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CANONICAL STRUCTURE IN $SO(4)_q$ AND RELATIONS TO $E(3)_q$ and $SO(3,1)_q$

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Abstract :

Starting with the $SU(2)_q \otimes SU(2)_q$ basis and using q-C.G. coefficients the closest possible q - deformation of the classical canonical Gelfand-Zetlin formalism is constructed explicitly for $SO(4)_q$. Simple introduction of q-brackets for each factor of the classical matrix elements, valid for $SU(n)_q$, is shown to need significant and unexpected modifications for $SO(n)_q$ even when n = 4. Contraction to $E(3)_q$ and continuation to $SO(3,1)_q$ are studied, making explicit the possibilities and the problems that arise. Periodic representations for q a root of unity are commented upon briefly.

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1. Introduction

For q=1, irreducible representations of the chain of algebras SO(n) ⊃ SO (n-1) ⊃ SO(n-2) ⊃.. have, long ago, been explicitly constructed [1] in the canonical basis. With these as starting points, explicit representations were constructed [2] for Euclidean and generalized Lorentz groups, using specific contractions and deformations,

$$SO(n+1) \xrightarrow{\text{contraction}} E(n) \text{ (or ISO (n))} \xrightarrow{\text{deformation}} SO(n,1)$$

What remains of such possibilities for $q \neq 1$? Here, we begin the exploration by studying, for the simplest non-trivial case $SO(4)_q$, the status of the canonical generators. Already, at this livel, special features, with possibilities and problems, are encountered. These are recapitulated in the concluding remarks (Sec.7) after the necessary derivations.

For unitary algebras, $U(n+1)_q$, $IU(n)_q$, $U(n,1)_q$, all were studied systematically in the canonical basis [3,4,5]. For other algebras only periodic representations were constructed [6] using special techniques for q a root of unity.

For unitary algebras one can start (for q generic or root of unity) by replacing each factor in the numerators and the denominators of the classical Gelfand-Zetlin matrix elements by the corresponding q-bracket ($(x) \rightarrow [x] \equiv (q^x - q^{-x}) / (q - q^{-1})$). This "minimal q-deformation" works because of the following crucial features. The Chevalley generators, in terms of which the Hopf algebra is formulated when $q \neq 1$, form directly, at the classical level, a subset of the canonical generators statisfying, for U(n),

$$[A^{i_1}_{j_1}, A^{i_2}_{j_2}] = \delta_{i_1 j_2} A^{i_2}_{j_1} - \delta_{j_1 i_2} A^{i_1}_{j_2}$$
$$(i_1, j_1 ; i_2, j_2 = 1,...,n)$$

The generators A^{i+1}_i $\left(A^i_{i+1}\right)$ are the familiar Chevalley generators e_i (f_i), the Serre-relations being classical double commutators. Another important fact is that the Cartan generators A^i_i are all diagonalized in the Gelfand-Zetlin basis. So $q^{\pm\left(A^i_i - A^{i+1}_{i+1}\right)}$ can be obtained directly. Then one proceeds to construct q-commutators and the Hopf algebra. *Explicit matrix element* for irreducible representations are obtained and the contraction and deformation prescriptions of [2] turn out to be

compatible with the Hopf structure.

For SO(n), the canonical classical generators satisfy (with the hermitian convention), along with antisymmetry,

$$[J_{i_1j_1}, J_{i_2j_2}] = i\,(\delta_{i_1i_2}J_{j_1j_2} + \delta_{j_1j_2}J_{i_1i_2} - \delta_{i_1j_2}J_{j_1i_2} - \delta_{j_1i_2}J_{i_1j_2})$$
$$(i_1, j_1\ ;\ i_2, j_2 = 1,\ldots,n)$$

These, even at the classical level, lack the simple correspondance with the Chevalley generators, typical of the unitary case. Moreover *only one* of the Cartan generators (J_{12}) is diagonalized in the G-Z basis. Thus starting with the classical G-Z matrix elements no simple prescription (such as the minimal deformation (x) $\rightarrow$ [x] described for SU(n)) is available. One has to face, among others, the problem of inverting non-diagonal matrices if one wants to make contact with the standard Hopf algebra for $SO(n)_q$.

For $SO(4)_q$ the direct product basis using

$$SO(4)_q \approx SU(2)_q^{(1)} \otimes SU(2)_q^{(2)}$$

provides a starting point to construct the basis which is as close to the classical cononical basis as possible. Classically, one diagonalises the $\vec{J}^{\,2}$ corresponding to

$$\vec{J} = \vec{J}^{(1)} + \vec{J}^{(2)}$$

and then

$$\vec{K} = \vec{J}^{(2)} - \vec{J}^{(2)}$$

completes the set. The basic features of the generators $\vec{K}$ is that they change the j values (corresponding to $\vec{J}$) as

$$j \rightarrow (j+1, j, j-1)$$

The states ($j \pm 2$, $j \pm 3$, ...) do <u>not</u> appear for linear combinations of ($K_\pm$, K_o). *What are the operators (for $q \neq 1$), expressed in terms of $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$ which have this spectrum ? What exactly are their actions, namely, the explicit matrix elements ?* The answer depends on the properties of the q-C.G. coefficients as will be shown in the following sections.

2. Recapitulation of the classical canonical basis for SO(4).

We summarize the relevant classical results for SO(4). This will introduce our notations and conventions and will provide ready comparison with the $SO(4)_q$ results to follow.

We start with the direct product states

$$|j_1\, m_1, j_2\, m_2> = |\, j_1\, m_1> \otimes |\, j_2\, m_2>$$

with standard SU(2) matrix elements

$$J_o^{(i)}\, |j_i, m_i> = m_i |\, j_i, m_i> \quad (i=1,2) \tag{2.1}$$
$$J_\pm^{(i)}\, |j_i, m_i> = ((j_i \mp m_i)\,(j_i \pm m_i + 1))^{1/2} |\, j_i, m_i \pm 1)$$

Introduce the basis, using C.G. coefficients,

$$|(j_1\, j_2)\, jm> = \sum_{m_1+m_2=m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |\, j_1\, m_1, j_2\, m_2> \tag{2.2}$$

and the generators

$$J_o = J_o^{(1)} + J_o^{(2)},\ J_\pm = J_\pm^{(1)} + J_\pm^{(2)} \tag{2.3}$$
$$K_o = J_o^{(1)} - J_o^{(2)},\ K_\pm = J_\pm^{(1)} - J_\pm^{(2)}$$

The matrix elements are (suppressing $(j_1\, j_2)$ in the coupled states)

$$J_o |\, jm> = m\, |\, jm> \tag{2.4}$$
$$J_\pm |\, jm> = ((j \mp m)\,(j \pm m + 1))^{1/2}\, |\, jm \pm 1>$$

and with

$$|h_2| \le j \le h_1\ ,\ h_2 = j_1 - j_2,\ h_1 = j_1 + j_2$$

$$\begin{aligned} K_o |\, jm> = {} & ((j+m+1)\,(j-m+1))^{1/2}\, a\,(j)\, |\, j+1, m> \\ & + ((j+m)\,(j-m))^{1/2}\, a\,(j-1)\, |\, j-1, m> \\ & m\, b\,(j)\, |\, j, m> \end{aligned} \tag{2.5}$$

$$\begin{aligned} K_\pm |\, jm> = {} & \mp ((j \pm m+1)\,(j \pm m+2))^{1/2}\, a\,(j)\, |\, j+1, m \pm 1> \\ & \pm ((j \mp m)\,(j \mp m-1)^{1/2}\, a(j-1)\, |j-1, m \pm 1> \\ & + ((j \mp m)\,(j \pm m+1))^{1/2}\, b(j)\, |\, j, m \pm 1> \end{aligned} \tag{2.6}$$

where

$$a(j) = \left\{ \frac{((h_1 + 1)^2 - (j+1)^2)\,((j+1)^2 - h_2^2)}{(2j+3)\,(2j+1)\,(j+1)^2} \right\}^{1/2} \tag{2.7}$$

$$b(j) = \left\{ \frac{(h_1 + 1)\,h_2}{j\,(j+1)} \right\} \tag{2.8}$$

(Our K_o coressponds to - iI_{43} of [1] along with an extra phase factor $(i)^j$ for the states. We have real symmetric maxtrix for K_o and $(K_+)^+ = K_-$).

