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CLIFFORD, DIRAC, [illegible] MAJORANA ALGEBRAS, AND THEIR [illegible]ENTATIONS*

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Abstract: We show that the [illegible]gebra is an algebra in five dimensions. It has [illegible]tionally been confused with the two [illegible] algebras in four dimensions, which we have [illegible]ified as the "Majorana algebra" and the Clifi[illegible]ra in Minkowski space-time. A careful discus[illegible]the subtle inter-relationship between th[illegible]e algebras is achieved by employing a basis of [illegible]tial forms. In addition, we provide f[illegible]st time a 4x4 complex matrix representatio[illegible]fford algebra in Minkowski spacetime, an[illegible] it to the matrix representations of the [illegible] Majorana algebras. A remark on Eddington'[illegible]rs" is included.

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705

1. Introduction.

In this paper we propose and prove the following point: what has traditionally been regarded as the "Dirac algebra" in fact refers to three entirely distinct algebras.

In addition to the Dirac algebra D [1], we have identified and named the other two algebras as (i) the "Majorana algebra" M [2,3,4], and (ii) the Clifford algebra [5,6,7] in Minkowski spacetime [3,4,8,9]. These three algebras have long been regarded as isomorphic. We demonstrate that they are in fact distinct; first by algebraic construction, and secondly by comparing the actual matrix representations.

The Dirac algebra D is the well-known algebra defined by all combinations of the complex 4x4 Dirac matrices [1]. The Majorana algebra M is that algebra defined by all combinations of the real 4x4 matrices given by Majorana [2]. We construct both algebras as Clifford algebras, [5,6,7] utilising the "vee" product of differential forms introduced previously in [3,4,8]. We show that D is isomorphic to the Clifford algebra $A^{2,3}$ in the five-dimensional deSitter space with metric $g(2,3) = (-1,-1,-1,+1,+1)$, while M is isomorphic to the Clifford algebra $A^{3,1}$ in a four-dimensional space with metric $g(3,1) = (+1,+1,+1,-1)$. This construction explicitly demonstrates that D and M are in fact entirely distinct algebras.

In addition, both the Dirac and Majorana algebras have been assumed to be isomorphic to the Clifford algebra in Minkowski spacetime $A^{1,3}$, since Dirac obtained his matrices after factoring the four-dimensional wave operator [1]. In fact, the Clifford algebra $A^{1,3}$ corresponding to spacetime with the Minkowski metric $g(1,3) = (-1,-1,-1,+1)$ is not well-known to physicists, and is entirely distinct from both the Dirac and Majorana algebras. In this paper, we provide for the first time a 4x4 complex matrix representation of this algebra, and directly contrast it with the matrix representations of the Dirac and Majorana algebras.

The presentation is as follows: We begin by reviewing our construction of Clifford algebras in terms of differential forms, duality, and the vee product [3,4,8]. We construct all the algebras in spaces of four dimensions, and identify these algebras with either M, or N_4, using the "vee group" structure of reference [4]. Then, algebras in spaces of five dimensions are constructed, among them the Dirac algebra D. Two new algebras appear, which are isomorphic to $N_4 \oplus N_4$ and $M \oplus M$, respectively. We show how the relationship between the algebras D, M and N_4 can be thought of as a complexification: $D = M \oplus iM = N_4 \oplus iN_4$, where i is identified with the vee product of forms, and possesses the algebraic properties of the complex unit $i = \sqrt{-1}$.

Following this, we construct the matrix representations of these algebras, We review the determination [6] of the representation space of each Clifford algebra. This fundamental work is not yet well-known to physicists, perhaps because the matrix representations were not explicitly given. We review and combine results on the construction of representation matrices [13,14,15], and give the explicit matrix representations of N_4, M and D. The matrix representation of N_4 appears here for the first time. It is intimately related to the matrix representations of the deSitter group [16,17].

The Clifford algebra $A^{1,3}$ in Minkowski spacetime has been the subject of two more extensive studies [8,9], where the applications to physics in spacetime were presented. In that work, a matrix representation was not introduced, since the novelty and utility of the formalism is the representation-free method of calculation. In reference [9] we show how anti-symmetric fields in spacetime can be multiplied and divided associatively, using the notation of traditional vector algebra. For related, but distinct studies of N_4, see [10,11].

There exists a related algebra constructed by Eddington [12] in terms of his "E-numbers". This algebra has traditionally been regarded as distinct from the Dirac algebra. We show that they are in fact isomorphic.

In summary we show that the Dirac, Majorana, and the Clifford algebras in Minkowski spacetime which traditionally have been regarded as isomorphic, are in fact distinct. The Eddington "E-numbers" traditionally regarded as distinct to the Dirac algebra, are shown to be isomorphic to it.

The ultimate aim of our study is to clarify precisely that algebraic structure which possesses an intrinsic relationship to physics in spacetime [8,9,18]. A necessary first step is the clearing up of the relationship between the various algebras traditionally used for the description of physics in spacetime. We believe that this is accomplished in the present communication.

