4

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THE SPECTRUM OF AN ANISOTROPIC RING-SHAPED OSCILLATOR VIA WIGNER-HEISENBERG SUPER-OSCILLATOR TECHNIQUE *

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The Wigner-Heisenberg super-oscillator technique extended here for two dimensions is effectively applied for an easier spectral resolution of the anisotropic ring-shaped oscillator potential $ar^2+b\,(r\sin\vartheta)^{-2}+s\,(r\cos\vartheta)^{-2}+d\,r^2\sin^2\vartheta+e\,r^2\cos^2\vartheta$. Connections of the present method with the $SO(2,1)\otimes SO(2,1)$ approach of earlier work and also with the two dimensional supersymmetric quantum mechanical algebra for this system are pointed out.

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1. Introduction

Spectral problems invoving ring-shaped oscillators have rceived recent attention [1,2,3] in this journal. Specifically Boschi Filho and Vaidya [1] have discussed the SO(2,1)⊗SO(2,1) dynamical algebra associated with the anisotropic ring-shaped oscillator potential

$$V(r) = a\, r^2 + b\, (r\sin\theta)^{-2} + s\, (r\cos\theta)^{-2} + d\, r^2 \sin^2\theta + e\, r^2\cos^2\theta \qquad (1)$$

(where a, b, s, d and e are positive constants) and have calculated the Green function, energy spectrum and wave functions. This potential has as particular cases the ring-shaped oscillator discussed by Quesne [4] and the double ring-shaped oscillator for which the spectral resolution was obtained by Carpio-Bernido and Bernido using the path integral technique [2] and the SO(2,1)⊗SO(2,1) algebraic approach [3]. In this letter we employ a much simpler algebraic technique owing to two of the present authors (Jayaraman and Rodrigues [5]) based on a super-realized general oscillator algebra of Wigner-Heisenberg (WH) [6-12] but extended here effectively for two dimensions (2D). Our method in fact is a super-linearization of the SO(2,1)⊗SO(2,1) approach of refs. [1-3] and has the advantage of providing an easier spectral resolution of the quantum mechanical problem of a particle confined to potential (1). The highlight of the present method is, as will be seen, the simple use of only creation and annihilation operators -the super Wigner ladder operators- at every stage for the complete spectral resolution. In the end we point out briefly how the super Wigner ladder operators constitute a super-linearization (in the sense of taking a square root) of the SO(2,1) operators of refs. [1-3]. We also make manifest the realization of the charge operators of the supersymmetric quantum mechanical (SUSYQM) algebra [13] associated with (1) in terms of the super Wigner ladder operators.

2. The Super Wigner-Heisenberg Algebra

Employing circular cylindrical coordinates ρ, ϕ and z, the time-independent Schrödinger equation associated with the potential (1) becomes

$$H\,\psi(z,\rho,\varphi) = E\,\psi(z,\rho,\varphi), \quad \psi(z,\rho,\varphi) = \chi(z)\,R(\rho)\exp(im\varphi), \tag{2}$$

$$H = H_1 + H_2,\ E = E_1 + E_2,\ H_1\chi(z)=E_1\chi(z),\ H_2R(\rho)=E_2R(\rho), \tag{3}$$

$$H_1 = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial z^2} + \frac{1}{2}M\omega_1^2 z^2 + \frac{\hbar^2}{2Mz^2}\left(\mu_1^2-\frac{1}{4}\right), \tag{4}$$

$$H_2 = -\frac{\hbar^2}{2M}\left(\frac{\partial}{\partial\rho} + \frac{1}{2\rho}\right)^2 + \frac{1}{2}M\omega_2^2\rho^2 + \frac{\hbar^2}{2M\rho^2}\left(\mu_2^2-\frac{1}{4}\right), \tag{5}$$

where the constants $\mu_i\,(i=1,2)$ are defined by $\mu_1=\frac{1}{2}\left(1+\frac{8Ms}{\hbar^2}\right)^{1/2}$ and $\mu_2=\frac{1}{2}\left(m^2+\frac{2Mb}{\hbar^2}\right)^{1/2}$ while $\frac{1}{2}M\omega_1^2=(e+a)$ and $\frac{1}{2}M\omega_2^2=(a+d)$.