The well-known algebra is (with upper or lower signs)

$$\begin{aligned} &[J_o, J_\pm] = \pm J_\pm\,,\ [J_+, J_-] = 2\,J_o \\ &[K_o, K_\pm] = \pm J_\pm\,,\ [K_+, K_-] = 2\,J_o \\ &[K_o, J_\pm] = \pm K_\pm\,,\ [K_\pm, J_\mp] = 2\,K_o \\ &[J_o, K_\pm] = \pm K_\pm\,,\ [J_\pm, K_\pm] = o \end{aligned} \tag{2.9}$$

The two Casimir operators are

$$C_1 = (J_+ J_- + J_o^2) + (K_+ K_- + K_o^2) \tag{2.10}$$

$$C_1 = \frac{1}{2}(J_+ K_- + J_- K_+) + (J_o K_o)$$

and

$$C_1 \mid j,m > = ((h_1 + 1)^2 + h_2^2 - 1) \mid j,m >$$

$$C_2 \mid j,m > = (h_1 + 1)\,h_2\, j,m > \tag{2.11}$$

3. *Structures in $SO(4)_q$*

Start again with a direct product basis, but this time of $SU(2)_q^{(1)} \otimes SU(2)_q^{(2)}$ such that

$$q^{\pm J_o^{(i)}} \mid j_1\, m_i > = q^{\pm m_i} \mid j_i\,, m_i > (i = 1,2) \tag{3.1}$$

$$J_\pm^{(i)} \mid j_i\, m_i > = ([j_i \mp m_i]\,[j_i \pm m_i + 1])^{1/2} \mid j_i\,, m_i \pm 1 >$$

where $$[x] \equiv (q^x - q^{-x}) / (q - q^{-1}).$$

In this section we consider generic $q \neq e^{i\frac{2\pi}{n}}$ $(n = 1, 2, 3, \ldots)$.

Let
$$| (j_1 j_2) jm > \sum_{m_1+m_2=m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q | j_1,m_1, j_2,m_2 > \tag{3.2}$$

where we have used q- C.G. coefficients [7]. (Substitute q^{-1} for $q^{1/2}$of [7] to fit with our conventions).

Define
$$q^{\pm J_o} = q^{\pm} (J_o^{(1)} + J_o^{(2)}) \tag{3.3}$$
$$J_\pm = J_\pm^{(1)} \; q^{-} J_o^{(2)} + J_\pm^{(2)} \, q^{-} J_o^{(1)} \tag{3.4}$$

when (suppressing again $(j_1 j_2)$ on the l.h.s. of (3.2))
$$q^{\pm J_o} | jm > = q^{\pm m} | jm > \tag{3.5}$$
$$J_\pm | Jm > = ([j \mp m] \, [j \pm m+1])^{1/2} | j, m \pm 1 > \tag{3.6}$$

These, of course, satisfy, as is well-known
$$q^{J_o} J_\pm = J_\pm \, q^{J_o \pm 1} \tag{3.7}$$
$$[J_+, J_-] = [2J_o] \tag{3.8}$$

Now we come to the problem underlined at the end of Sec. 1. The key result is estabilshed in the Appendix.

Defining
$$X \equiv q^{J_o^{(2)} - J_o^{(1)}} \tag{3.9}$$
it is shown it the Appendix that
$$X | jm > = - \frac{(q-q^{-1})}{(q^{j+1}+q^{-j-1})} ([j+m+1] \, [j-m+1])^{1/2} \, \alpha (j) | j+1,m >$$
$$- \frac{(q-q^{-1})}{(q^{j}+q^{-j})} ([j+m] \, [j-m])^{1/2} \, \alpha (j-1) | j-1,m >$$
$$- \frac{1}{2} (f_-^{(j)} (q^m - q^{-m}) - f_+^{(i)} (q^m + q^{-m})) | j,m > \tag{3.10}$$

where
$$\alpha(j) = \left\{ \frac{([h_1+1]^2 - [j+1])^2 \, ([j+1]^2 - [h_2]^2)}{[2j+3] \, [2j+1] \, [j+1]^2} \right\}^{1/2} \tag{3.11}$$
$$(h_1 = j_1 + j_2, \; h_2 = j_1 - j_2)$$

and

$$f_{\pm} = \frac{(q^{h_1+1} \pm q^{-h_1-1})\,(q^{h_2} \pm q^{-h_2})}{(q^{j+1} \pm q^{-j-1})\,(q^{j} \pm q^{-j})} \tag{3.12}$$

Note that $\alpha(j)$ is obtained from a(j) of (2.7) by

$$(x) \rightarrow [x]$$

and

$$f_{-}(j) \equiv \beta(j) = \frac{[h_1 + 1]\,[h_2]}{[j]\,[j+1]} \tag{3.13}$$

is obtained analogously from b(j).

A crucial feature is the appearence of f_+ assuring

$$X \overset{q \rightarrow 1}{\rightarrow} \mathbb{1}\,.$$

The operator X^{-1} is discussed in the Appendix. For $q \rightarrow q^{-1}$ (3.4) the roles of X and X^{-1} are interchanged.

The other operators with the spectrum

$$j \rightarrow (j+1, j, j-1)$$

can now be obtained from X *and* $(J_{\pm}, J_o)$

Of particular interest are $A_{\pm}$, $B_{\pm}$ defined through

$$J_{\pm} X - q^{\mp 1} X J_{\pm} = \pm (q-q^{-1})\, A_{\pm}\, q^{-J_o \mp 1} \tag{3.14}$$
$$J_{\pm} X - q^{\pm 1} X J_{\pm} = \mp (q-q^{-1})\, B_{\pm}\, q^{J_o \pm 1}$$

From (3.3), (3.4), (3.9) and (3.14) one obtains

$$A_{\pm} = J_{\pm}^{(1)}\, q^{J_o^{(2)}}$$
$$B_{\pm} = J_{\pm}^{(2)}\, q^{-J_o^{(1)}} \tag{3.15}$$

The matrix elements of $A_\pm$, $B_\pm$ are obtained from (3.5), (3.6) and (3.10). *These, along with X, lead to a particularly simple algebra and coproducts.* This will be presented later. Let us immediately note the matrix elements of

$$K_\pm = A_\pm - B_\pm \tag{3.16}$$

and

$$L_\pm = A_\pm + B_\pm \tag{3.17}$$

One obtains (with α (j) of (3.11) and β (j) of (3.13))

$$\begin{aligned} K_\pm \mid jm > = & \mp ([j\pm m+1]\,[j\pm m+2])^{1/2}\,\alpha\,(j) \mid j+1, m\pm 1 > \\ & \pm ([j\mp m]\,[j\mp m-1])^{1/2}\,\alpha\,(j-1) \mid j-1, m\pm 1 > \\ & + ([j\mp m]\,[j\pm m+1])^{1/2}\,\beta\,(j) \mid j, m\pm 1 > \end{aligned} \tag{3.18}$$

Thus

$$K_\pm = J_\pm^{(1)}\, q^{J_o(2)} - J_\pm^{(2)}\; q^{-J_o(1)} \tag{3.19}$$

provide the "minimal q- deformation" of (2.6). They satisfy

$$[K_+, K_-] = [2J_o] \tag{3.20}$$

$$[K_\pm, J_\pm] = o \tag{3.21}$$

Also (with f_+ of (3.12))

$$\begin{aligned} L_\pm \mid jm = & - ([j\pm m+1]\,[j\pm m+2])^{1/2} \left(\frac{q^{j+1}-q^{-j-1}}{q^{j+1}+q^{j+1}}\,\alpha\,(j)\right) \mid j+1, m\pm 1 > \\ & - ([j\mp m]\,[j\mp m-1])^{1/2} \left(\frac{q^{j}-q^{-j}}{q^{j}+q^{j}}\,\alpha\,(j-1)\right) \mid j-1, m\pm 1 > \\ & - ([j\mp m]\,[j\pm m+1])^{1/2} f_+\,(j) \mid j, m\pm 1 > \end{aligned} \tag{3.22}$$

Note that for q = 1

$$L_\pm \mid jm > = J_\pm \mid, j,m > \tag{3.23}$$

Along with (J_+, J_-, J_o), (K_+, K_-, J_o), (L_+, L_-, J_o) form a third $SU(2)_q$ triplet satisfying

$$[L_+, L_-] = [2J_o] \tag{3.24}$$

The q-commutators with $q^{\pm J_o}$ are evidently as usual. Note that each term on the right of (3.19) separately has the same type of spectrum (namely, under their action $j \to j \pm 1, j$)). In (3.4) the two

terms on the right taken separately involes X^{-1}, an operator we want to introduce as late as possible in the formalism.

