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NUMBER PROJECTED QUASI–PARTICLE RANDOM PHASE APPROXIMATION MATRIX–ELEMENTS FOR DOUBLE β DECAY

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ABSTRACT

A general formalism for proton–neutron excitations in odd–odd nuclei within a particle number projected treatment of the quasi–particle Random–Phase Approximation is presented. It is a generalized and improved combination of a known formalism used to describe identical nucleon systems. The expressions of the matrix–elements for neutrinoless and two–neutrino double β decay modes are presented. By means of our projected QRPA version the numerical calculation requires only a little more computational effort than unprojected ones. The formalism is very well suited for application on calculations of observables, where the BCS superfluid parameters are different for initial and final states.

I. INTRODUCTION

In recent years, the goal of testing the symmetry properties of the standard model[1,2,3], has encouraged great effort in performing nuclear double β–decay calculations. There are many reviews in the literature[4] which give a complete explanation of this question.

Because of the dimensions of the problem, the proton–neutron version of the quasi–particle Random Phase Approximation (QRPA) has been widely applied to the double β–decay of medium and heavy nuclei[5]. This approximation causes an explicit violation of the SU(4) symmetry and it is suggested that the extreme sensitivity of the $\beta\beta$–decay amplitude to the proton–neutron coupling is a consequence of the violation of the particle number conservation. It is well established that number projection is essential in BCS Tamm–Dancoff spectroscopy calculations[6].

Only recently, Civitarese et. al[7–8] performed $\beta\beta$–decay rate calculations in a fully number–projected, QRPA framework. They extended the number conserving version of QRPA for like quasi–particle pairs studied in Ref. (9) to unlike ones for medium and heavy nuclei. It is found that the particle number projection yields non–vanishing matrix elements in the neighbourhood of critical values of the proton–neutron coupling interaction.[7] Calculations of $\beta\beta$–decay observables within this approximation[10] show that the number projection is important for light nuclei, too.

In the present paper, in order to develope a proton–neutron number projection QRPA formalism, we use the generating function technique introduced by Ottaviani and Savoia[11] in TDA for identical nucleon systems. Within the version for nuclear closure approximation proposed in Ref.12, the explicit formulas for the nuclear matrix elements for two–neutrino and neutrinoless modes are presented.

II. FORMALISM

The total Hamiltonian is conveniently expressed as

$$H = H_p + H_n + H_{pn} \ , \tag{1}$$

where H_p and H_n describe the effective Hamiltonian in proton and neutron space, respectively, while H_{pn} denotes the effective interaction between protons and neutrons. The quantities H_p , H_n and H_{pn} , in second quantized form, are given by

$$H_t = \sum_t (e_t - \lambda_t)\, c_t^\dagger\, c_t + \frac{1}{4} \sum_{t's} \langle t_1\, t_2 | V | t_3\, t_4 \rangle_{\mathcal{A}}\, c_{t_1}^\dagger\, c_{t_2}^\dagger\, c_{t_4}\, c_{t_3} \tag{2}$$

and

$$H_{pn} = \sum_{pp'nn'} \langle pn\, | V |\, p'n' \rangle_{\mathcal{A}} : c_p^\dagger\, c_n^\dagger\, c_{n'}\, c_{p'} : \ , \tag{3}$$

where the subscripts t(t) stand for p(p) or n(n), depending on whether H_p or H_n is being considered. Here $t \equiv t,\ m_t$, with $t \equiv (n_t\, l_t\, j_t)$ and $m_t \equiv m_{j_t}$, and all the remaining notation has the standard meaning: e_t is the single particle energy (s.p.e.), λ_t is the chemical potential, $c_t^\dagger$ (c_t) are the single particle creation (annihilation) operators, the label $\mathcal{A}$ denotes the matrix elements with respect to antisymmetric states and the symbol : : denotes a normal product of fermion operators.

