

$$\frac{1}{2}\phi_{,x}^2 - \nu_{,x}\phi_{,x} - \lambda^2 = 0, \tag{10}$$

$$-\phi_{,xx} + \nu_{,xx} + \nu_{,x}^2 + \phi_{,x}^2 - \nu_{,x}\phi_{,x} - 2\lambda^2 = 0. \tag{11}$$

The linear, free, dilaton solution is of course of the form, $\phi \propto x$. Equation (9) then gives,

$$\phi = -\sqrt{\frac{2}{3}}\lambda x + \text{constant}, \tag{12}$$

while (10) puts the metric in the form,

$$ds^2 = -e^{2\sqrt{\frac{2}{3}}\lambda x}dt^2 + dx^2. \tag{13}$$

Given there is a free dilaton solution, string theories hint that we should look for a black hole. The general solution of (9) is of the form, $\phi = -\frac{2}{3}\ln\left(A\cosh\left(\sqrt{\frac{3}{2}}\lambda x\right) + B\sinh\left(\sqrt{\frac{3}{2}}\lambda x\right)\right)$. We now set $B = 0$, (this can always be done if $A >\mid B \mid$).Then,

$$\phi = -\frac{2}{3}\ln\left(\cosh\sqrt{\frac{3}{2}}\lambda x\right) + \phi_0. \tag{14}$$

The absolute value of the dilaton increases at each spatial infinity, $x \to \infty$ and $x \to -\infty$. Equation (10) yields the metric,

$$ds^2 = -\tanh^2\left(\sqrt{\frac{3}{2}}\lambda x\right)\cosh^{\frac{4}{3}}\left(\sqrt{\frac{3}{2}}\lambda x\right)dt^2 + dx^2. \tag{15}$$

The metric (15) has singularities at $x = 0$, $x \to +\infty$ and $x \to -\infty$. However, they are merely coordinate singularities since the scalar curvature, $R = -\frac{4}{3}\lambda^2\frac{\sinh^2\sqrt{\frac{3}{2}}\lambda x}{\cosh^2\sqrt{\frac{3}{2}}\lambda x}$, has a regular behavior at these ends. In fact, equations (14) and (15) describe the geometry external to the horizon at $x = 0$. To bypass the coordinate singularity at $x = 0$, and to show that (15) is a black hole, we display its maximal analytical extension [10]. First, if we specify the intermediary coordinate $r = \sqrt{\frac{3}{2}}\frac{1}{\lambda}\cosh^{\frac{2}{3}}\left(\sqrt{\frac{3}{2}}\lambda x\right)$, we can put (15) in the Schwarzschild gauge, $ds^2 = -\left(a^2r^2 - \frac{1}{ar}\right)dt^2 + \frac{dr^2}{a^2r^2 - \frac{1}{ar}}$,where $a \equiv \sqrt{\frac{2}{3}}\lambda$. We can then define a second intermediary coordinate, $r_* = \frac{1}{a}\left[\frac{1}{6}\ln(ar-1)^2 - \frac{1}{6}\ln\left(a^2r^2 + ar + 1\right) + \frac{1}{\sqrt{3}}\arctan\frac{2ar+1}{\sqrt{3}}\right]$. This puts the metric in the conformal gauge, $ds^2 = -\left(a^2r^2(r_*) - \frac{1}{ar(r_*)}\right)\left(-dt^2 + dr_*{}^2\right)$. To expose the maximal analytical extension we can now define the Kruskal null coordinates, $U = -\frac{1}{\lambda}\sqrt{\frac{2}{3}}e^{-\sqrt{\frac{3}{2}}\lambda(t-r_*)}$ and $V = \frac{1}{\lambda}\sqrt{\frac{2}{3}}e^{\sqrt{\frac{3}{2}}\lambda(t+r_*)}$. In Kruskal coordinates the metric takes the form

$$ds^2 = -\frac{1 + \frac{(ar-1)}{(a^2r^2+ar+1)^{\frac{1}{2}}}}{ar}\frac{\left(a^2r^2 + ar + 1\right)^{\frac{3}{2}}}{e^{\sqrt{3}\arctan\frac{2ar+1}{\sqrt{3}}} - \frac{3\lambda^2}{2}UV}dUdV, \tag{16}$$