We have suitable q - deformation of the classical $K_{\pm}$ in (3.18). What about K_o ?

The operators (see Appendix)

$$\frac{2}{(q-q^{-1})}(1 - X) \tag{3.25}$$

$$-\frac{1}{(q-q^{-1})}(X - X^{-1}) = [J_o^{(1)} - J_o^{(2)}] \tag{3.26}$$

both have K_o of (2.5) as the classical (q = 1) limit. But of particular interest to us are the operators $K_o^{(\pm)}$ defined as

$$\begin{aligned} K_o^{(+)} = (K_o^{(-)})^+ &= (q^{J_o}[J_+, K_-] - q^{-J_o}[K_+, J_-])(q^{2J_o} - q^{-2J_o})^{-1} \\ &= [J_o^{(1)} - J_o^{(2)}] - (q - q^{-1}) J_-^{(1)} J_+^{(2)} \end{aligned} \tag{3.27}$$

One obtains

$$\begin{aligned} K_o^{(\pm)} \mid j,m > &= q^{\mp (j+1)}([j\pm m+1])[j-m+1])^{1/2}\, \alpha(j) \mid j+1, m > \\ &+ q^{\mp j}([j+m])[j-m])^{1/2}\, \alpha(j-1) \mid j-1, m > \\ &+ [m]\, \beta(j) \mid j,m > \end{aligned} \tag{3.28}$$

(concerning the doubling of the classical K_o under q-deformation, see sec 7). These have evidently (2.5) as the classical limit. The operators $K_o^{(+)}$ and $K_o^{(-)}$ coincide for q=1. For $q \neq 1$ they are related through hermitian conjugacy and also through $q \rightarrow q^{-1}$. The q-factors in (3.28) can be eliminated by defining

$$\hat{K}_o = v^{\mp 1/2} K_o^{(\pm)} v^{\pm 1/2} \tag{3.29}$$

where

$$v \mid j,m > = q^{-j(j+1)} \mid j,m > \tag{3.30}$$

in an interesting operator appearing in Ribbon Hopf Algebra [8]. $\hat{K}_o$ is just the "minimal q-deformation" of K_o of (2.5).

The operators $K_o^{(\pm)}$ are particularly suitable for making the structures of the q-Casqimirs transparent.

Define

$$C_1 = (q+q^{-1})^{-1} \left\{ K_+ K_- + K_- K_+ + q\, K_o^{(+)} K_o^{(-)} + q^{-1} K_o^{(-)} K_o^{(+)} + (q^2+q^{-2})\, D \right\} \quad (3.31)$$

where $$D \mid j,m > = \frac{1}{2} \left\{ (J_+ J_- + J_- J_+) + (q + q^{-1})\, [J_o]^2 \right\} \mid jm > = [j]\,[j+1] \mid jm > \quad (3.32)$$

and $$C_2 = \frac{1}{2} (K_+ J_- + K_- J_+) + \frac{1}{2} [J_o](qK_o^{(+)} + q^{-1} K_o^{(-)}) \quad (3.32)$$

Then

$$C_1 \mid jm > = ([h_1+1]^2 + [h_2]^2 - 1) \mid jm > \quad (3.33)$$

$$C_2 \mid jm > = [h_1+1]\,[h_2] \mid jm > \quad (3.34)$$

Comparison with (2.10) and 2.11) is direct.

Since the matrix for X^{-1} (see Appendix) is complicated, let us see first how far one can go without bringing it in.

The classical system (Sec.2) can be obtained as a limit starting with $(q^{\pm J_o}, A_\pm, B_\pm, X)$ (3.35) noting particularly (3.18), (3.23) and (3.25). One has also following closed system of q-commutators and coproducts, obtainable fairly simply form (3.9) and (3.15) quite consistently with the matrix elements (3.10), (3.18) and (3.22).

$$\begin{aligned}
&X A_\pm = q^{\mp 1} A_\pm X, \quad q^{J_o} A_\pm = q^{\pm 1} A_\pm q^{J_o} \\
&X B_\pm = q^{\pm 1} B_\pm X, \quad q^{J_o} B_\pm = q^{\pm 1} B_\pm q^{J_o} \\
&[A_+, A_-], = \frac{1}{(q-q^{-1})} (q^{2J_o} - X^2) \\
&[B_+, B_-] = \frac{1}{(q-q^{-1})} (X^2 - q^{-2J_o}) \\
&q^{-1} A_+ B_+ - q B_+ A_+ = o \\
&q A_- B_- - q^{-1} B_- A_- = o \\
&[A_+, B_-] = o = [A_-, B_+] \\
&q^{J_o} X = X q^{J_o}
\end{aligned} \quad (3.36)$$

From these the necessary relations can be obtained for $K_\pm$, $K_o^{(\pm)}$. They exhibit correct classical limits. The coproducts are given by (3.9) and (3.15), using the standard ones for the generators of $SU(2)_q^{(1)}$ and $SU(2)_q^{(2)}$, as

$$\Delta X = X \otimes X\,, \quad \Delta q^{J_o} = q^{J_o} \otimes q^{J_o}$$

$$\Delta A_{\pm} = A_{\pm} \otimes X + q^{J_o} \otimes A_{\pm}$$
$$\Delta B_{\pm} = B_{\pm} \otimes q^{-J_o} + X \otimes B_{\pm} \tag{3.37}$$

These can be directly verified to be consistent with (3.36). The counits are

$$\varepsilon A_{\pm} = o, \varepsilon B_{\pm} = o$$
$$\varepsilon q^{\pm J_o} = 1, \varepsilon X = 1 \tag{3.38}$$

The operator X^{-1} *appears if the status of a fully pseudo-triangular Hopf algebra is sought with antipodes.*

One can add to the proceeding the evident relations for X^{-1}, namely

$$X^{-1} A_{\pm} = q^{\pm 1} A_{\pm} X^{-1}$$
$$X^{-1} B_{\pm} = q^{\mp 1} B_{\pm} X^{-1}$$
$$\Delta X^{-1} = X^{-1} \otimes X^{-1}$$
$$\varepsilon X^{-1} = 1 \tag{3.39}$$

Now the antipodes, satisfying all necessary constraints, can be written as

$$S(q^{\pm J_o}) = q^{\mp J_o}$$
$$S(X^{\pm 1}) = X^{\mp 1}$$
$$S(A_{\pm}) = -q^{-J_o} A_{\pm} X^{-1}$$
$$S(B_{\pm}) = -X^{-1} B_{\pm} q^{J_o} \tag{3.40}$$

One can write coproducts in terms of

$$q^{\pm J_o}, X, X^{-1}, K_{\pm} = A_{\pm} - B_{\pm} \text{ and } J_{\pm} = (A_{\pm} q^{-J_o} + B_{\pm} q^{J_o}) X^{-1} \tag{3.41}$$

But they are much more complicated than (3.37).