In order to calculate the matrix elements of the Hamiltonian (1), we introduce the z–dependent canonical transformation:

$$d_p^\dagger = \rho_p^{1/2}\,(u_p c_p^\dagger - z_p \bar{v}_p c_{\bar{p}})\,; \qquad \rho_p^{-1} = u_p^2 + z_p^2\,\bar{v}_p^2\ , \tag{4a}$$

and

$$d_n^\dagger = \rho_n^{1/2}\,(\bar{u}_n c_n^\dagger - z_n v_n c_{\bar{n}})\,; \qquad \rho_n^{-1} = \bar{u}_p^2 + z_n^2\,v_n^2\ , \tag{4b}$$

for protons and neutrons, respectively, with

$$c_{\bar{t}} = (-)^{t+m_t}\, c_{t,-m_t} \tag{5}$$

The unbarred u_t, v_t and barred $\bar{u}_t$, $\bar{v}_t$ parameters are obtained by solving the BCS equations for the initial (N,Z) nucleus and for the final $(N-2,\ Z+2)$ nucleus, respectively. It is easy to see that

$$\left\{d_p^\dagger\,,\, d_{p'}\right\} = \delta_{pp'}\ , \tag{6a}$$

$$\left\{d_n^\dagger\,,\, d_{n'}\right\} = \delta_{nn'}\ , \tag{6b}$$

and

$$d_p\,|0^+\,,\, z_p\, z_n> \;=\; d_n\,|0^+\,,\, z_p\, z_n> \;= 0 \tag{7}$$

where

$$|0^+;\, z_p\, z_n> \;=\; \prod_{p>0}\left(u_p + z_p\,\bar{v}_p\, c_p^\dagger\, c_{\bar{p}}^\dagger\right) \prod_{n>0}\left(\bar{u}_n + z_n\, v_n\, c_n^\dagger\, c_{\bar{n}}^\dagger\right)|0>\ , \tag{8a}$$

and $|0>$ represents the particle vacuum.

The generating wave function (eq.8a), can be also rewritten as

$$|0^+;z_p,z_n> \;=\; \sum_{p=0}^{p_{max}} \sum_{n=0}^{n_{max}} z_p^n\, z_n^n\, |0_N^+> \tag{8b}$$

where $|0_N^+>$ with $N = 2n + 2p$, is a wave function with fixed numbers of 2p–protons and 2n–neutrons.

By means of the inverse transformations obtained from (4), and neglecting the residual interactions between identical quasiparticles the Hamiltonian (1) reads:

$$H \;=\; H_p^{00} + H_p^{11} + H_p^{20} + H_p^{02} + H_n^{00} + H_n^{11} + H_n^{20} + H_n^{02} + H_{pn}^{22} + H_{pn}^{04} + H_{pn}^{40}\ , \tag{9}$$

with

$$H_p^{00} \;=\; \sum_p \rho_p\, z_p^2 \left[\bar{v}_p^2 \left(\varepsilon_p - \lambda_p - \tfrac{1}{2} z_p^2\, \tilde{\mu}_p\right) - \tfrac{1}{2} u_p\, \bar{v}_p\, \tilde{\Delta}_p\right]\ , \tag{10a}$$

$$H_p^{11} \;=\; \sum_p \rho_p \left[\left(\varepsilon_p - \lambda_p - z_p^2\, \tilde{\mu}_p\right)\left(u_p^2 - z_p^2\, \bar{v}_p^2\right) + 2\, z_p^2\, u_p\, \bar{v}_p\, \tilde{\Delta}_p\right] d_p^\dagger\, d_p\ , \tag{10b}$$

$$H_p^{20} \;=\; \left[H_p^{02}\right]^\dagger \;=\; \sum_p \rho_p \left[\left(\varepsilon_p - \lambda_p - z_p^2\, \tilde{\mu}_p\right) u_p\, \bar{v}_p - \tfrac{1}{2}\left(u_p^2 - z_p^2\, \bar{v}_p^2\right) \tilde{\Delta}_p\right]$$

$$\times \left[d_p^\dagger\, d_{\bar{p}}^\dagger + d_{\bar{p}}\, d_p\right] \tag{10c}$$