Noting the complete symmetry of the two eigenvalue problems in (3-5) under the formal exchange of the variables $(x_1=z)\leftrightarrow(x_2=\rho)$, $\left(\frac{\partial}{\partial x_1}=\frac{\partial}{\partial z}\right)\leftrightarrow\left(\frac{\partial}{\partial x_2}=\frac{\partial}{\partial\rho}+\frac{1}{2\rho}\right)$, $\omega_1\leftrightarrow\omega_2$ and $\mu_1\leftrightarrow\mu_2$ with $\left(-i\frac{\partial}{\partial x_i}\right)^\dagger=\left(-i\frac{\partial}{\partial x_i}\right)$, $(i=1,2)$, we provide below a unified treatment of the cases $i=1$ and 2. Following the operator technique of Jayaraman and Rodrigues [5], we begin with the following super-realized mutually adjoint operators:

$$A_i^{\pm} \equiv A_i^{\pm}\left(\mu_i+\frac{1}{2}\right)=\frac{1}{\sqrt{2M\hbar\omega_i}}\,\Sigma_i^i\left\{\pm\hbar\frac{\partial}{\partial x_i} \mp \frac{\hbar}{2x_i}(1+2\mu_i)\Sigma_3^i - M\omega_i x_i\right\} = \left\{A_i^{\mp}\right\}^\dagger,\ (i=1,2), \tag{6}$$

$$\left[A_i^{\epsilon},\ A_j^{\epsilon'}\right]_- = 0, \quad (i\neq j;\ i,j=1,2;\ \epsilon,\epsilon'=+,\ -). \tag{7}$$

In (6,7), $\underline{\Sigma}^i=(\Sigma_1^i,\ \Sigma_2^i,\ \Sigma_3^i)$, $(i=1,2)$ constitute two sets of mutually commuting Pauli spin matrices that provide the fermionic coordinates for the cases $i=1$

and 2 respectively.

It is readily checked that A_i^+ and A_i^- are indeed the creation and annihilation operators for the spectra of the 1D Wigner Hamiltonians

$$\mathcal{H}_i(\mu_i+\tfrac{1}{2}) = \tfrac{1}{2}\hbar\omega_i\left[A_i^+ , A_i^-\right]_+ = \begin{pmatrix} \mathcal{H}_{i-}(\mu_i-\tfrac{1}{2}) & 0 \\ 0 & \mathcal{H}_{i+}(\mu_i-\tfrac{1}{2}) \end{pmatrix}, \tag{8}$$

$$\left[\mathcal{H}_i(\mu_i+\tfrac{1}{2}),\ A_i^\pm\right]_- = \pm\hbar\omega_i A_i^\pm, \quad (i=1,2). \tag{9}$$

In (8), the Hamiltonians of the bosonic (i.e fermionic number $N_i=\frac{1}{2}(1-\Sigma_3^i)\to 0$) and fermionic (i.e fermionic number $N_i=\frac{1}{2}(1-\Sigma_3^i)\to 1$) sectors are respectively given (for i=1 and 2) by the equalites:

$$\mathcal{H}_{1-}(\mu_1-\tfrac{1}{2}) = H_1,\ \mathcal{H}_{2-}(\mu_2-\tfrac{1}{2}) = H_2\ , \tag{10}$$

$$\mathcal{H}_{1+}(\mu_1-\tfrac{1}{2}) = \mathcal{H}_{1-}(\mu_1+\tfrac{1}{2}),\quad \mathcal{H}_{2+}(\mu_2-\tfrac{1}{2}) = \mathcal{H}_{2-}(\mu_2+\tfrac{1}{2}). \tag{11}$$

Thus we have embedded the Hamiltonians H_1 and H_2 of equations (4) and (5) respectively in the bosonic sectors of the 1D Wigner Hamiltonians $\mathcal{H}_i(\mu_i+\frac{1}{2})$, (i=1,2), of (8).

Further, the ladder operators $A_i^\pm$ satisfy a generalized super Wigner commutation relation in the form

$$\left[A_i^- , A_i^+\right]_- = \left\{1+(2\mu_i+1)\Sigma_3^i\right\}, \quad (i=1,2), \tag{12}$$

with

$$\left[\Sigma_3^i\right]^2=1,\quad \left[\Sigma_3^i,\ A_i^\pm\right]_+ = 0 \to \left[\Sigma_3^i,\ \mathcal{H}_i(\mu_i+\tfrac{1}{2})\right]_- = 0,\ (i=1,2). \tag{13}$$

The (anti-)commutation relations given by (8), (9), (12) and (13) constitute a super WH-algebra which is a general oscillator algebra.