where r is given implicitly as a function of U and V. Near the horizon, the metric is given by, $ds^2 \simeq -3\sqrt{3}\frac{dUdV}{e^{\frac{3x}{2\sqrt{3}}} - \frac{3}{2}\lambda^2 UV}$, which is immediatly recognizable as a black hole metric. This was expected, since at the horizon, $x \simeq 0$, the metric (15) takes the form $ds^2 \simeq -\tanh^2\left(\sqrt{\frac{3}{2}}\lambda x\right)dt^2 + dx^2$ which is similar to the black hole in 2D string theory. The true singularity, at $ar = 0$, obeys $UV = \frac{2}{3\lambda^2}e^{\frac{3x}{2\sqrt{3}}}$. This equation yields the usual two-branched horizontal hyperbolae, the future branch representing

the black hole singularity and the past branch a naked singularity. Also, at $ar \to \infty$, one has $UV = -\frac{2}{3\lambda^2} e^{\frac{3\pi}{2\sqrt{3}}}$, which are two vertical hyperbolae. In figure 1 we draw the corresponding Kruskal diagram. The causal structure of this manifold may then be interpreted in the usual way, like the Schwarzschild solution.

Region I can be interpreted as the exterior gravitational solution. The black hole is in region II. No signal leaves region II. Upon time reversal, region II transforms into region IV which is called a white hole. We may regard region III as the $x < 0$ copy of region I ($x > 0$). Then $x = 0$ corresponds to the Einstein-Rosen wormhole Regions V and VI, above and below the singularities, have recently received some attention [11,2], although it is still uncertain what is the exact meaning of extending a solution past the geometrical singularity.

The black hole solution (14) and (15) is asymptotic to the linear dilaton spacetime (12) and (13). Now, asymptotically there is a timelike killing vector $\frac{\partial}{\partial t}$. Thus we expect the existence of the conserved quantity $T_{0a}\xi^a$, where ξ^a is the normalized killing vector. Using standard techniques one can calculate the ADM mass at $x \to \infty$, given by,

$$M_{total} = \frac{\lambda}{\sqrt{6}} e^{-3\phi_0} + \left[\frac{8\sqrt[3]{2}}{9} \lambda^2 e^{-\phi_0} x \right]_\infty , \tag{17}$$

where ϕ_0 is the value of the dilaton at $x = 0$, (see equation (14)). The second term on the right hand side has an infinite value and can be interpreted as the infinite 'cosmological' mass of this 2D universe. The first term on the right side can then be regarded as the mass of the black hole, M_{BH}, given by,

$$M_{BH} = \frac{\lambda}{\sqrt{6}} e^{-3\phi_0}. \tag{18}$$

Analogously to the 2D string theory, the mass of the black hole is linked with the value of the dilaton at the horizon. Starting with the free dilaton solution one can create a black hole by throwing in a massive particle from infinity.

In order to include quantum field effects in the classical geometry of the black hole we must compute the Hawking temperature. The Euclideanized solution is periodic in imaginary time with period $\frac{2\pi}{\sqrt{\frac{3}{2}}\lambda}$. This is characteristic of a thermal state emitting radiation at temperature,

$$T = \sqrt{\frac{3}{2}}\frac{\lambda}{2\pi}. \tag{19}$$

This is independent of the mass, a result which has been reported in other 2D theories, and which we will be interpreting below in the context of a 4D planar spacetime [12]. A step further towards a full quantum gravitational treatment should include back reaction of the radiation on geometry by using the trace anomaly relation [3], but that is a matter for another place.

How can we relate these solutions with the 4D world? By construction, solutions (14) and (15) can be paste together to yield,

$$\begin{aligned} ds^2 = &-\tanh^2\left(\sqrt{\frac{3}{2}}\lambda x\right)\cosh^{\frac{4}{3}}\left(\sqrt{\frac{3}{2}}\lambda x\right)dt^2 + dx^2 \\ &+\cosh^{\frac{4}{3}}\left(\sqrt{\frac{3}{2}}\lambda x\right)e^{-2\phi_0}\left(dy^2+dz^2\right), \end{aligned} \tag{20}$$

which is a planar black hole in General Relativity!

A mass M in 2D is a surface density σ in 4D. From (6) we can infer that they are connected through $M \sim \sigma e^{-2\phi}$. Now, if we increase $-\phi_0$, we are changing the gauge of our planar coordinates, $y \to e^{-\phi_0}y$, $z \to e^{-\phi_0}z$, or in other words, our units of length are being decreased. An aereal unit is transformed into a smaller

area, so σ increses, and therefore M increases. This 4D viewsight clears up the dependence of the 2D mass on ϕ_0.