$$H_n^{00} \;=\; \sum_n \rho_n\, z_n^2 \left[v_n^2 \left(\varepsilon_n - \lambda_n - \tfrac{1}{2} z_n^2\, \tilde{\mu}_n\right) - \tfrac{1}{2} \bar{u}_n\, v_n\, \tilde{\Delta}_n\right]\ , \tag{10d}$$

$$H_n^{11} \;=\; \sum_n \rho_n \left[\left(\varepsilon_n - \lambda_n - z_n^2\, \tilde{u}_n\right)\left(\bar{u}_n^2 - z_n^2\, v_n^2\right) + 2\, z_n^2\, \bar{u}_n\, v_n\, \tilde{\Delta}_n\right] d_n^\dagger\, d_n\ , \tag{10e}$$

$$H_n^{20} = \left[H_n^{02}\right]^\dagger = \sum_n \rho_n \left[\left(\varepsilon_n - \lambda_n - z_n^2 \tilde{\mu}_n\right) \bar{u}_n v_n - \frac{1}{2}\left(\bar{u}_n^2 - z_n^2 v_n^2\right) \tilde{\Delta}_n\right]$$

$$\times \left[d_n^\dagger d_{\bar{n}}^\dagger + d_{\bar{n}} d_n\right] , \tag{10f}$$

$$H_{pn}^{22} = \sum_{pp'nn} \left(\rho_p \rho_n \rho_{p'} \rho_{n'}\right)^{1/2} \left[<pn|V|p'n'>_{\mathcal{A}} \left(u_p \bar{u}_n u_{p'} \bar{u}_{n'} + \right.\right.$$

$$\left.\left. z_p^2 z_n^2 \bar{v}_p v_n \bar{v}_{p'} v_{n'}\right) - <p\bar{n}'|V|p'\bar{n}>_{\mathcal{A}} \left(z_n^2 u_p v_n u_{p'} v_{n'} + z_p^2 \bar{v}_p \bar{u}_n \bar{v}_{p'} \bar{u}_{n'}\right)\right]$$

$$d_p^\dagger d_n^\dagger d_{n'} d_{p'} , \tag{10g}$$

$$H_{pn}^{40} = \left[H_{pn}^{04}\right]^\dagger = \sum_{pnp'n'} \left(\rho_p \rho_n \rho_{p'} \rho_{n'}\right)^{1/2} <pn|V|p'n'>_{\mathcal{A}} z_p z_n u_p \bar{u}_n \bar{v}_{p'} v_{n'}$$

$$d_p^\dagger d_n^\dagger d_{\bar{n}'}^\dagger d_{\bar{p}'}^\dagger . \tag{10h}$$

The gap parameters $\tilde{\Delta}$ and the self–energies $\tilde{\mu}$ are given by

$$\tilde{\Delta}_p = -\frac{1}{2} \hat{p}^{-1} \sum_{p'} \hat{p}' u_{p'} \bar{v}_{p'} \rho_{p'} G(pp\,p'p';0) , \tag{11a}$$

$$\tilde{\Delta}_n = -\frac{1}{2} \hat{n}^{-1} \sum_{n'} \hat{n}' \bar{u}_{n'} v_{n'} \rho_{n'} G(nn\,n'n';0) , \tag{11b}$$

$$\tilde{\mu}_p = -\hat{p}^{-1} \sum_{p'} \hat{p}' \bar{v}_{p'}^2 \rho_{p'} F(pp\,p'p';0) , \tag{12a}$$

$$\tilde{\mu}_n = -\hat{n}^{-1} \sum_{n'} \hat{n}' \, v_{n'}^2 \, \rho_{n'} \, F(nn\,n'n';0) \quad . \tag{12b}$$

The quantities F(abcd;I) and G(abcd;I) are, respectively, the particle–hole and particle–particle antisymmetrized matrix elements of the interaction[14] and $\hat{t} = (2j_t+1)^{1/2}$.