Useful q-commutators for $K_o^{(\pm)}$ are

$$K_+ J_- - J_- K_+ = q^{-J_o} K_o^{(+)} + q^{J_o} K_o^{(-)}$$
$$q^{\pm 1} K_o^{(\pm)} K_+ - K_+ K_o^{(\pm)} = q^{\pm J_o} J_+$$
$$K_o^{(\pm)} J_+ - q^{\pm 1} J_+ K_o^{(\pm)} = K_+ q^{\mp J_o} \tag{3.42}$$

and their hermitian conjugates (interchanging + and - in J and K and the order in each product). Remarkably enough (compare (3.31) and (3.33)

$$\left\{(q-q^{-1})^{-1}\,(q\,K_o^{(+)}\,K_o^{(-)} - q^{-1}\,K_o^{(-)}\,K_o^{(+)}) + (q+q^{-1})\,D - [J_o]^2\right\}|\,jm> = C_1\,|\,jm> \tag{3.43}$$

The operators $K_o^{(\pm)}$ (or $\widehat{K}_o$ and v) have close relations to the classical K_o. But they do not seem to be suitable (along with $q^{\pm J_o}$, $J_\pm$, $K_\pm$) for constructing coproducts. For this reason (3.36) and (3.37) were presented. Having used (3.37) and (3.39) one can then construct in their terms Δ's of $J_\pm$, $K_\pm$, $K_o^{(\pm)}$ leading often to involved expressions. After constructing the product representation using (X, $A_\pm$, $B_\pm$) one can , of course, construct the other operators ($K_\pm$, $K_o^{(\pm)}$) directly and easily from them establishing again the closest correspondance with the classical picture. The $J_\pm$ operators can be obtained from the set used for coproduct but also very easily from the standard coproduct rules for the J's. Thus a complete structure for the products is always thus obtained for such a q-canonical basis.

4. q a root of unity

In previous papers [4] it was shown how G-Z matrix elements for SU(n) adapted to $SU(n)_q$ contain in them periodic representations for

$$q = e^{i\frac{2\pi}{n}}\ (\ n = 3,4,...) \tag{4.1}$$

For the representations to be well defined the domains of the parameters (ordered integers for generic q) had to be altered introducing "**fractional parts**", the term being used in a broad sense for both real, proper fractions and imaginary parts. There fractional parts had to be unequal for parameters of the same or adjacent rows in a G-Z basis state [4]. A sriking feature for $q^n = 1$ is that the n-th power of all the generators belong to a suddenly inflated centre. The eigenvalues of these new invariant operators acting on a G-Z state depend on the invariant fractional parts introduced. The fractional parts are invariant since only the real integer parts of the parameters change under the action of the generators.

Here the corresponding picture for $SO(4)_q$ will be briefly indicated. The study will be quite incomplete. We hope to present a more thorough investigation for $SO(n)_q$ elsewhere.

First we note from (3.36),

$$X^n A_\pm = q^{\pm n} A_\pm X^n \tag{4.2}$$

$$[A_+^n, A_-] = -\frac{1}{(q-q^{-1})} A_+^{n-1} \left\{ q^{2J_o} (1+q^{-2} + q^{-4} + \ldots + q^{-2(n-1)}) - X^2 (1 + q^2 + q^4 + \ldots q^{2(n-1)}) \right\} \tag{4.3}$$

and so on.

Hence it is easy to show that

$$X^{\pm n},\ A_\pm^n,\ B_\pm^n$$

all belong to the centre.

In [4] periodicity was introduced by identifying states that lead to identical matrix elements for the generators. Under the action of the n-th power of a generator only such states survived, leading to a consistant situation. We indicate an analogous mechanism here for a particularly simple case.

Let $q^3 = 1$

Now,

$$K_+^3 \,|\, jm > = \sum_{i=-3}^{3} M_i \,|\, j+i, m+3 > \tag{4.4}$$

where the matrix elements M_i can be obtained by iterating (3.18). One can show that $M_{+2} = o$ due to the identity (with β of (3.13))

$$\beta(j) + \beta(j+1) + \beta(j+2) = o \tag{4.5}$$

for $q^3 = 1$.

Also $M_{+1} = o$ due to the identity (using (3.11), (3.13) and $q^3 = 1$)

$$\beta(j+1)\,\beta(j+1) + \beta(j)\,\beta(j+1) + \beta(j)\,\beta(j) + \alpha(j+1)\,\alpha(j+1) + \alpha(j)\,\alpha(j) + \alpha(j-1)\,\alpha(j-1) = o \tag{4.6}$$

Similarly M_{-2} and M_{-1} can also be shown to vanish.

Hence for $q^3 = 1$

$$K_+^3 \mid jm > = M_3 \mid j+3, m+3 > + M_o \mid j, m+3 > + M_{-3} \mid j-3, m+3 > \tag{4.7}$$

$$\approx \mid jm > \tag{4.8}$$

since each state on the right in (4.7) is equivalent to $\mid jm >$ in the sense previously mentioned, namely each respond to the action of the generators like $\mid jm >$ giving the same matrix elements. The matrix elements M_3 ,M_o, M_{-3} can be shown to depend finally only on invariant parameters (h_1, h_2 and fractional parts of j and m). The fractional parts of j,m are to be chosen such that the generators are not nilpotent to achieve full periodicity [4]. Otherwise one can have partial periodicity [4].A full investigation of periodic and partially periodic representation will not be undertaken here.

Periodicity can be investigated also by using the direct product ($SU(2)_q \otimes SU(2)_q$) basis. *But we explored directly the canonical parametrization which may show the way for $SO(p)_q$ (p>4)* where certain special tricks possible for $SO(4)_q$ are no longer available.

5. Contraction to $E(3)_q$

Explicit, irreducible representations were constructed [2] for non-semisimple, inhomogeneous classical algebras starting both form SU(n) and SO(n). For the unitary case, the construction was generalized to $q \neq 1$ giving I $SU(n)_q$ representations [3, 4, 5]. Here we study the structure that emerges when an anologous contraction is applied to the canonical formalism for $SO(4)_q$. For the non-compact case the representations will of course be infinite dimensional for generic q. Let $h_1 \to \infty$, h_2 remaining finite and let us define from (3.18)

$$P_\pm \mid jm > = \lim_{h_1 \to \infty} (\lambda\, [h_1 + 1]^{-1} K_\pm) \mid j,m > \tag{5.1}$$

where λ is some arbitrary real constant.
Then

$$\begin{aligned} P_\pm \mid jm > = \mp \lambda & \left\{ \frac{[j\pm m+1]\,[j\pm m+2]\,([j+1]^2-[h_2]^2)}{[2j+3]\,[2j+1]\,[j+1]^2} \right\}^{1/2} \mid j+1, m\pm 1 > \\ & \pm \lambda \left\{ \frac{[j\mp m]\,[j\mp m-1]\,([j]^2-[h_2]^2)}{[2j+1]\,[2j-1]\,[j]^2} \right\}^{1/2} \mid j+1, m\pm 1 > \\ & + \lambda\, ([j\mp m]\,[j\pm m+1])^{1/2} \frac{[h_2]}{[j]\,[j+1]} \mid j, m \pm 1 > \end{aligned} \tag{5.2}$$

For q=1, one gets $P_{\pm}$, translations of classical E(3)[2]. The states can be fully parametrized as $|(\lambda h_2)\, jm>$. But as before we use a simpler notation. The J's remain, of course, as before, namely,

$$q^{\pm J_o} \,|\, jm > = m|jm > \tag{5.3}$$

$$J_{\pm} \,|\, jm > = ([j\mp m]\,[j\pm m+1])^{1/2}\, |j,m \pm 1 >$$

Defining, from (3.28),

$$P_o^{(\pm)} \,|\, jm > = \lim_{h_1 \to \infty} (\lambda\, [h_1+1]^{-1}\, K_o^{(\pm)}) \,|\, jm > \tag{5.4}$$

one obtains

$$P_o^{(\pm)} \,|\, j,m > = q_o^{\mp(j+1)}\, \lambda \left\{ \frac{[j+m+1]\,[j-m+1]\,([j+1]^2-[h_2]^2)}{[2j+3]\,[2j+1]\,[j+1]^2} \right\}^{1/2} |\, j+1, m >$$

$$+ q^{\pm j}\, \lambda \left\{ \frac{[j+m]\,[j-m]\,([j]^2-[h_2]^2)}{[2j+1]\,[2j-1]\,[j]^2} \right\}^{1/2} |\, j-1, m >$$

$$+ \lambda \frac{[m]\,[h_2]}{[j]\,[j+1]}\, (j,m > \tag{5.5}$$

$P_o^{(\pm)}$ are hermitian conjugates and are also related through $q \to q^{-1}$ coinciding for q = 1 to the classical P_o of E(3) [2].