2. Construction of Clifford Algebras as "Vee"-Groups.

In our analysis, we will employ the differential form realisation of Clifford algebras introduced in references [3,4,8]. This enables us to compare the structure of the three algebras of immediate interest without resorting to specific matrix representations, which will be introduced later. We review here the basic definitions of the "vee"-structure necessary in the analysis of this paper.

Consider the differential basis oneforms $\sigma^\alpha = dx^\alpha$, of any flat space of dimension n. There are 2^n basis forms of all ranks, which are constructed using the exterior product $\wedge$ [19].

$$\{1, \sigma^\alpha, \sigma^\alpha \wedge \sigma^\beta, \ldots, \omega^n\} \quad \alpha = 1, \ldots n; \alpha \neq \beta \tag{1}$$

We have labelled the volume element in n dimensions as ω^n.

$$\omega^n = \sigma^1 \wedge \sigma^2 \wedge \cdots \wedge \sigma^n . \tag{2}$$

Define a metric scalar form $g^{\alpha\beta}$, which is diagonal, and has entries either +1 or -1 [19].

$$g^{\alpha\beta} = (\sigma^\alpha, \sigma^\beta) \tag{3}$$

Define a multiplication v, called "vee", [3,4,8] between a basis r-form and a basis (s-r)-form as the sum of permutations of basis forms in (1) with successive contractions (3).

Definition:

$$(\sigma^{\lambda_1} \wedge \cdots \wedge \sigma^{\lambda_r}) \vee (\sigma^{\lambda_{r+1}} \wedge \cdots \wedge \sigma^{\lambda_s}) = \frac{1}{r!(s-r)!} \sum_{\Pi_r} \sum_{\Pi_{s-r}} \sum_{\Pi_s} \sum_{k=0}^{s} (-1)^{\Pi_r} (-1)^{\Pi_{s-r}} (-1)^{\Pi_s} \cdot g^{\lambda_1\lambda_2} \cdots g^{\lambda_{2k-1}\lambda_{2k}} \sigma^{\lambda_{2k+1}} \wedge \cdots \wedge \sigma^{\lambda_s} \tag{4}$$

Here, $(-1)^{\pi_r}$ is the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & r \\ \lambda_1 & \lambda_2 & \cdots & \lambda_r \end{pmatrix} \tag{5}$$

Definition (4) is in practice quite simple to use; see the examples in references [3,4,9]. In references [3,4] we have shown that the algebra of forms defined by (1) and (4) is in fact isomorphic to the real Clifford algebra corresponding to the space defined by the metric (3).

By direct calculation, one can tabulate the vee product between all the distinct basis forms in (1). Each basis form in (1) possesses a unique inverse which is the form itself, up to a sign; hence the set of forms (1) form a finite group under "vee". This group is called the "vee"-group [3,4], and is of order 2^{n+1} (since we must count the forms in (1) along with their negatives as separate elements). We can identify each "vee" group of forms (1) in the space (3) with a known finite group. This has been done for all distinct spaces (3) in [3,4], and serves to provide a classification of all Clifford algebras in terms of the corresponding "vee"-groups. All the interesting properties of the Clifford algebras follow essentially from this group structure.

A key result of our work is the following "duality theorem", which reduces the notion of duality from the traditional index operation [19] to a purely algebraic manipulation. Duality in our framework is achieved by multiplication by an element in the algebra. This is expressed in the "duality theorem" [3,4]

Theorem 1: The dual of any form f is the vee product of that form with the volume element.

$$\left(\underset{r}{\times} f\right) = \pm\, \omega^r \vee f \tag{6}$$

The sign depends on r, on the signature and dimension n of the base space, and on the rank of f. (Note that $r \leq n$). The r-dimensional dual is accomplished by vee multiplying a form with the r-volume element.

What is novel in this formalism is the fact that we can take duals of the same form in different dimensions, interchange them, combine them, and separate them into their corresponding subspaces, just by simple algebraic manipulation. This utility is of particular importance in separating the space dual in three dimensions from the spacetime dual when working in four-dimensional spacetime. For examples on the application of the duality theorem, see [3,4,9].

As an application of the duality, we can label all the higher rank forms in terms of the duals of corresponding lower rank forms. For instance, $\sigma^1 \wedge \sigma^2 \wedge \sigma^4$ is just the four-dual of the form σ^3, which can be written via theorem (6) as $\sigma^1 \wedge \sigma^2 \wedge \sigma^4 = \omega^4 \vee \sigma^3$.

The algebraic manipulation of duals using the duality theorem (equation 6) depends on the algebraic properties of the volume elements. These were originally given by Clifford [5]. We recall them here as theorems 2 and 3.