The QRPA phonons for a proton–neutron mode with angular momentum I are approximated by the expansion

$$\Gamma^\dagger(\alpha I) = \sum_{pn} \left[X(pnI;\alpha)\, D^\dagger(pnI) - Y(pnI;\alpha)\, D(pn\bar{I})\right] \quad , \tag{13}$$

in a finite set of basis operators

$$D^\dagger(pnI) = \mathcal{N}(pnI)\ (\rho_p\, \rho_n)^{1/2} \left[d_p^\dagger\, d_n^\dagger\right]^I \quad , \tag{14}$$

with the normalization factor $\mathcal{N}$ obtained from the orthogonal condition:

$$\langle 0_N^+ | D(p'n'I)\, D^\dagger(pnI) | 0_N^+ \rangle = \delta_{pp'}\, \delta_{nn'} \quad . \tag{15}$$

The forward– and backward–going amplitudes $X(pnI;\alpha)$ and $Y(pnI;\alpha)$ as well as the eigenvalues $\omega_{\alpha I}$ are obtained, as in the usual QRPA formalism[13], from the equation of motion:

$$\hat{I}^{-1}\langle 0_N^+|\left[\Gamma(\alpha\bar{I}),\, H, \Gamma(\beta I)\right]^0|0_N^+\rangle = \omega_{\alpha I}\ \delta_{\alpha\beta}\ , \tag{16a}$$

and

$$\hat{I}^{-1}\langle 0_N^+|\left[\Gamma(\alpha\bar{I}),\, \Gamma^\dagger(\beta I)\right]^0|0_N^+\rangle = \delta_{\alpha\beta}\ . \tag{16b}$$

The generating function (8) has the property that matrix elements between projected states of any operator $\hat{O}$ preserving the particle number can be derived by

$$<0_N^+|\hat{O}|0_N^+> = \left[\frac{1}{2\pi i}\right]^2 \oint dz_p\, z_p^{-2p-1} \oint dz_n\, z_n^{-2n-1} <0^+;z_p,z_n|\hat{O}|0^+;z_p,z_n> \tag{17}$$

where the contour of integration includes the origin. Then, the equation of motion (16) can be rewritten as:

$$\begin{bmatrix} A(I) & B(I) \\ -A(I) & -B(I) \end{bmatrix} \begin{bmatrix} X(\alpha I) \\ Y(\alpha I) \end{bmatrix} = \omega_{\alpha I} \begin{bmatrix} X(\alpha I) \\ Y(\alpha I) \end{bmatrix} \tag{18}$$

with the matrix elements given by

$$A(pn,p'n';I) = \hat{I}^{-1} \left[\frac{1}{2\pi i}\right]^2 \oint dz_p\, z_p^{-2p-1} \oint dz_n\, z_n^{-2n-1}$$

$$\times\ \langle 0^+;\, z_p,z_n|\left[D(pn\bar{I}),\, H, D^\dagger(p'n'I)\right]^0|0^+;\, z_p,z_n\rangle\ , \tag{19}$$

$$B(pn,p'n';I) = -\hat{I}^{-1} \left[\frac{1}{2\pi i}\right]^2 \oint dz_p\, z_p^{-2p-1} \oint dz_n\, z_n^{-2n-1}$$

$$\times\ \langle 0^+;\, z_p,z_n|\left[D(pn\bar{I}),\, H, D\,(p'n'\bar{I})\right]^0|0^+;\, z_p,z_n\rangle\ . \tag{20}$$