The contraction procedure along with (3.20), (3.21), (3.42) leads to

$$[P_+, P_-] = o,$$

$$q^{\pm 1}\, P_o^{(\pm)} P_+ - P_+ P_o^{(\pm)} = o$$

$$P_+ J_- - J_- P_+ = q^{-J_o} P_o^{(+)} + q^{J_o} P_o^{(-)}$$

$$P_+ J_+ - J_+ P_+ = o$$

$$P_o^{(\pm)} J_+ - q^{\pm 1} J_+ P_o^{(\pm)} = P_+\, q^{\mp J_o} \tag{5.6}$$

The second relation shows the *non-commutativity* of $P_o^{(\pm)}$ with P_+ and similary with P_- from the conjugate. (See also (5.9) below).

Other relations of interest can be obtained from these using conjugation, as from (3.42) for the K's. The results for $J_{\pm}$, $q^{\mp J_o}$ (namely (3.7), (3.8)) remain inchanged.

For the Casimirs one now has the following generalizations of the classical results [2],

$$C_1 |\, jm > \equiv (q+q^{-1})^{-1} \{ P_+P_- + P_-P_+ + q\, P_o^{(+)} P_o^{(-)} + q^{-1} P_o^{(-)} P_o^{(+)} \} \,|jm > = \lambda^2 \,|\, jm > \tag{5.7}$$

$$C_2 | jm > \equiv \frac{1}{2} \{ P_+ J_- + P_- J_+ + [J_o] (q P_o^{(+)} + q^{-1} P_o^{(-)}) \} | jm > = \lambda [h_2] | jm > \tag{5.8}$$

Corresponding to (3.43) one now has

$$(q P_o^{(+)} P_o^{(-)} - q^{-1} P_o^{(-)} P_o^{(+)}) | jm > = (q-q^{-1}) \lambda^2 | jm > = (q-q^{-1}) C_1 | jm >$$

which, from (5.7), implies the further non-commutativity

$$P_o^{(+)} P_o^{(-)} - P_o^{(-)} P_o^{(+)} = (q-q^{-1}) P_+ P_- \tag{5.9}$$

This vanishes trivially for q = 1. One can verify (5.9) directly from (5.2) and (5.5). One can introduce, as in (3.29) and (3.30), $\widehat{P}_o$ through

$$\widehat{P}_o = v^{\mp\frac{1}{2}} P_o^{(\pm)} v^{\pm\frac{1}{2}} \tag{5.10}$$

where $v | j,m > = q^{-j(j+1)} | j,m >$

Defining, using (3.10),

$$Y | jm > = \lim_{h_1 \to \infty} (\lambda [h_1 + 1]^{-1} X) | jm > \tag{5.11}$$

one has

$$Y | jm > = - \lambda \frac{(q-q^{-1})}{(q^{J+1}+q^{-J-1})} \left\{ \frac{[j+m+1]\,[j-m+1]\,([j+1]^2-[h_2]^2)}{[2j+3]\,[2j+1]\,[j+1]^2} \right\}^{1/2} | j+1, m >$$

$$- \lambda \frac{(q-q^{-1})}{(q^j+q^{-j})} \left\{ \frac{[j+m]\,[j-m]\,([j]^2-[h_2]^2)}{[2j+1]\,[2j-1]\,[j]^2} \right\}^{1/2} | j-1, m >$$

$$- \frac{\lambda}{2} (q-q^{-1}) \left\{ \frac{[h_2]\,[m]}{[j]\,[j+1]} + \varepsilon \frac{(q^{h2} + q^{-h2})\,(q^m+q^{-m})}{(q^j + q^{-j})\,(q^{j+1}+q^{-j-1})} \right\} | j,m > \tag{5.12}$$

with $\varepsilon = \pm 1$ for $q \gtrless 1$.

Before one had

$$\lim_{q \to 1} X | jm > = | j,m > \tag{5.13}$$

now

$$\lim_{q \to 1} Y \mid jm > = o \tag{5.14}$$

Hence without detailed study it is already evident that Y^{-1} cannot be defined consistently for all values of generic q. (For q a root of unity, when moreover all the matrices are finite, the situation will be studied elsewhere).

We have thus the following situation. The matrix elements of the set of operators, forming a closed system,

$$(q^{\pm J_o}, J_\pm, P_\pm \text{ and } P_o^{(\pm)} \text{ or } \widehat{P}_o \text{ and } v) \tag{5.15}$$

lead to the correct classical limit giving general irreducible representations of E(3). But the Hopf algebra ((3.36) to (3.40)) does not survive the contraction due to Y^{-1} not being well defined among other things. Suitable coproducts have not been obtained directly terms of (5.15), though the relations (5.6) and (5.9) are relatively simple. Thus the formalism does not contain all the desirable properties. Contructing product representations at the level of $SO(4)_q$, then constructing the K's and then contracting them to the P's one obtains a prescription for products of $E(3)_q$, which is evidently not an intrinsic one. Here we have attempted to construct representations of $E(3)_q$ *for all values of q*. This should be compared to the case where an algebra is proposed for $q \to 1$, the ratio of q and ε, the contraction parameter, remaining finite [9]. There it has been found that , choosing the basis adequately, the generators can be expressed as fairly simple non-linear functions of the classical ones. This will be shown elsewhere.

6. Continuation to $SO(3,1)_q$

It is known that classical SO(4) and SO(3,1) irreducible representations are related through

$$(h_1 + 1) \to i\nu \tag{6.1}$$

and

$$(K_\pm, K_o) \to i\,(K_\pm, K_o)$$

in (2.5) to (2.8). One has now from (2.7) and (2.8) (writing μ for h_2 and suitably defining square roots)

$$i\,a\,(j) \rightarrow \left\{ \frac{(\nu^2 + (j+1))^2\,((j+1)^2 - \mu^2)}{(2j+3)\,(2j+1)\,(j+1)^2} \right\}^{1/2} \tag{6.2}$$

$$i\,b\,(j) \rightarrow \frac{\nu\mu}{j\,(j+1)} \tag{6.3}$$

A different phase convention is diplayed in [2]. The unitary representations of SO(3,1) are infinite dimensional and the change in (2.9) are

$$[K_+, K_-] = -\,2\,J_o\,,\ [K_o, K_\pm] = \mp\,J_\pm \tag{6.4}$$

The casimirs are now (for states $|\,(\nu\mu), jm>$ denoted as $|\,jm>$)

$$C_1 |\,jm> \equiv \{(K_+K_- + K_o^2) - (J_+J_- + J_o^2)\}\,|jm> = (\nu^2 - \mu^2 + 1)\,|\,jm> \tag{6.5}$$

$$C_2\,jm> \equiv \{\,\frac{1}{2}\,(J_+\,K_- + J_-\,K_+) + J_o\,K_o\,\}|\,jm> = \nu\mu\,|\,jm> \tag{6.6}$$

For $\nu^2 \geq o$ one has the principal series representations. For $\mu = o$, $o < -\,\nu^2 < 1$ one has the supplementary series. The spectrum of μ (or h_2) is given by integers or half integers.

For $q \neq 1$, if one sets in $\alpha\,(j)$ of (3.11) and $\beta(j)$ of (3.13),

$$[h_1 + 1] \rightarrow i\,[\nu] \tag{6.7}$$

and, multiplying the matrix elements of the K's as before by i to make them real, then from (3.18) (writing again μ for h_2),

$$K_\pm\,|\,jm> = \mp\,([j\pm m+1]\,[j\pm m+2])^{1/2} \left\{ \frac{([\nu]^2 + [j+1])^2\,([j+1]^2 - [\mu]^2)}{[2j+3]\,[2j+1]\,[j+1]^2} \right\}^{1/2} |\,j+1, m\pm 1>$$

$$\pm\,([j \mp m]\,[j \mp m-1])^{1/2} \left\{ \frac{([\nu]^2 + [j]^2)\,([j]^2 - [\mu]^2}{[2j+1]\,[2j-1]\,[j]^2} \right\}^{1/2} |\,j-1, m\pm 1>$$

$$+\,([j \mp m]\,[j \pm m+1])^{1/2}\,\frac{[\nu][\mu]}{[j]\,[j+1]}\,|\,j, m \pm 1> \tag{6.8}$$

and one has

$$[K_+ , K_-] = - [2 J_o] \tag{6.9}$$

with a change of sign in (3.20).

For $K_o^{(\pm)}$ the α and β factors are altered in (3.28) as above and the changes in the q-commutators (3.42) are evident, such as

$$q^{\pm} K_o^{(\pm)} K_+ - K_+ K_o^{(\pm)} = - q^{\pm J_o} J_+ \tag{6.10}$$

and so on.