Theorem 2: The volume element commutes with all the elements of the vee-group in that space if the dimension of the space n is odd; and anticommutes when n is even. If σ is any member of (1), then

$$\omega^n \vee \sigma = (-1)^{n+1}\, \sigma \vee \omega^n \tag{7}$$

Theorem 3: The square of a volume element in the vee multiplication is given by:

$$(\omega^n)^2 = \omega^n \vee \omega^n = (-1)^{n(n-1)/2} \det g = \pm 1 . \tag{8}$$

Here, g is the metric of the base space (3). The proofs are direct.

The implications of theorem 2 are far reaching in terms of representation theory, since that theorem can be utilised to give the following structural result:

Theorem 4: Clifford algebras in n dimensions are generated by n mutually anticommuting elements when n is odd; and by n+1 mutually anticommuting elements when n is even.

$$\text{generators: } \{\sigma^1, \ldots, \sigma^n\}\ n\text{=odd};\ \{\sigma^1, \ldots, \sigma^n, \omega^n\}\ n\text{=even}. \tag{9}$$

Theorem 4 [illegible] the fact that in even-dimensional [illegible] volume element must be counted among the generators. [illegible] representation theory the geometrical information is lost, the volume element is indistinguishable from the one-form generators after they are represented as matrices. This point has implications which are discussed at length in the subsequent sections.

3. Clifford Algebras in Four Dimensions.

We proceed to construct the Clifford algebras in four dimensions using the differential form realisation of the preceeding section.

In four dimensions, the one-form basis is $\{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ and from these, we construct all higher rank forms via the exterior product. There are $2^4 = 16$ such forms of ranks 0,1,2, 3, and 4 respectively [8,9,19]. It is possible to display all these forms in the following manner, which illustrates the three-dimensional subalgebra explicitly.

$$\{1, \sigma^i, \sigma^4, \sigma^i \wedge \sigma^j, \sigma^i \wedge \sigma^4, \omega^3, \sigma^i \wedge \sigma^j \wedge \sigma^4, \omega^4\}$$

$$i = 1,2,3; \quad i \neq j. \tag{10}$$

Here there are two volume elements distinctly labelled as such; that in three space ω^3, and that in four dimensions ω^4.

By making use of the duality theorem 1, which expresses higher rank forms as the vee product of low rank forms and the two volume elements ω^3 and ω^4, we can write the [illegible] (10) more compactly in terms of the vee product as:

$$\{1, \sigma^i, \sigma^4, \omega^3 \vee \sigma^i, \sigma^i \vee \sigma^4, \omega^3, \omega^4 \vee \sigma^i, \omega^4\} \tag{11}$$

Each element of this set (11) is directly equivalent to the corresponding element in the above set (10).

The underlying vee group structure can be determined only when we choose a specific metric (3). In four dimensions, there are five distinct metrics, designated by their signature as g(p,q). These are explicitly g(4,0), g(3,1), g(2,2), g(1,3) and g(0,4). In general,

$$g(p,q) = (\pm 1, \pm 1, \pm 1, \pm 1) \qquad (12)$$

where there are p positive and q negative entries, and $p+q=4$. The order of the signs is immaterial.

There is no a priori reason that spaces of the same dimension but with different metric signature $g(p,q)$ should have isomorphic Clifford algebras. It turns out that there are in fact isomorphisms among the Clifford algebras in a base space of the same dimension, but distinct signature. These isomorphisms have been determined in [3,4,6].

The result is that there are only <u>two</u> nonisomorphic algebras in a base space of four dimensions, and these are the Majorana algebra M, and the Clifford algebra in Minkowski spacetime N_4 [3,4]. Their correspondence to the four-dimensional spaces of distinct signature is as follows. Denoting as $A^{p,q}$ the Clifford algebra in a space with metric of signature $g(p,q)$, the algebras in four dimensions are:

$$A^{4,0} = N_4,\ A^{3,1} = M,\ A^{2,2} = M,\ A^{1,3} = N_4,\ A^{0,4} = N_4 \qquad (13)$$

This illustrates the isomorphisms between Clifford algebras in four dimensional spaces with distinct signature. For details on the determination of these isomorphisms, see [3,4].

4. <u>Clifford Algebras in Five Dimensions</u>: <u>The Dirac Algebra</u>.