For the Hamiltonian (9), in a straightforward way, we have obtained the explicit formulas:

$$A(pn,p'n';I) = \left\{\delta_{pn,p'n'}\left[R_{11}^{2p}(pp)\, I_n^{2n}(n) + R_{11}^{2n}(nn)\, I_p^{2p}(p)\right] + R_{22}^{2p,2n}(pn,p'n')\right\} \mathcal{N}^2(pnI) \tag{21}$$

and

$$B(pn,p'n';I) = R_{40}^{2p,2n}(pn,p'n')\, \mathcal{N}^2(pnI) \tag{22}$$

where

$$R_{11}^{2p}(pp) = \left(\varepsilon_p - \lambda_p\right)\left[u_p^2\, I_p^{2p}(pp) - \bar{v}_p^2\, I_p^{2p-2}(pp)\right] + \hat{p}^{-1} \sum_{p'} \hat{p}'\Big\{\bar{v}_{p'}^2\Big[u_p^2\, I_p^{2p-2}(p'pp) -$$

$$\bar{v}_p^2\, I_p^{2p-4}(p'pp)\Big]\, F(p'p'pp;0) - 2\, u_{p'}\, \bar{v}_{p'}\, u_p\, \bar{v}_p\, I_p^{2p-2}(p'pp)\, G(p'p'pp;0)\Big\} \quad , \tag{23a}$$

$$R_{11}^{2n}(nn) = \left(\varepsilon_n - \lambda_n\right)\left[\bar{u}_n^2\, I_n^{2n}(nn) - v_n^2\, I_n^{2n-2}(nn)\right] + \hat{n}^{-1} \sum_{n'} \hat{n}'\Big\{v_{n'}^2\Big[\bar{u}_n^2\, I_n^{2n-2}(n'nn) -$$

$$v_n^2\, I_n^{2n-4}(n'nn)\Big]\, F(n'n'nn;0) - 2\, \bar{u}_{n'}\, v_{n'}\, \bar{u}_n\, v_n\, I_n^{2n-2}(n'nn)\, G(n'n'nn;0)\Big\} \quad , \tag{23b}$$

$$R_{22}^{2p,2n}(pn,p'n') = \Big[u_p\, \bar{u}_n\, u_{p'}\, \bar{u}_{n'}\, I_p^{2p}(pp')\, I_n^{2n}(nn') + \bar{v}_p\, v_n\, \bar{v}_{p'}\, v_{n'}\, I_p^{2p-2}(pp')$$

$$I_n^{2n-2}(nn')\Big]\, G(pn\, p'n';I) + \Big[u_p\, v_n\, u_{p'}\, v_{n'}\, I_p^{2p}(pp')\, I_n^{2n-2}(nn') - \bar{v}_p\, \bar{u}_n\, \bar{v}_{p'}\, \bar{u}_{n'}$$

$$I_p^{2p-2}(pp')\, I_n^{2n}(nn')\Big]\, F(pn\, p'n';I) \quad , \tag{23c}$$

$$R_{40}^{2p,2n}(pn,p'n') = \Big[\big(\bar{v}_p\,\bar{u}_n\,u_{p'}\,v_{n'} + u_p\,v_n\,\bar{v}_{p'}\,\bar{u}_{n'}\big)\,F(pn\,p'n';I) - \big(u_p\,\bar{u}_n\,\bar{v}_{p'}\,v_{n'} + \bar{v}_p\,v_n\,u_{p'}\,\bar{u}_{n'}\big)\,G(pn\,p'n';I)\Big]\,I_p^{2p-1}(pp')\,I_n^{2n-1}(nn') \quad , \tag{23d}$$

and the normalization

$$\mathcal{N}(pnI) = \big(\langle 0_N^+ \mid D(pnI)\,D^\dagger(pnI) \mid 0_N^+\rangle\big)^{-1/2} = \big(I_p^{2p}(p)\,I_n^{2n}(n)\big)^{-1/2} \ . \tag{23e}$$

The residuum integrals are defined as

$$I_p^K(pp'\cdots r) = \frac{1}{2\pi i}\oint dz_p\,z_p^{-K-1}\big(\rho_p\,\rho_{p'}\ldots\rho_r\big)\prod_{p''>0}\big(u_{p''}^2 + z_p^2\,\bar{v}_{p''}^2\big) \quad , \tag{24a}$$

$$I_n^K(nn'\cdots s) = \frac{1}{2\pi i}\oint dz_n\,z_n^{-K-1}\big(\rho_n\,\rho_{n'}\ldots\rho_s\big)\prod_{n''>0}\big(\bar{u}_{n''}^2 + z_n^2\,v_{n''}^2\big) \quad , \tag{24b}$$

where the integration path encloses the origin but no other singularities.