But where f_+, given by (3.12), appears (as for X and $L_\pm$) the situation is different. One has

$$f_+ = \frac{(q^{h_1+1}+q^{-h_1-1})\ (q^{h_2}+q^{-h_2})}{(q^{j+1}+q^{-j_1-1})(q^j+q^{-j_1})} = \frac{\sqrt{< h_1+1>^2 + 4}\ (q^{h_2}+q^{-h_2})}{(q^{j+1}+q^{-j-1},)\ (q^j+q^{-j})} \tag{6.11}$$

where $$< h_1 + 1> \equiv (q^{h_1+1} - q^{-h_1-1}) = (q- q^{-1})\ [h_1 + 1] \tag{6.12}$$

Following the prescription leading to (6.8) and (6.9) one should substitute using

$$f_+ \rightarrow \frac{\sqrt{<\nu>^2- 4}\ (q^{\mu}+q^{-\mu})}{(q^{j+1}+q^{-j-1})\ (q^j+q^{-j})} \tag{6.13}$$

where $<\nu > \equiv (q^{\nu} - q^{-\nu})$

Thus for $<\nu >^2 <4$ f_+ is imaginary and $f_+ = i$ for $q = 1$. (6.14)

This range cannot be excluded for a satisfactory formulation. Thus the set $(X^{\pm 1}, A_\pm, B_\pm)$ leads to *complex representations* of SO(3,1). A simplification occurs for the very special case $<\nu >^2 = 4$ where $f_+ = o$.

The structure of the coproducts (3.37), it taken over as it is, also induces complexity when $(X, A_\pm, B_\pm)$ are multiplied by i after (6.7). On the otherland $(q^{\pm J_o}, J_\pm, K_\pm, K_o^{(\pm)})$ constructed according to our prescription do give a q-deformation of the classical unitary SO(3,1) representations, with

$$C_1 | jm > \equiv \frac{1}{(q+q^{-1})} \{ (K_+K_- + K_- K_+ + qK_o^{(+)}K_o^{(-)} + q^{-1} K_o^{(-)}) K_o^{(+)} - (q^2 + q^{-2}) D \} | jm >$$

$$= ([v]^2 - [\mu]^2 + 1) | jm > \qquad (6.14)$$

$$C_2 | jm > \equiv \frac{1}{2} \{ (K_+J_- + K_- J_+ + [J_o] (qK_o^{(+)} + q^{-1}K_o^{(-)}) \} | jm > = [v] [\mu] | jm > \qquad (6.15)$$

The incompatibility of the reality constraints with the Hopf algebra leading from (3.36) to (3.40) remains the unsatisfactory feature. Several studies of quantum Lorentz algebras exist [10, 11, 12, 13]. Here our aim has been to construct explicitly irreducible representations and to study the consequences of directly q-deforming the canonical classical matrix elements for the principal and the supplementary series and correspondingly the algebra of the Lorentz generators, (leading to $K_o^{(\pm)}$, $K_\pm$).

7. Remarks

For $SO(n)_q$, and if possible for $E(n-1)_q$ and $SO(n-1,1)_q$ our aim is to go beyond writing relations among a set of generators to explicit construction of irreducible representations. The matrix elements on a space of properly parametrized states can themselves be taken to define the q-deformed algebra, yielding all the necessary properties including the spectrum of each Casimir operator. For the unitary algebras this aim has been fairly completely realized [3,4,5]. For the orthogonal case the classical G-Z states and the matrix elements [1] and contraction and deformation of SO(n) to E(n-1) and SO(n-1, 1) respectiveley [2] are already available for q = 1. For reasons indicated in Sec.1 they cannot be directly adapted to $q \neq 1$ using (as for the unitary case) what we have called the "minimal q-deformation" namely, direct insertion of q-brackets in the classical matrix elements.

Exploiting the availability of the direct product basis $(SU(2)_q^{(1)} \otimes (SU(2)_q^{(2)})$ for $SO(4)_q$ and using the properties of the q-C.G. coefficients we have explicitly constructed on the q-canonical basis (diagonalizing the q-analogue of $(\vec{J}^{(1)}+\vec{J}_q^{(2)})^2$) the matrix elements of

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$$X = q\, J_o^{(2)} - J_o^{(1)},$$

$$K_\pm = J_\pm^{(1)}\, q^{J_o^{(2)}} - J_\pm^{(2)}\, q^{-J_o^{(1)}}$$

$$K_o^{(\pm)} = [J_o^{(2)} - J_o^{(2)}] - (q-q^{-1})\, J_\mp^{(1)}\, J_\pm^{(2)} \; (= v^{\pm 1/2}\, \hat{K}_o\, v^{\mp 1/2}\,) \qquad (7.1)$$

where (with the convention (3.4) for $J_{\pm}$) they are given by (3.10), (3.18), (3.28), (3.29) and (3.30). (They key result is derived in the Appendix for X). *This is the closest one can come to a minimal q-deformation of the classical results mentioned above.* The basic feature we have tried to preserve is that, under the action of the K's

$$j \rightarrow j+1, j, j-1$$

as for the classical case. (In general all states, from j_1+j_2 to $|j_1 - j_2|$ appear). The differences that arise are imposed by the starting point, namely the direct product basis, and are presumably unavoidable for ultimate consistency.

In the direct product basis on has *four diagonal operators*

$$q^{\pm J_o^{(1)}},\ q^{\pm J_o^{(2)}} \tag{7.2}$$

In the q-canonical basis one has correspondingly

$$q^{\pm J_o} \text{ and the non-diagonal pair } K_o^{(\pm)}. \tag{7.3}$$

In (7.2) inversion and $q \rightarrow q^{-1}$coincide. In (7.3) for the non-diagonal pair $K_o^{(\pm)}$ *only the aspect* $q \rightarrow q^{-1}$ *has been carried over.*This is the basic reason for doubling of the classical K_o to $K_o^{(\pm)}$ (which can be then replaced by $\widehat{K}_o$ and v as in (3.29)). The K's in (7.1) are the closest to the canonical K's ($\vec{K} = \vec{J}^{(1)}-\vec{J}^{(2)}$). For a satisfactory formulation of the Hopf algebra we had however to consider the operators

$$X^{\pm 1}\ ,\ A_{\pm},\ B_{\pm}$$

corresponding to

$$q^{\pm(J_o(2)} - J_o^{(2)})\ ,\ J_{\pm}^{(1)}\ qJ_o^{(2)},\ J_{\pm}^{(2)}\ q^{-J_o^{(1)})} \tag{7.4}$$

respectively ((3;36) to (3.40)).

In these (apart from the complications in inverting X) the most striking departure from the "minimal q-deformation" is the appearence of the term f_+ of (3.12), not directly a ratio of products of q-brackets. This again is imposed by the properties of the q-C.G. coefficients. This has consequences throughout afterwards. For $E(3)_q$ and for $SO(3,1)_q$ the associated features (see (5.14) and (6.13)

and the remarks that follow them) have been discussed in Sec 5 and Sec 6 respectively. Such features have prevented us from carrying over the Hopf algebra structure of Sec. 3 (with suitable evident modifications) to these cases. Having thus pointed out the difficulties, we would like to emphasize that a quite interesting structure for *non-commutative translations* was obtained for $E(3)_q$ (the P's of (5.6) and (5.9)) for arbitrary q with explicit matrix elements. This was the aim announced for our general program [3,4,5] concerning contact with non-commutative geometries. (The doubling of the classical P_o to $P_o^{(\pm)}$ (or the appearence of the operator v) corresponds to the remarks following (7.3)).

One particularly interesting possibility of canonical aproach is a *unified freatment of generic and root of unity q's* as illustrated for unitary algebras [4,5] by constructing ,periodic and partially periodic representations. Here we have only briefly indicated this possibility (Sec.4) for $SO(4)_q$. The next desirable step would be to carry out this program for canonical $SO(5)_q$ and to compare the results with those of [6].

It is a pleasure to thank Enrico Celeghini for many discussions throughout this work. I thank Daniel Arnaudon for helpful remarks.