In five dimensions, we follow essentially the same construction of Clifford algebras employed in the case of four dimensions, in the previous section. The number of basis elements is now $2^5=32$, which is double that in four dimensions. The basis one-forms are $\{\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$. Following our general construction, the complete set of basis forms can be explicitly written as follows:

$$\{1, \sigma^\mu, \sigma^5, \sigma^\mu\wedge\sigma^\nu, \sigma^\mu\wedge\sigma^5, \sigma^\mu\wedge\sigma^\nu\wedge\sigma^\lambda, \sigma^\mu\wedge\sigma^\nu\wedge\sigma^5, \omega^4, \sigma^\mu\wedge\sigma^\nu\wedge\sigma^\lambda\wedge\sigma^5, \omega^5\}$$

$$\mu,\nu,\lambda = 1,2,3,4;\ \mu\neq\nu\neq\lambda \qquad (14)$$

This is a rather cumbersome set to deal with. However, just as in the preceeding case, the duality theorem simplifies the labelling considerably, since we can express the five-duals of forms as the vee product with the five-dimensional volume element. We obtain the following equivalent set to (14).

$$\{1, \sigma^\mu, \sigma^5, \sigma^\mu\wedge\sigma^\nu, \sigma^\mu\wedge\sigma^5, \omega^4\vee\sigma^\mu, \sigma^\mu\wedge\sigma^\nu\wedge\sigma^5, \omega^4, \omega^5\vee\sigma^\mu, \omega^5\} \qquad (15)$$

The manipulations with the five-volume element ω^5 are almost trivial, since from theorem (2), ω^5 commutes with all the basis forms in (15). Theorem (3) gives the square of ω^5 in the vee multiplication as either +1, or -1, depending on the particular signature of the base space. In the case of the Dirac algebra $A^{2,3}$ it is equal to -1, therefore the volume element ω^5 can be manipulated just like the complex unit $i = \sqrt{-1}$. This is a very important point, whose consequences are examined further below.

It appears that there are <u>three</u> distinct algebras in five dimensions, one of which is isomorphic to the familiar Dirac algebra D [3,4]. The other two are isomorphic to a double copy of the Majorana algebra M, and a double copy of the Clifford algebra in Minkowski spacetime N_4 [3,4]. We can identify these algebras with each particular metric as follows. In the five dimensional spaces with metric g(p,q), the corresponding Clifford algebras $A^{p,q}$ are:

$$A^{5,0}=N_4 \oplus N_4,\ A^{4,1}=D,\ A^{3,2}=M \oplus M,\ A^{2,3}=D,\ A^{1,4}=N_4 \oplus N_4,\ A^{0,5}=D. \tag{16}$$

Thus, the Dirac algebra is shown to be isomorphic to the Clifford algebras in the spaces with metric g(4,1), g(2,3), and g(0,5).

One can make the observation that the Dirac algebra is identified with precisely those spaces expected from theorem 3. The identification $\omega^5 \leftrightarrow i$ is valid only when $(\omega^5)^2=-1$, and by theorem 3, this is only true in the spaces (4,1), (2,3) and (0,5). In subsequent discussion, D will be geometrically identified with the algebra $A^{2,3}$. Even though the other two algebras $A^{4,1}$ and $A^{0,5}$ are isomorphic to $A^{2,3}$, the multiplication rules for the <u>individual</u> forms (15) are distinct in each case.

We now give the relationship of the Dirac algebra to the Clifford algebras in four dimensions. That is achieved via an identity easily verified with the vee-product rules in $A^{2,3}$.

$$\omega^4 \vee \omega^5 = -\sigma^5 \tag{17}$$

Using this identity, we can rewrite the elements (15) of

the Dirac algebra as the collection of two similar sets, as follows:

$$\{1, \sigma^\mu, \sigma^\mu \wedge \sigma^\nu, \omega^4 \vee \sigma^\mu, \omega^4\} + \omega^5 \vee \{1, \sigma^\mu, \sigma^\mu \wedge \sigma^\nu, \omega^4 \vee \sigma^\mu, \omega^4\}$$

$$\mu,\nu = 1,2,3,4\ ;\ \mu \neq \nu. \tag{18}$$

The elements in each set of brackets are precisely the elements (11) of the algebra in four dimensions. We have remarked that the properties of the volume element ω^5 are analogous to the complex unit, hence this rewriting appears as nothing more than a complexification. Since the algebras in four dimensions are either N_4 or M, the Dirac algebra can therefore be thought of as a "complexification" of either the algebra N_4 or the algebra M.

This analysis has determined the relationship between the three algebras to be the following:

$$D = M \oplus \omega^5 \vee M = N_4 \oplus \omega^5 \vee N_4 \tag{19}$$

which can be written as a complexification:

$$D = M \oplus iM = N_4 \oplus iN_4 \tag{20}$$

Note that, while $M \neq N_4$, the "complexification" of each algebra produces new elements in such a way that the sets $M \oplus iM$ and $N_4 \oplus iN_4$ are equivalent.

5. The Construction of Matrix Representations.

This section collects and extends certain general results from representation theory needed to construct the matrix representations of the algebras D, M and N_4.

We begin by identifying the smallest Clifford algebras with known associative algebras. By direct construction using the vee-groups [3,4], it is easy to verify that the first few Clifford algebras are isomorphic to the following associative algebras, (see also [6,7]).

$$A^{0,0}=R,\ A^{0,1}=C,\ A^{0,2}=H \tag{21}$$

These three algebras R (Reals), C (Complex) and H Quaternions) are all associative, normed, division algebras. These properties identify these algebras as fields.