Numerically, instead of the residuum integral $I^k(ab...w)$ one can to evaluate sums [15]

$$L^k\,(ab....w) = \frac{1}{M}\,\{1 + 2\sum_{m=1}^{(M-1)/2}\mathrm{Real}\Big[z^{-k}(\rho_a\rho_{b'}\cdots\rho_w)\prod_{t>0}\big(u_t^2 + z^2 v_t^2\big)\Big] \ . \tag{24c}$$

where $z = \exp(i\pi m/M)$, with M a large odd–number.

The nuclear matrix element for a $\beta\beta$–decay is of the form[4]:

$$\mathcal{F}_N = \left| \mathcal{M}^{GT} - (g_A/g_V)^2 \mathcal{M}^F \right|^2 , \tag{25}$$

where $\mathcal{M}^{GT}$ and $\mathcal{M}^F$ are, respectively, the Gamow–Teller (GT) and Fermi(F) transition amplitudes.

In the two neutrino case, the amplitude $0^+ \to 0^+$ $2\nu\beta\beta$ is given by[12]

$$\mathcal{M}_{2\nu}(I) = \sum_{\alpha} <0_f^+ \| \mathcal{O}_+(I) \| \alpha I^\dagger> <\alpha I^\dagger \| \mathcal{O}_+(I) \| 0_i^+> / D(\alpha I) \tag{26}$$

where $\mathcal{M}_{2\nu}(I=1) \equiv \mathcal{M}^{GT}_{2\nu}$ and $\mathcal{M}_{2\nu}(I=0) \equiv \mathcal{M}^F_{2\nu}$. The 0_f^+ and 0_i^+ stand for the ground state of the final and initial nuclei, respectively, and $\mathcal{O}_\pm(I)$ are the one–body charge–exchange operators, expressed in second–quantization as:

$$\mathcal{O}_\pm(I) = \sum_{t's} <t_1|O(I)t_\pm|t_2> c^\dagger_{t_1} c_{t_2} \tag{27}$$

where $O(I=0) = 1$ and $O(I=1) = \vec{\sigma}$ with $<p|t_+|n> = 1$. The energy denominator $D(\alpha I)$, in natural units $(\hbar = c = m_e = 1)$, reads:

$$D(\alpha I) = E_{\alpha I} - E_i + \frac{1}{2} Q_{\beta\beta} + 1 \tag{28}$$

where E_i is the initial energy and $Q_{\beta\beta}$ the Q–value of the decay, and the sum in eq.(26) extends over a complete set of intermediate states $|\alpha I^+>$.

Using the relations (4a) and (4b) the charge–exchange operators (27) take the following forms:

$$\mathcal{O}_+(I) = \sum_{pn} \left(\Lambda_+^0 (pnI)[d_p^\dagger d_n^\dagger]^I + \Lambda_-^0 (pnI)[d_n d_p]^I \right) , \tag{29a}$$

$$\mathcal{O}_-(I) = \sum_{pn} \left(\Lambda_+^0 (pnI)[d_n d_p]^{\bar{I}} + \Lambda_-^0 (pnI)[d_p^\dagger d_n^\dagger]^I \right) , \tag{29b}$$

where

$$\Lambda_+^0 (pnI) = \hat{I}^{-1} (\rho_p \rho_n)^{1/2} z_n u_p v_n \langle p \| O(I) \| n \rangle , \tag{30a}$$

$$\Lambda_-^0 (pnI) = -\hat{I}^{-1} (\rho_p \rho_n)^{1/2} z_p \bar{v}_p \bar{u}_n \langle p \| O(I) \| n \rangle . \tag{30b}$$