3. Spectral Resolution

While the positive definite form (8) gets sharpened, by virtue of (12), to either of the following forms

$$\mathcal{H}_i(\mu_i+\tfrac{1}{2}) = \hbar\omega_i\left\{ A_i^+ A_i^- + \tfrac{1}{2}\left[1 + (2\mu_i+1)\Sigma_3^i\right]\right\} \tag{14a}$$

$$= \hbar\omega_i\left\{ A_i^- A_i^+ - \tfrac{1}{2}\left[1 + (2\mu_i+1)\Sigma_3^i\right]\right\}, \tag{14b}$$

the ladder relations (9) together with (14a) lead to the direct determination of the ground state energies $\epsilon_i^{(0)}$ and the corresponding Wigner ground state wave functions $\Psi_i^{(0)}$ by the simple application of the annihilation conditions $A_i^-\Psi_i^{(0)}=0$, $(i=1,2)$, thus obtaining explicitly:

$$\Psi_1^{(0)}(x_1) = \chi^{(0)}(z)\begin{bmatrix}1\\0\end{bmatrix} \propto z^{\mu_1+1/2}\exp\left\{-\frac{M\omega_1}{2\hbar}z^2\right\}\begin{bmatrix}1\\0\end{bmatrix}, \tag{15a}$$

$$\varepsilon_1^{(0)} = (1+\mu_1)\hbar\omega_1; \tag{15b}$$

$$\Psi_2^{(0)}(x_2) = R^{(0)}(\rho)\begin{bmatrix}1\\0\end{bmatrix} \propto \rho^{\nu}\exp\left\{-\frac{M\omega_2}{2\hbar}\rho^2\right\}\begin{bmatrix}1\\0\end{bmatrix}, \tag{16a}$$

$$\varepsilon_2^{(0)} = (1+\mu_2)\hbar\omega_2. \tag{16b}$$

Once $\varepsilon_i^{(0)}$ has been determined the complete energy spectrum

$$\varepsilon_i^{(n_i)} = \hbar\omega_i[(1+\mu_i)+n_i], \quad (i=1,2;\ n_i=0,1,2,\ldots), \tag{17}$$

of $\mathcal{H}_i(\mu_i+\frac{1}{2})$ trivially follows from the step-up operation provided by A_i^+, which also determines the whole set of eigenfunctions $\Psi_i^{(n_i)}$ starting from $\Psi_i^{(0)}$. Explicitly the following spinorial solutions are easily obtained for the

even Wigner quanta i.e $n_1=2m_1$ $(m_1=0,1,2,\ldots)$:

$$\Psi_1^{(2m_1)}(x_1)=\Psi_1^{(2m_1)}(z)=(A_1^+)^{2m_1}\Psi_1^{(0)}(z) \propto z^{\mu_1+1/2}\exp\left\{-\frac{M\omega_1}{2\hbar}z^2\right\}L_{m_1}^{(\mu_1)}\left(\frac{M\omega_1}{\hbar}z^2\right)\begin{bmatrix}1\\0\end{bmatrix}, \quad (18)$$

$$\Psi_2^{(2m_2)}(x_2)=\Psi_2^{(2m_2)}(\rho)=(A_2^+)^{2m_2}\Psi_2^{(0)}(\rho) \propto \rho^{\mu_2}\exp\left\{-\frac{M\omega_2}{2\hbar}\rho^2\right\}L_{m_2}^{(\mu_2)}\left(\frac{M\omega_2}{\hbar}\rho^2\right)\begin{bmatrix}1\\0\end{bmatrix}, \quad (19)$$

where $L_m^{(\alpha)}(\rho^2)$ are generalized Laguerre polynomials [14].

Note that (18) and (19) are also simultaneous eigenfunctions respectivelly of the fermion number operator $N_i=\frac{1}{2}(1-\Sigma_3^i)$, $(i=1,2)$, for fermion number zero that defines the bosonic sectors of the respective 1D Wigner Hamiltonians. Since the mutually commuting parts H_1 and H_2 of the anisotropic ring-shaped oscillator Hamiltonian H (vide equations (2-5) and (10)) have been embedded into the bosonic sectors of the 1D Wigner Hamiltonians $\mathcal{H}_i(\mu_i+1/2)$, equation (8), it follows directly that the complete spectra of H_i $(i=1,2)$ and $H=H_1+H_2$ get determined to be

$$E_1^{(m_1)}=\varepsilon_1^{(2m_1)}=\varepsilon_{1-}^{(m_1)} = \hbar\omega_1(1 + \mu_1+ 2m_1), \quad (m_1=0,1,2,\ldots), \quad (20a)$$

$$\psi_1^{(m_1)} = \chi^{(m_1)}(z) \propto z^{\mu_1+1/2}\exp\left\{-\frac{M\omega_1}{2\hbar}z^2\right\}L_{m_1}^{(\mu_1)}\left(\frac{M\omega_1}{\hbar}z^2\right), \quad (m_1=0,1,2,\ldots); \quad (20b)$$

$$E_2^{(m_2)}=\varepsilon_2^{(2m_2)}=\varepsilon_{2-}^{(m_2)} = \hbar\omega_2(1 + \mu_2+ 2m_2), \quad (m_2=0,1,2,\ldots), \quad (21a)$$

$$\psi_2^{(m_2)} = R^{(m_2)}(\rho) \propto \rho^{\mu_2}\exp\left\{-\frac{M\omega_2}{2\hbar}\rho^2\right\}L_{m_2}^{(\mu_2)}\left(\frac{M\omega_2}{\hbar}\rho^2\right), \quad (m_2=0,1,2,\ldots); \quad (21b)$$