By using the vee-group structure, it is possible to verify the following identities relating different Clifford algebras. These were explicitly given in reference [4].

$$A^{0,n} \otimes A^{2,0} = A^{n+2,0}$$
$$A^{n,0} \otimes A^{0,2} = A^{0,n+2} \tag{22}$$

Over any field K, we can define a matrix algebra, denoting as K(n) the nxn matrices with values from the field K. This matrix algebra can be used to construct matrix representations of some associative algebra. Since all Clifford algebras are associative, they possess matrix representations.

A result of considerable importance is the determination of the representation space of each Clifford algebra; i.e. what matrix field K(n) (or combinations thereof) corresponds

to the lowest irreducible representation of a given Clifford algebra. This has been done for the Clifford algebras $A^{0,n}$ and $A^{n,0}$ in the classic paper of Atiyah, Bott and Shapiro [6]. Their result indicates that one can construct all Clifford algebras as matrix algebras over the fields R, C, and H. A method of obtaining the representation matrices was not however given.

By successive application of identities (22), and the elementary properties of matrix algebras, one can construct all Clifford algebras as matrix algebras over R, C, and H [6]. The results are written as follows:

$$A^{1,0}=R \oplus R,\ A^{2,0}=R(2),\ A^{3,0}=C(2),\ A^{4,0}=H(2),\ A^{5,0}=H(2) \oplus H(2)$$
$$A^{0,1}=C,\ A^{0,2}=H,\ A^{0,3}=H \oplus H,\ A^{0,4}=H(2),\ A^{0,5}=C(4) \tag{23}$$

It is possible to use similar methods to determine the special noncompact case

$$A^{3,1} = R(4). \tag{24}$$

This completes the discussion of the results of Atiyah, Bott, and Shapiro [6]. We stop at five dimensions since that is sufficient for this paper, but it should be pointed out that this process can be extended indefinitely to derive the periodicity of eight dimensions (see [4,6]).

By comparing the above results to the classification of Clifford algebras in references [3,4], we obtain an identification between the Clifford algebras constructed as vee-groups, and the present construction as matrix algebras over R, C, and H. For the algebras of interest, compare (13) and (16) with (23) and (24) to obtain the result:

$$M=R(4),\ N_4=H(2),\ D=C(4) \tag{25}$$

The important point is that we have thus determined the representation space of the Clifford algebras of interest. From (25) we see that the Majorana algebra can be represented by 4x4 real matrices; the Clifford algebra in Minkowski space-time N_4 by 2x2 Quaternion-valued matrices, and the Dirac algebra by 4x4 Complex matrices. The first and last cases are confirmed by the representations that are well known in Physics.

Matrix representations of the Clifford algebras $A^{n,0}$ in even-dimensional spaces were first discussed by Jordan and Wigner [14]. Pauli gave his celebrated theorems for the special case of the Dirac matrices in [15]. He utilized earlier work of Schur [13].

We present here analogues of those Pauli theorems that are generalised to any Clifford algebra. The fundamental theorem is:

Theorem 6: Any two sets of anticommuting matrices representing a Clifford algebra in the <u>same</u> space (and signature) are related by a similarity transformation.

This statement is equivalent to the Universality theorem of Clifford algebras [6].

<u>Theorem 7</u>: All the representation matrices have zero trace, except the unit.

The proof is given in Pauli [15]. See also Good [20].

<u>Theorem 8</u>: (a) In spaces of even dimension, any matrix that commutes with all the representation matrices must be the unit matrix (up to a constant).

(b) In spaces of odd dimension, there are two inequivalent matrices that commute with all the other matrices;

these are the unit, and the matrix which represents the volume element.

This theorem is known under a variety of guises. In reference [4], we determined the center of each vee group defining a Clifford algebra, using theorems 2 and 3. The matrix representation of the center of each Clifford algebra gives a proof of theorem 8. It is interesting to note that when a Lie algebra is constructed from a Clifford algebra, theorem 8(a) survives in the celebrated Lemma of Schur. For the connection between Clifford and Lie algebras, see [3,4].

A distinct means of discussing the matrix representations is to use Kroenecker products of spinors. This method is due to Cartan [21,22] and Brauer and Weyl [23], who did not however give a method for explicitly writing down the representations. An alternate method was provided by Freudenthal [24]. For a discussion of these methods, see [25]. The connection between these methods, and the methods discussed herein will be the subject of a separate communication.