In our formalism the $\langle 0_N^+|$ component of the wave function (8b) is the ground–state of the final and initial nuclei in eq.(26). Using the relation (17), the transition matrix elements are given by

$$\langle \alpha I^+ \| \mathcal{O}_\pm(I) \| 0^+ \rangle = \frac{\langle 0_N^+ | [\Gamma(\alpha \bar{I}), \mathcal{O}_\pm(I)]^0 | 0_N^+ \rangle}{\langle 0_N^+ | 0_N^+ \rangle^{1/2}} = -\hat{I}\, \Lambda_\pm(\alpha I) , \tag{31}$$

with

$$\Lambda_+(\alpha I) = \hat{I}^{-1} \left[\sum_{pn} \langle p \| \mathcal{O}_+(I) \| n \rangle \Lambda_+(pnI, \alpha)\, \mathcal{N}(pnI) \right] \mathcal{N}_0 , \tag{32a}$$

$$\Lambda_-(\alpha I) = \hat{I}^{-1} \left[\sum_{pn} \langle p \| \mathcal{O}_-(I) \| n \rangle \Lambda_-(pnI, \alpha)\, \mathcal{N}(pnI) \right] \mathcal{N}_0 , \tag{32b}$$

where

$$\Lambda_+(pnI,\alpha) = u_p\, v_n\, X(pnI,\alpha)\, I_p^{2p}(p)\, I_n^{2n-1}(n) + \bar{v}_p\, \bar{u}_n\, Y(pnI,\alpha)\, I_p^{2p-1}(p)\, I_n^{2n}(n) \ , \tag{33a}$$

$$\Lambda_-(pnI,\alpha) = \bar{v}_p\, \bar{u}_n\, X(pnI,\alpha)\, I_p^{2p-1}(p)\, I_n^{2n}(n) + u_p\, v_n\, Y(pnI,\alpha)\, I_p^{2p}(p)\, I_n^{2n-1}(n) \ , \tag{33b}$$

and

$$\mathcal{N}_0 = \left(\langle 0_N^+ \mid 0_N^+ \rangle\right)^{-1/2} = \left(I_p^{2p}\, I_n^{2n}\right)^{-1/2} \ . \tag{34}$$

Within our QRPA method, the energy denominator $D(\alpha I)$ reads:

$$D(\alpha I) = E_{\alpha I} - \tilde{E}_0 \equiv \omega_{\alpha I} \ , \tag{35}$$

where

$$\tilde{E}_0 = \mathcal{N}_0 \left\{ I_n^{2n} \sum_p \left[\left[\bar{v}_p^2 \left(\varepsilon_p - \lambda_p\right) - \tfrac{1}{2} u_p\, \bar{v}_p\, \tilde{\Delta}_p \right] I_p^{2p-2}(p) - \bar{v}_p^2\, \tilde{\mu}_p\, I_p^{2p-4}(p) \right] + \right.$$

$$\left. + I_p^{2p} \sum_n \left[\left[v_n^2 \left(\varepsilon_n - \lambda_n\right) - \tfrac{1}{2} \bar{u}_n\, v_n\, \tilde{\Delta}_n \right] I_n^{2n-2}(n) - v_n\, \tilde{\mu}_n\, I_n^{2n-4}(n) \right] \right\} \ , \tag{36}$$

which allows us to rewrite (26) in the form

$$\mathcal{M}_{2\nu}(I) = \hat{I}^2 \sum_\alpha \Lambda_+(\alpha I)\, \Lambda_-(\alpha I) \,/\, \omega_{\alpha I} \ . \tag{37}$$

In the case of the zero neutrino decay mode $(0\nu\beta\beta)$, we have the same matrix element $\mathcal{F}_N$ given by (25), and the $0^+ \to 0^+$ amplitudes are approximated by[16]:

$$\mathcal{M}_{0\nu} = \left\langle 0_f^+ \left| \sum_{1,2} H(r)\, t_+(1)\, t_+(2)\, \mathcal{O}_{12} \right| 0_i^+ \right\rangle \tag{38}$$

where $\mathcal{O}_{12}^{F} = 1$ and $\mathcal{O}_{12}^{GT} = \vec{\sigma}_1 \cdot \vec{\sigma}_2$, the neutrino potential $H(r) = \phi(Dr)/r$, with $r = |\vec{r}_1 - \vec{r}_2|$ and $D = \langle E_j \rangle - E_i + \frac{1}{2} Q\beta\beta + 1$; here E_j is the energy of the intermediate state.