Appendix

Matrix elements of $q^{\pm(J_o^{(2)}-J_o^{(1)})}$ *in the canonical basis.*

Start with the basis state (3.2). Using the orthogonality of the q-C.G. coefficients

$$q^{(J_o^{(2)}-J_o^{(1)})} \mid (j_1 j_2)\, jm > =$$

$$\sum_{j'}\left(\sum_{m_1+m_2=m}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q q^{(m_2-m_1)} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix}_q\right) \mid (j_1 j_2)\, j'm > \qquad \text{(A1)}$$

(For the convention with $q \to q^{-1}$ in (3.4) we start with $q^{-(J_o^{(2)}-J_o^{(1)})}$).

For $m = j$, $j' = j + p$ (p=o, 1, 2,...). From [7] (with $q^{1/12} \to q^{-1}$ to fit with our conventions)

$$< (j_1\, j_2)\, j'j \mid q^{J_o^{(2)}-J_o^{(1)}} \mid (j_1\, j_2)\, jj >$$

$$= q^{j+j_1(j_1+1)-j_2(j_2+1)+\frac{1}{2}j'(j'+1)}$$

$$x\left(\frac{[2j+1]\,!\,[j_1+j_2-j]\,!\,[j'+j]\,!\,[j_1+j_2-j']\,[2j'+1]}{[j_1-j_2+j]\,!\,[-j_1+j_2+j]!\,]\,[j_1+j_2+j+1]\,![j_1-j_2+j]\,!\,]\,[j_1-j_2+j']\,!\,[j_1+j_2+j'+1]\,!}\right)^{1/2}$$

$$x\left(\sum_{z,\,m_1}(-1)^z \frac{q^{-z(j'+j+1)}}{[z]\,!\,[j'-j-z]\,!}\, q^{-2m_1(j+2)} \frac{[j_1+m_1+z]\,!\,[j_2+j'-m_1-z]\,!}{[j_1-m_1-z]\,!\,[j_2-j'+m_1+z]\,!}\right) \qquad \text{(A2)}$$

where $[n]\,! = [n]\,[n-1]\,...\,[2]\,[1]$, $[o]\,! \equiv 1$ and only non-negative values of n are considered.

Setting

$$\begin{aligned} &j_1 - m_1 - z = s,\ j' = j + p \\ &a = 2j_1, = (j_1 + j_2) + (j_1 - j_2) \\ &b = j - (j_1 - j_2) + p \\ &c = (j_1 + j_2) - j - p \end{aligned} \qquad \text{(A3)}$$

the double sum in (A.2) becomes

$$\sum_{m_1,z} (\ldots) = q^{-j_1(2j+4)} S_p F_p \tag{A4}$$

where $$S_p = \sum_{z=o}^{p} (-1)^z \frac{q^{z(3-p)}}{[z]\ !\ [p-z]\ !} \tag{A5}$$

and $$F_p = \sum_s q^{s(a+b-c+4-2p)} \frac{[a-s]\ !\ [b+s]\ !}{[s]\ !\ [c-s]\ !} \tag{A6}$$

The range of s being over non-negative values of the arguments of the factorials.

The *key result* is

$$S_p = \delta_{o,p} + (1-q^2)\, \delta_{1,p} \tag{A7}$$

i,e, $S_p = o$ for $p = 2,3,\ldots$

For eventual use let us note the recursion relations for the two parameter case

$$S_{(m,p)} = \sum_{z=o}^{p} \frac{(-1)^z\, q^{z(m-p)}}{[z]\ !\ [p-z]\ !} \tag{A8}$$

One can show that

$$(1-q^{m-1-2p})\, S_{(m,p)} = [p+1]\, S_{(m,p+1)} \tag{A9}$$

$$(1-q^{m+1-2p})\, S_{(m+2,p)} = (1-q^{m+1})\, [S_{(m,p)} \tag{A10}$$

Setting m = 3, one see from (A9)

$$S_{(3,2)} = S_{(3,3)} \ \ldots = o \tag{A11}$$

leading to (A7).

Hence from (A4) and (A7),

$$\sum_{m_1,z} (\ldots) = q^{-j_1(2j+4)} (F_o + (1-q^2) F_1) \tag{A12}$$

Now $$F_1 = \sum_s q^{s(a+b-c+2)} \frac{[a-s]!\,[b+s]!}{[s]!\,[c-s]!}$$

$$= q^{(b+1)c} \frac{[a-c]!\,[b]!\,[a+b+1]!}{[c]!\,[a+b-c+1]!} \tag{A13}$$

This corresponds to the identity A.5 of [14] or to (the corrected version) of A.6 of [15]. Using recursion relations between F_p and F_{p+1}, and simplifying one can show

$$F_o = q^{(b+1)c} \left\{ q^{a+c+1} [b+1] + q^{-b-1}[a-c+1] \right\} \frac{[a-c]!\,[b]!\,[a+b+1]!}{[c]!\,[a+b-c+2]!} \tag{A14}$$

From these results, substituting for a,b,c and simplifying,

$$< (j_1, j_2)\, j+1\, j \mid q^{(j_o^{(2)} - j_o^{(1)})} \mid (j_1, j_2)\, jj >$$

$$= -(q-q^1) \left(\frac{([j_1+j_2+1]^2 - [j+1]^2)\,([j+1]^2 - [j_1-j_2]^2)}{[2j+3]\,[2(j+1)]^2} \right)^{1/2} \tag{A.15}$$

$$< (j_1\, j_2)\, jj \mid q^{(j_o^{(2)} - j_o^{(1)})} \mid (j_1\, j_2)\, jj > = \frac{(-1)}{(q-q^{-1})\,[2(j+1)]} \left\{ (q^{2j_1+1} + q^{-2j_1-1})\, q^{-j-1} - (q^{2j_2+1} + q^{-2j_2-1})\, q^{j+1} \right\} \tag{A16}$$

In the above derivation, for simplicity, we set m=j. But the fact that the matrix elements of

$$X \equiv q^{(j_o^{(2)} - j_o^{(1)})} \tag{A.17}$$

are zero for $j' \geq j+2$ cannot depend on m. Moreover the operator X on space of the states $\mid jm >$ is a real symmetric matrix given by (A1). The symmetry between j and j' implies that non-zero matrix elements correspond to

$$j' = J+1,\ j,\ j-1 \tag{A18}$$

The explicit m-depence is obtained in a compact form (rather than from A1) from the relations

$$X^2 J_{\pm} + J_{\pm} X^2 - (q+q^{-1}) X J_{\pm} X = o \tag{A19}$$

$$J_{\pm}^2 X + X J_{\pm}^2 - (q+q^{-1}) J_{\pm} X J_{\pm} = o \tag{A20}$$

Using, however, the result (A18).

One obtains (suppressing the symbols $(j_1\, j_2)$ in the states) the non-zero elements

$$< j+1, m \mid X \mid j,m > = <j,m \mid X\, j+1, m >$$

$$= ([j+m+1]\,[j-m+1])^{1/2}\, f\,(j_1, j_2, j) \tag{A21}$$

$$< j,m \mid X \mid j,m > = f_1\,(j_1, j_2, j)\, q^m + f_2\,(j_1, j_2, j)\, q^{-m} \tag{A22}$$

where f_1 ,f_2 , f must satisfy

$$(q-q^{-1})^2 f_1(j)\, f_2(j) = [2j-1]\, f^2\,(j-1) - [2j+3]\, f^2(j) \tag{A23}$$

$$q^j\,(f_1(j) - q^2 f_1(j+1)) + q^{-j}\,(f_2\,(j) - q^{-2} f_2(j+1)) = o \tag{A24}$$

Comparing (A15) and (A21) one gets, immediately,

$$f_{(j)} = -\,(q-q^{-1}) \left\{ \frac{([h_1+1]^2 - [j+1]^2)\,([j+1]^2 - [h_2]^2)}{[2j+1]\,[2j+3]\,[2(j+1)]^2} \right\}^{1/2} + \tag{A25}$$

Comparing (A16) and (A22)

$$f_1(j)\, q^j + f_2(j)\, q^{-j} = -\,\frac{1}{(q-q^{-1})}\,\frac{1}{[2(j+1)]} \left\{ A_1\, q^{-j-1} - A_2\, q^{j+1} \right\} \tag{A26}$$

where $$h_1 \equiv j_1 + j_2\,,\; h_2 \equiv j_1 - j_2 \tag{A27}$$

$$A_1 \equiv q^{2j_1+1} + q^{-2j_1-1},\; A_2 \equiv q^{2j_2+1} + q^{-2j_2-1}$$

so that $$A_1 \mp A_2 = (q^{h_1+1} \mp q^{-h_1-1})\, q^{h_1} \mp q^{-h_2}) \tag{A28}$$

Starting with m = - j one gets, quite analogously to (A16), the combination

$$f_1(j)\, q^{-j} + f_2(j)\, q^{j}$$

and finally

$$f_1(j_1, j_2, j) = \frac{1}{(q-q^{-1})^2\,[2j]\,[2(j+1)]}\,(A_2\,(q^{2j+1} + q^{-2j-1}) - A_1\,(q+q^{-1})) \qquad \text{(A29)}$$

$$f_2(j_1, j_2, j) = \frac{1}{(q-q^{-1})^2\,[2j]\,[2(j+1)]}\,(A_1\,(q^{2j+1} + q^{-2j-1}) - A_2\,(q+q^{-1})) \qquad \text{(A30)}$$

It has been verified that our f, f_1, f_2 indeed satisfy (A23) and (A24).