6. Matrix Representations of the Majorana Algebra.

The Majorana algebra M has been shown in section 3 to be isomorphic to the Clifford algebra $A^{3,1}$ (13). Therefore, the basis oneforms generating M are $\sigma^1, \ldots, \sigma^4$, with squares equal to (+1,+1,+1,-1). However, by theorem 4 , the total number of mutually anticommuting generators is five, since we must also include the volume element ω^4 (9). Hence, the generators of M are:

$$\{\sigma^1, \sigma^2, \sigma^3, \sigma^4, \omega^4\} \tag{26}$$

From theorem 3 we see that $(\omega^4)^2 = -1$ in this case, hence the generators (26) have corresponding squares,

$$\{+1, +1, +1, -1, -1\} \tag{27}$$

The structure of the Majorana algebra is known from its underlying vee-group. In particular, there exists the volume element ω^3, which by theorem 3 has square equal to -1. Furthermore, from theorem 2 , ω^3 commutes with $\sigma^1, \sigma^2, \sigma^3$. A [illegible] application of the vee product (4) shows that ω^3 anti- [illegible] σ^4 and ω^4. This determines that part of the [illegible] structure that needs to be reproduced by the representation.

The matrix representations of the Majorana algebra can be constructed by combining the results of preceeding sections. From (25), we know that they must be 4x4 real matrices. Furthermore, there must be five mutually anticommuting matrices, with squares equal to (27), and in addition, a matrix which commutes with those matrices corresponding to $\sigma^1, \sigma^2, \sigma^3$ and anticommutes with those corresponding to σ^4 and ω^4 . Using

theorems 7 and 8 , it is not difficult to find sets of matrices which satisfy the above requirements. Any two such sets are by theorem 6 related by a similarity transformation.

In displaying the matrix representations, use will be made of the familiar Pauli matrices, which are denoted by:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{28}$$

Use will also be made of the real symplectic matrix denoted by $\mathbb{J} = i\tau_2$, which has square equal to -1. The unit matrix in two dimensions will be denoted as $\mathbb{1}$.

Two equivalent matrix representations of the Majorana algebra are displayed in table I. Representation (a) is the one originally given by Majorana in [2]. Representation (b) appears here for the first time, and is included for comparison purposes. We have specifically indicated the correspondence between the generators (26) and the representation matrices. The matrix corresponding to the element ω^3 has been identified for completeness.

The representation space R(4) of the Majorana algebra (25) is precisely the real part of (and exactly one-half as large as) C(4), the representation space of the Dirac algebra. This provides an independent corroboration of the geometric relationship between the Dirac and Majorana algebras $D = M \oplus iM$ (20), derived in section 4. Note, however, that one cannot simply split the Dirac matrices into real and complex parts to obtain representations of the Majorana algebra.

TABLE I. Matrix Representations of the Majorana Algebra.

Generators	Representation (a)	Representation (b)
σ^1	$\begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
σ^2	$\begin{pmatrix} 0 & -\mathbb{J} \\ \mathbb{J} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\mathbb{J} \\ \mathbb{J} & 0 \end{pmatrix}$
σ^3	$\begin{pmatrix} -\tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$
σ^4	$\begin{pmatrix} 0 & \mathbb{J} \\ \mathbb{J} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \tau_3 \\ -\tau_3 & 0 \end{pmatrix}$
ω^3	$\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbb{J} & 0 \\ 0 & \mathbb{J} \end{pmatrix}$
ω^4	$\begin{pmatrix} \mathbb{J} & 0 \\ 0 & -\mathbb{J} \end{pmatrix}$	$\begin{pmatrix} 0 & \tau_1 \\ -\tau_1 & 0 \end{pmatrix}$

. Matrix Representations of the Dirac Algebra.

The matrix representations of the Dirac algebra D are too well known to need comment. However, it is only by constructing these representations using the formalism of this paper that we can display the geometric structure of the Dirac algebra. And in this manner we can clear up the relationship between the three algebras D, M and N_4.

In section 4 we derived the result that the Dirac algebra D is isomorphic to the Clifford algebra $A^{2,3}$. This algebra is generated by the set of five anticommuting elements

$$\{\sigma^1, \sigma^2, \sigma^3, \sigma^4, \sigma^5\} \tag{29}$$

which have squares in the vee multiplication respectively equal to:

$$\{-1, -1, -1, +1, +1\} \tag{30}$$

The volume element ω^5 in this case commutes with all the elements of the algebra. The algebraic properties of the volume elements in D have been discussed in detail in section 4.

From (25), we know that the matrix representations of the Dirac algebra are 4x4 complex matrices (25). Two such equivalent sets are well known, and are listed in Table III. The generators σ^ρ, $\rho = 1,\ldots,5$ correspond directly to the usual γ^ρ. Representation (a) is the standard one in use [26], while representation (b) is due to Kramers [27] and is known as the "spinor" representation [28].

In particular, we have identified the matrices representing the volume elements ω^3, ω^4, and ω^5, thereby clarifying the geometrical structure. We have identified γ^5

as the generator associated with the fifth coordinate x^5, and not as the volume element W^4 or W^5 (pseudoscalar), with which it has traditionally been associated. This is a very important distinction which has consequences discussed in the following section.