The matrix elements of a two–body operator $\mathcal{O}_{12}$ in (38) is defined as

$$\langle \mathcal{O}_{12} \rangle = \frac{1}{2} \sum_{\substack{pnp'n' \\ I\pi\alpha I'}} \langle 0_f^+ \| [c_{p'}^{\dagger} c_{\bar{n}'}]^I \| \alpha I^{\pi} \rangle \langle \alpha I^{\pi} \| [c_p^{\dagger} c_{\bar{n}}]^I \| 0_i^+ \rangle$$

$$(-)^{n+p'+I+I'} \hat{I}'^2 \begin{Bmatrix} p & n & I \\ n' & p' & I' \end{Bmatrix} \langle pp'I' | t_+(1)\, t_+(2)\, \mathcal{O}_{12} | nn'I' \rangle \quad . \tag{39}$$

In our QRPA formalism, the matrix elements of the one–body transition densities are given, analogously to (31), by

$$\langle \alpha I^{\pi} \| [c_p^{\dagger} c_{\bar{n}}]^I \| 0^+ \rangle = \hat{I}\, \Lambda_-(pnI,\alpha)\, \mathcal{N}(pnI)\, \mathcal{N}_0 \quad , \tag{40a}$$

and

$$\langle 0^+ \| [c_p^{\dagger} c_{\bar{n}}]^I \| \alpha I^{\pi} \rangle = \hat{I}\, \Lambda_+(pnI,\alpha)\, \mathcal{N}(pnI)\, \mathcal{N}_0 \quad . \tag{40b}$$

Finally, in order to apply the QRPA formalism to double β–decay calculation we need to solve the BCS equations.

There are two procedures for the variation method, where

$$\frac{\delta \langle 0^+ | H | 0^+ \rangle}{\langle 0^+ | 0^+ \rangle} = 0 \quad , \tag{41}$$

is imposed. The PBCS approximation, where the variation (41) is performed before the number projection with the BCS vacuum in our formalism is expressed by

$$|0^+> = \prod_{p>0} \left(\bar{u}_p + v_p\, c_p^\dagger\, c_{\bar{p}}^\dagger\right) \prod_{n>0} \left(u_n + \bar{v}_n\, c_n^\dagger\, c_{\bar{n}}^\dagger\right) |0> \quad , \tag{42}$$

gives a condition on λ_p and λ_n, respectively

$$2\, u_p\, \bar{v}_p\, (e_p - \lambda_p - \tilde{\mu}_p) = (u_p^2 - \bar{v}_p^2)\, \tilde{\Delta}_p \quad , \tag{43a}$$

and

$$2\, \bar{u}_n\, v_n\, (e_n - \lambda_n - \tilde{\mu}_n) = (\bar{u}_n^2 - v_n^2)\, \tilde{\Delta}_n \quad . \tag{43b}$$

In the FBCS approximation the variation (41) is evaluated after the projection on the wave funtion $|0_N^+>$ with a fixed number N of particles.

III. CONCLUSION

In this work we developed a proton–neutron number projected version of the quasi–particle Random–Phase Approximation, based on the generating function technique.

The extensive formulas are given. The formalism is very well suited for applications on β and $\beta\beta$–decay calculations.

Our formalism differs from that evaluated by Civitarese et. al.[7] in the choice of the vacua (10a) and, consequentely, in the form of the excited states $|J_i^\pi\rangle$ in intermediate (N–1,Z+1) nucleus. This causes the absence of the overlap $\langle J_i^\pi | J_k^\pi \rangle$ in our expressions for the matrix–elements of $\beta\beta$–decay rates.

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