The independence of the Hawking temperature on the 2D mass can also be explained. Changing our units of length on the planar 2-surface doesnot alter the units on the orthogonal spacetime surface. Thus the event horizon in Schwarzschild-like coordinates is at $r_H = \sqrt{\frac{3}{2}}\frac{1}{\lambda}$, and its surface gravity is $k = \sqrt{\frac{3}{2}}\lambda = \frac{3}{2r_H}$. Since these are not altered by such a change of units, there is nothing to change the 2D temperature. A crude dimensional argument shows that a 4D mass m, must be linked to λ by $m = \frac{1}{\alpha\lambda}$ where $\alpha \geq 0$ is a constant. Therefore in the 4D world one would have $T \sim \frac{1}{m}$, restoring our expectations.

We have found and commented on two solutions only, namely the free dilaton and the black hole. But, of course, there are many other solutions. Equations (6) and (7) have an in-built symmetry which transforms vacuum static solutions into homogeneous (time-dependent) solutions by making, $\lambda \to i\lambda$, $t \to i\chi$ and $x \to i\tau$. In addition, from 4D planar General Relativity we know we have the Taub, the planar Kasner, and the Horský-Novotný solutions [4], which are also solutions in this 2D theory. All of these have interesting causal structures. Out of these, the most fundamental is maybe Taub's which has so far eluded a clear interpretation. In both backgrounds, either in 2D or in 4D planar symmetry, we can now interpret the Taub solution as the spacetime which takes over when the geometrical singularity of the planar black hole is approached. In this context, one is carried into the viewpoint that the Taub solution is an approximation to the Schwarzschild spherical solution near the singularity, inside the event horizon. Although the topology of the 2D spatial surfaces of the orbits of the group of isometries as well as the groups themselves are very different, (in the planar case being $S^1 x S^1$ or R^2 with the group being composed of two translations and one rotation, and in the spherical case

being S^2 with the $SO(3)$ group), an observer in that region could set up some semi-local coordinate system, (not as local as a quasi-Minkowskian observer, at least on a trajectory orthogonal to the 2D spatial surfaces), where Taub spacetime approximates the Schwarzschild solution.

Another interesting solution in this 2D theory is the one-particle (delta-function) solution with horizons [13], which corresponds in 4D to the Ipser dust wall [14]. In this connection we mention that a solution with horizons in a 2D theory was found by Brown, Henneaux and Teitelboim [15]. It is a one-particle solution and relates to the Vilenkin wall [16]. It is an object which doesnot belong to the theory we have been presenting, since the theory itself cannot admit 'transversal' pressures. In fact this particle solution is an object of the $R = T$ theory in 2D [17]. One could continue to list several other possible 2D solutions containing matter: the multiple particle solution [18], the string (cosmological) solution, and the 'beadcollar' string solution (i.e., a string sprinkled with particles) [13]. One could also try gravitational collapse in 2D. A relation between the collapse of dust and the collapse of null radiation could be found, as it is suggested by the spherical collapse in 4D [19]. This would also allow to test cosmic censorship and the formation of naked singularities.

String theories in 1+1 dimensions are being used to gain insight towards a quantum treatment of the graviton. One problem that is always raised is, how the 2D results connect to the 4D world. In this letter we have built a bridge that provides such a link. We have constructed a 2D theory, with a structure similar to the string theory, which is formally and directly related to 4D planar symmetry in General Relativity. The other part emerges when we are able to relate generic features, (such as the existence of black holes), from within these type of 2D theories,

perhaps using the Brans-Dicke parameter ω. In any case we should start to fully understand 4D plane-symmetric systems.

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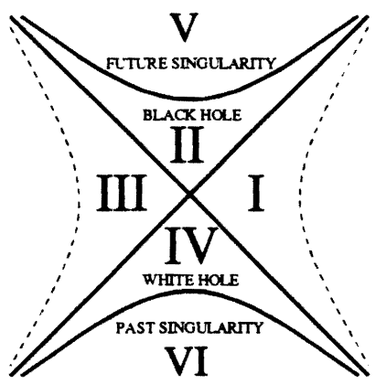

Figure 1. The Kruskal diagram for the maximal analytical extension of the black hole solution. Region I and III describe two identical but space-inverted regions, asymptotic to the free dilaton and connected through a wormhole. Regions II and IV describe two identical but time-reversed regions, representing the black hole and the white hole, respectively. No signal leaves region II. Regions V and VI are beyond the singularity and do not have a clear interpretation.

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