Another matrix representation of the Dirac algebra was given by Sir A. S. Eddington, which he called the "E-numbers" [12]. Eddington apparently did not realise that his "E-numbers" were isomorphic to the Dirac algebra. The Eddington matrices were given as a representation of the Clifford algebra $A^{0,5}$. We know that $A^{0,5}$ is isomorphic to $A^{2,3}$, the usual Dirac algebra (16); hence, the "E" matrices can be utilised to give a representation of the Dirac algebra D.

It is instructive to show precisely how the Dirac matrices, originally constructed to describe an algebra in four dimensions, actually represent an algebra in <u>five</u> dimensions. The original matrices of Dirac [1] are known as the Dirac matrix representation [29], and all have squares equal to +1. Those matrices provide a representation of the algebra $A^{4,0}$. From (13) we see that this algebra is in fact isomorphic to N_4; however we <u>cannot</u> utilise $A^{4,0}$ to describe vector fields in Minkowski space, since it does not have the correct metric. This follows because the group isomorphies (13) do <u>not</u> preserve the geometric properties of each individual algebra.

The traditional method of describing vector fields in Minkowski spacetime is to complexify the algebra $A^{4,0}$ of the original Dirac matrices in order to accomodate the Minkowski metric. This is done in two different ways.

TABLE II. <u>The Matrix Representations of the Dirac Algebra</u>.

Generators	Representation (a)	Representation (b)
σ^k, $k=1,2,3$	$\begin{pmatrix} 0 & \tau_k \\ -\tau_k & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\tau_k \\ \tau_k & 0 \end{pmatrix}$
σ^4	$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
σ^5	$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$.
W^3	$\begin{pmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i\mathbb{1} \\ -i\mathbb{1} & 0 \end{pmatrix}$
W^4	$\begin{pmatrix} 0 & i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}$
W^5	$\begin{pmatrix} i\mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{pmatrix}$	$\begin{pmatrix} i\mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{pmatrix}$

First, one can introduce a pure imaginary time component for each vector, to realise the (complexified) algebra $A^{3,1} \oplus iA^{3,1}$. The resulting algebra is just the Dirac algebra, since $A^{3,1}=M$ (13), and $M \oplus iM=D$ (20). This method corresponds to the "Dirac-Pauli-ict method" [29].

A second method is to use imaginary space components, which is equivalent to inserting a factor of i in the space bases. This procedure results in the complexified algebra $A^{1,3} \oplus iA^{1,3}$, which is also equal to the Dirac algebra. In this case, $A^{1,3}=N_4$ (13), and $N_4 \oplus iN_4=D$ (20). This is precisely how the "standard representation" [26] is related to the original Dirac matrices [1,29].

In the following section, we discuss how it is possible to describe vector fields in Minkowski spacetime <u>without</u> complexifying the four-dimensional algebra into the five-dimensional Dirac algebra.

8. Matrix Representations of the Clifford Algebra in Minkowski Spacetime.

In this section, we isolate for the first time the 4x4 complex matrix representations of this algebra, which has been traditionally confused with the Dirac algebra.

The Clifford algebra in Minkowski spacetime $A^{1,3}=N_4$ is constructed in a similar manner as the Majorana algebra (section 3). N_4 is generated by the five anticommuting elements (26), with the important difference that the squares of the corresponding elements are equal to

$$\{-1,-1,-1,+1,-1\} \tag{31}$$

The volume element ω^3 retains the same algebraic properties as in the Majorana algebra, except that in this case, its square is equal to +1.

In section 5, we determined that the matrix representations of the algebra N_4 inhabit the space of 2x2 quaternion matrices (25). We have constructed two (equivalent) representations of N_4 in terms of quaternions. Recall the definition of the quaternion units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, satisfying the cyclic identity

$$\mathbf{ij} = -\delta_{ij} + \varepsilon_{ijk}\mathbf{k} \tag{32}$$

It is well known that we can represent the quaternions as 2x2 complex matrices. A complex representation of the quaternions can be given in terms of the Pauli matrices τ_k, k=1,2,3 as:

$$\mathbf{i} = i\tau_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \mathbf{j} = \mathbf{J} = i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} = -i\tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{33}$$

Hence, our quaternion representations may be presented in a more practical manner as 4x4 complex matrices. These are denoted in Table III as representations (a) and (b).

An examination of either representation (a) or (b) (Table III) will confirm the fact that they do not span the representation space C(4), 4x4 complex matrices. This follows because H(2), the representation space of N_4, is only one-half as large as C(4). Quite independently, we have shown using geometrical methods that the algebra N_4 is one-half the size of the Dirac algebra D (20). Since D spans C(4) (25), we thus have an independent justification of relationship (20) in representation space.

We note that representation (b) (Table III) can be related to a representation of the Clifford algebra $A^{0,4}$ given in [16,17], via the isomorphism $A^{0,4}=A^{1,3}=N_4$ (13).