$$E^{(m_1,m_2)} = E_1^{(m_1)} + E_2^{(m_2)} = \hbar\omega_1(1 + \mu_1 + 2m_1) + \hbar\omega_2(1 + \mu_2 + 2m_2), \tag{22a}$$

$$\psi^{(m_1,m_2)}(z,\rho,\phi)=\chi^{(m_1)}(z)R^{(m_2)}(\rho)\exp(im\varphi), \quad (m_{1,2}=0,1,2,\ldots;m=0,\pm1,\pm2,\ldots). \tag{22b}$$

The energy spectra (22a) and the energy eigenfunctions (22b) (upto a normalization) coincide with those obtained in ref. [1] using the SO(2,1)⊗SO(2,1) algebra approach for the Green function.

Actually, the WH-(super)algebra method of this letter amounts to a super-linearization of the SO(2,1) Lie algebra generators of references [1-3]. In fact starting from the (anti-)commutation relations (8) and (9) involving $\mathcal{H}_i(\mu_i+\frac{1}{2})$ and $A_i^{\pm}$, it follows directly that the operators $K_i^{\pm}=\frac{1}{2}\left[A_i^{\pm}\right]^2$ and $K_3^i=\mathcal{H}_i(\mu_i+\frac{1}{2})$ satisfy the SO(2,1)⊗SO(2,1) caracteristic commutation relations

$$\left[K_i^-,\ K_j^+\right]_- = 2K_3^i\delta_{ij},\quad \left[K_3^i,\ K_j^{\pm}\right]_- = \pm K_i^{\pm}\delta_{ij},\quad (i,j=1,2), \tag{23}$$

which get separated into distinct sets of Lie algebra relations on respective projections to the bosonic and fermionic sector Hamiltonians of $\mathcal{H}_i(\mu_i+\frac{1}{2})$, $(i=1,2)$.

We now briefly indicate the construction of SUSYQM algebra associated with (1) based on its super WH-algebra association detailed in this letter. The symmetrized form of the Hamiltnian $\mathcal{H}_i(\mu_i+\frac{1}{2})$, equation (8), in terms of the Wigner ladder operators $A_i^{\pm}$, the general super-quantum rule for $A_i^{\pm}$ given by (12) and the anti-commutativity of Σ_3^i with $A_i^{\pm}$ in (13) lead in fact to a straightforward construction of the following SUSY Hamiltonian H_{SUSY}^i, $(i=1,2)$, associated with the asymmetric ring-shaped potential (1):

$$\begin{aligned} H_{SUSY}^i &= \mathcal{H}_i(\mu_i+\tfrac{1}{2}) - \tfrac{1}{2}\hbar\omega_i\Sigma_3^i\left[A_i^-,\ A_i^+\right]_- = \tfrac{1}{2}\left[A_i^-,\ A_i^+\right]_+ - \tfrac{1}{2}\hbar\omega_i\Sigma_3^i\left[A_i^-,\ A_i^+\right]_- \\ &= \left[Q_i^-,\ Q_i^+\right]_+,\quad (i=1,2), \end{aligned} \tag{24a}$$

$$Q_i^- = \frac{1}{2}\hbar\omega_i(1-\Sigma_3^i)A_i^-, \qquad Q_i^+ = \frac{1}{2}\hbar\omega_i(1+\Sigma_3^i)A_i^+, \tag{24b}$$

$$(Q_i^-)^2 = (Q_i^+)^2 = 0, \quad \left[H_{SUSY}^i, Q_i^{\mp}\right]_- = 0, \quad (i=1,2), \tag{24c}$$

$$\left[Q_i^{\epsilon}, Q_j^{\epsilon'}\right]_- = 0 \quad (i\neq j;\ i,j=1,2;\epsilon,\epsilon'=\pm). \tag{24d}$$

4. Discussion

We have obtained an easier spectral resolution for the potential (1) via WH super-oscillator technique. A purely algebraic solution based on the 2D SUSY construction in (24) will also follow by the now well-known methods [15-17] which, however, entail in consideration of a hiearchy of SUSY Hamiltonians with shape-invariance property [15-17]. An advantage of the present approach is the fact that no further hierarchy of Hamiltonians need participate other than just the Wigner Hamiltonian and that the whole spectral resolution can be achieved, as shown here, by operations with super Wigner ladder operators only. Also comparing with the SO(2,1)⊗SO(2,1) approach of refs. [1-3], the super Wigner ladder operators of this work have been shown to provide a super-linearization of the appropriate SO(2,1) generators employed in these references and furnish as well a simplified treatment of the problem with just oscillator raising and lowering operations.

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