We have now the result

$$\begin{aligned} X \mid (j_1 j_2)\, jm > &= q^{(J_o^{(2)} - J_o^{(1)})} \mid (j_1 j_2)\, jm > \\ &= ([j+m+1]\,[j-m+1])^{1/2}\, f(j_1, j_2, j) \mid (j_1, j_2)\, j+1\ m > \\ &\quad + ([j+m]\,[j-m])^{1/2}\, f(j_1, j_2, j-1) \mid (j_1, j_2)\, j-1\ m > \\ &\quad + (f_1(j_1, j_2, j)\, q^{m} + f_2(j_1, j_2, j)\, q^{-m}) \mid (j_1\, j_2)\, jm > \end{aligned} \qquad \text{(A31)}$$

where f, f_1, f_2 are given by (A25), (A28) and (A29) respectively.

More generally one obtains the matrix elements of $q^{(c_1 J_o^{(1)} + c_2 J_o^{(2)})}$ by replacing in (A1) the factor $q^{-m_1+m_2}$ by $q^{c_1 m_1 + c_2 m_2}$. For $c_1 = -1$, $c_2 = 1$, we have shown that the values of j' are truncated as in (A18) through (A7).

For $c_1 = 1$, $c_2 = -1$, i,e, $X^{-1} = q^{(J_o^{(1)} - J_o^{(2)})}$, (A32)

in the double sum (A15) the S_p (which is, more explicitly $S_{(3,p)}$ according to (A8) is replaced by

$$S_{(-1,p)} = q^{-\frac{1}{2}p(p+1)}\,(q-q^{-1})^{p} \qquad \text{(A33)}$$

Thus the matrix elements no longer vanish due to this factor for $j' \neq j, j \pm 1$. The matrix elements are, of course, explicitly given by

$$< j'm \mid X^{-1} \mid jm > = \sum_{m_1+m_1 = m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q q^{(m_1-m_2)} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix}_q \qquad \text{(A34)}$$

This is a well-defined real, symmetric matrix. But it is useful to express X^{-1} in another way. Define (with $q \to q^{-1}$ in (3.4))

$$\bar{J}_{\pm} \equiv (J_{\pm})_{q^{-1}} = J_{\pm}^{(1)} q^{J_o^{(2)}} + J_{\pm}^{(2)} q^{-J_o^{(1)}} \tag{A35}$$

and the corresponding (j,m) states as

$$|(j_1, j_2)\, jm >> = \sum_{m_1+m_2=m} \left[\!\left[\begin{matrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{matrix} \right]\!\right]_q |\, j_1 m_1, j_2 m_2 > \tag{A36}$$

Then $< (j_1, j_2)\, j+p, m \,|\, X^{-1} |\, (j_1, j_2)\, jm >_q$

$$= \sum_{jj''} <(j_1,j_2)\, j+p,m \,|(j_1,j_2)\, j''m>><<(j_1,j_2)j''m \,|\, X^{-1}|(j_1, j_2)\, j'm >>_q << (j_1,j_2)j'm \,|\, (j_1,j_2)jm > \tag{A37}$$

But

$$<<(j_1,j_2)j''m \,|\, X^{-1}|\, (j_1, j_2)\, j'm >>_q = < (j_1,j_2)j''m \,|\, X|\, (j_1, j_2)\, j'm >_{q^{-1}} \tag{A38}$$

Hence defining (as in [14] with u = o in (3.6) and (3.7) there)

$$R = \sum_{j,m} |\, (j_1,j_2)\, jm >> <(j_1,j_2)\, j_m \,|$$

$$(X^{-1})_q = (R\, X\, R^{-1})_{q^{-1}} \tag{A39}$$

(Such a relation holds trivially when instead of $X^{\pm 1}$ one has $q^{\pm J_o}$). The states (A36) and the interchange of j_1 and j_2 associated with the R-matrix are related through

$$\left[\!\left[\begin{matrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{matrix} \right]\!\right] = (-1)^{j_1+j_2-j} \left[\begin{matrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{matrix} \right] \tag{A40}$$

The consitency of the definitions, concerning the presence or absence of q-factors, can be checked with

J_1 = h, arbitrary and $j_2 = \frac{1}{2}$ or 1, when X^{-1} can be easily constructed directly. One should note, in direct construction of the inverse of X, that (for a given m)

$$\det X \Big|_m^{=} \prod_{m_1+m_2=m} q^{-(m_1-m_2)} \tag{A41}$$

This value, evident in the $| j,m_1, j_2\ m_2 >$ basis, is not altered in the $| (j_1\ ,j_2)\ jm >$ basis, the transformation being orthogonal. Thus for

$$j_2 = \frac{1}{2},\ \det X |_m = \prod_{m_2=\pm\frac{1}{2}} q^{-(m-2m_2)} = q^{-2m} \tag{A42}$$

More generally, when m_2 can take all the values from j_2 to $- j_2$

$$\det\ X |_m = q^{-(2h_2+1)\ m} \tag{A43}$$

Thus explicit expressions for the matrix elements of X^{-1} are also obtained by multiphying the corresponding cafactors in X by $(\det X)^{-1}$ for a fixed m, namely

$$(\det X)^{-1} = \prod_{m_1+m_2=m} q^{m_1-m_2} \tag{A44}$$

As a particularly simple example let us give X for the case

$$j_1 = h, \text{ arbitrary an } j_2 = \frac{1}{2}$$

Then (for a given m)

$$X = \frac{1}{[2h+1]} \begin{vmatrix} (q^m+[2h]q^{-m}) & -(q-q^{-1}([h+m+\frac{1}{2}]\,[h-m+\frac{1}{2}])^{1/2}) \\ -(q-q^{-1}([h+m+\frac{1}{2}]\,[h-m+\frac{1}{2}])^{1/2}) & (-q^m+[2h+2]q^{-m}) \end{vmatrix} \tag{A45}$$

and

$$\det X = q^{-2m} \tag{A46}$$

Corrected versions of eqns (3.22) and (3.28)

$$L_{\pm} \mid jm = -([j\pm m+1]\,[j\pm m+2])^{1/2}\left(\frac{q^{j+1}-q^{-j-1}}{q^{j+1}+q^{-j-1}}\,\alpha(j)\right) \mid j+1, m\pm 1>$$
$$-([j\mp m]\,[j\mp m-1])^{1/2}\left(\frac{q^{j}-q^{-j}}{q^{j}+q^{-j}}\,\alpha(j-1)\right) \mid j-1, m\pm 1>$$
$$+([j\mp m]\,[j\pm m+1])^{1/2} f_{+}(j) \mid j, m\pm 1> \qquad (3.22)$$

$$K_0^{(\pm)} \mid j,m> = q^{\mp(j+1)}([j\pm m+1])[j-m+1])^{1/2}\,\alpha(j) \mid j+1, m>$$
$$+ q^{\pm j}([j+m])[j-m])^{1/2}\,\alpha(j-1) \mid j-1, m>$$
$$+ [m]\,\beta(j) \mid j,m> \qquad (3.28)$$

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