An observation worth making is the fact that the representations of N_4 in Table III can be obtained from the representations of the algebra $A^{4,0}=N_4$ in terms of the well-known "α and β matrices", in a way such that one does <u>not</u> complexify N_4 into D. (Compare the discussion in the preceeding section.) In this context, representation (a) can be derived from the Kramers representation [27,28], and representation (b) from the Dirac-Pauli representation [1,29].

In this paper, a total of four representations of the algebra $A^{1,3}=N_4$ will be given (and these are by no means all the possible representations). By Universality, all the representations are related by similarity transformations, and are therefore equivalent.

TABLE III. <u>Matrix Representations of the Clifford Algebra in Minkowski Spacetime</u>.

Generators	Representation (a)	Representation (b)
σ^k $k=1,2,3$	$\begin{pmatrix} i\tau_k & 0 \\ 0 & -i\tau_k \end{pmatrix}$	$\begin{pmatrix} 0 & i\tau_k \\ i\tau_k & 0 \end{pmatrix}$
σ^4	$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$
ω^3	$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
ω^4	$\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

Two more representations of N_4 can be obtained from the representations of the Dirac algebra given in Table II, as follows. Since the differential form realisation of $A^{2,3}=D$ preserves the geometric structure of the four-dimensional subalgebra $A^{1,3}=N_4$, then the set $\{\sigma^1,\sigma^2,\sigma^3,\sigma^4,\omega^4\}$ in either representation (a) or (b) of Table II provides a representation of N_4. This identification has been utilised in reference [8] to obtain a complete set of trace identities from the algebraic structure, which are useful in perturbation theory.

Comparing Tables II and III, we can make the following simple, yet vital observation: When one treats the i and the γ^5 in the expression for the volume element $\omega^4=i\gamma^5$ (Table II) as separate elements, then one necessarily obtains the complexification of N_4 (which is the Dirac algebra D). By using the representations of Table III, one cannot effect such a separation.

This argument underlines the fundamental point of our [illegible] that is the fact that we have isolated and identified an i as a distinct element of the algebra. This result follows exclusively from the use of the differential form basis, which singles out the volume element ω^5 as the i in the Dirac algebra. In the traditional use of complex matrix representations, one cannot distinguish between an i which is an element of the algebra, and any other i appearing in the representation space. Hence, the analysis of this paper is simply not possible with traditional methods.

We hope that this discussion has cleared up the intricate relationship between the different matrix representations of the Dirac algebra, and the Clifford algebra in Minkowski spacetime.

9. Conclusion.

One of the historical problems of theoretical physics was in finding the appropriate algebraic structure corresponding to spacetime with the Minkowski metric. For a long time, it had been thought that the complete solution was provided by Dirac, but this is not so. The algebra sought for is the Clifford algebra in Minkowski spacetime, which is not well known in physics. This algebra has lacked up until now an explicit matrix representation that distinguishes it from the Dirac algebra.

We have identified the confusion between these algebras as arising from the fact that the original Dirac matrices defined a compact algebra (their squares were all +1), and could not directly describe physics in Minkowski spacetime. Two solutions to the problem of describing physics in the noncompact Minkowski space were (a) the use of the "ict" metric, and (b) the construction of the "standard" representation of the Dirac matrices, (which differ by factors of i from the original ones). Both solutions doubled the original algebra from four dimensions to five dimensions. The Clifford algebra in Minkowski spacetime, which was in fact the algebra desired, was thus overlooked. Majorana noticed this point, but himself found not the Clifford algebra in Minkowski spacetime, but the other distinct algebra in four dimensions, which we have named in tribute to him.

In this paper, we have constructed the Dirac algebra D, the Majorana algebra M, and the Clifford algebra in Minkowski spacetime N_4, as separate and distinct algebras. The

relationship between these three algebras is a complexification, $D = M \oplus iM$, and $D = N_4 \oplus iN_4$, but $M \neq N_4$.

The results obtained have direct relevance to the description of physics in spacetime, since we are now able to geometrically identify the physical fields of a given theory with a particular space and a particular metric. This "geometrical" description of fields has been utilised in [8,9,18] to obtain nontrivial physical results.

In addition to the geometrical analysis, we have provided two distinct practical methods of calculating with these three algebras. First, the differential form realisation using duality and the divisible "vee" product provides a representation-free means of calculation. It is this formalism that has been used in all our previous work. Second, we have given the explicit 4x4 matrix representations of the three algebras D, M, and N_4. The representation of the algebra N_4 appears here for the first time. The relation between the algebras can be seen directly in the representation space.

We have also shown that the "E-numbers" of Eddington are just another representation of the Dirac algebra.

In conclusion, we have shown that three algebras traditionally regarded as isomorphic are in fact distinct; and a fourth algebra that was thought to be distinct is in fact isomorphic. We feel that the analysis of this paper has cleared up a very old confusion, which has immediate consequences to the description of physics in spacetime.

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