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# SEMI-CLASSICAL ASYMPTOTICS BEYOND ALL ORDERS FOR SIMPLE SCATTERING SYSTEMS

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**Abstract**

The semi-classical limit $\epsilon \to 0$ of the scattering matrix $S$ associated with the equation $i\epsilon \frac{d\varphi(t)}{dt} = A(t)\varphi(t)$ is considered. If $A(x)$ is an analytic $n \times n$ matrix whose eigenvalues are real and non degenerate for all $x \in \mathbf{R}$, the matrix $S$ is computed asymptotically up to errors $O(e^{-\kappa\epsilon^{-1}})$, $\kappa > 0$. Moreover, for the case $n = 2$ and under further assumptions on the behaviour of the analytic continuations of the eigenvalues of $A(x)$, the exponentially small off-diagonal elements of $S$ are given by an asymptotic expression accurate up to *relative* errors $O(e^{-\kappa\epsilon^{-1}})$. The adiabatic transition probability for the time dependent Schrödinger equation, the semi-classical above barrier reflection coefficient for the stationary Schrödinger equation and the total variation of the adiabatic invariant of a time dependent classical oscillator are computed asymptotically to illustrate our results.

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## 1. INTRODUCTION

Let us consider the following well-known equations.

The first one, is the time-dependent Schrödinger equation for a two-level system,

$$i\hbar\frac{d\psi(t)}{dt} = H(\epsilon t)\psi(t) \tag{1.1}$$

$t \in \mathbf{R}$, $\psi(t) \in \mathcal{H} = \mathbf{C}^2$ and $H(\epsilon t)$ is a $2\times 2$ self-adjoint linear operator with two distinct real eigenvalues. The parameter $\epsilon$ is positive and small. The second equation is the stationary one-dimensional Schrödinger equation

$$-\hbar^2\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{1.2}$$

$x \in \mathbf{R}$, $\psi(x) \in \mathbf{C}$ and $V(x)$ is a bounded real-valued function. The real parameter $E$ is chosen in such a way that

$$E > \sup_{x\in\mathbf{R}} V(x) \tag{1.3}$$

The third equation is the equation of motion of a classical oscillator whose frequency varies with time

$$\ddot{v}(t) = -\omega^2(\epsilon t)v(t), \quad v(0) = u_0, \quad \dot{v}(0) = u_1 \tag{1.4}$$

This equation is of the same type as (1.2) since we assume that the real valued function $\omega(t)$ is bounded and such that

$$\inf_{t\in\mathbf{R}} \omega^2(t) > 0 \tag{1.5}$$

For the first two equations we are interested in the behaviour of the solution for $t \to +\infty$ or $x \to +\infty$, when the behaviour for $t \to -\infty$ or $x \to -\infty$ is fixed. Moreover we want to analyze this scattering situation when $\epsilon$ tends to zero and $\hbar = 1$ for equation (1.1), the so-called adiabatic limit, or $\hbar$ tends to zero for equation (1.2), the so-called semi-classical limit. For the initial value problem (1.4), we consider the adiabatic invariant $J$ defined as twice the ratio of the energy to the frequency

$$J(t,\epsilon) = \frac{|\dot{v}(t)|^2 + \omega^2(\epsilon t)|v(t)|^2}{\omega(\epsilon t)} \tag{1.6}$$



in the limit $\epsilon \to 0$. More precisely, we are interested in its total variation during the whole evolution

$$\Delta J(\epsilon) \equiv J(+\infty,\epsilon) - J(-\infty,\epsilon)$$

With this respect, we consider (1.4) more as a scattering problem again than as an initial value problem. All three problems are very closely related. Let $x = \epsilon t$ be a rescaled time for equations (1.1) and (1.4). Then equation (1.1) becomes with $\varphi(x) = \psi(t(x))$ and $\hbar = 1$

$$i\epsilon\frac{d\varphi(x)}{dx} = H(x)\varphi(x) \tag{1.7}$$

On the other hand, defining $u(x) = v(t(x))$ and

$$\varphi(x) = \begin{pmatrix} u(x) \\ i\epsilon\frac{du(x)}{dx} \end{pmatrix} \tag{1.8}$$

equation (1.4) is equivalent to

$$i\epsilon\frac{d\varphi(x)}{dx} = \begin{pmatrix} 0 & 1 \\ \omega^2(x) & 0 \end{pmatrix}\varphi(x), \quad \varphi(0) = \begin{pmatrix} u_0 \\ iu_1 \end{pmatrix} \tag{1.9}$$

Similarly, with

$$\varphi(x) = \begin{pmatrix} \psi(x) \\ i\epsilon\frac{d\psi(x)}{dx} \end{pmatrix} \tag{1.10}$$

and setting $\hbar = \epsilon$, equation (1.2) becomes

$$i\epsilon\frac{d\varphi(x)}{dx} = \begin{pmatrix} 0 & 1 \\ E - V(x) & 0 \end{pmatrix}\varphi(x) \tag{1.11}$$

Thus the three equations (1.7), (1.9) and (1.11) are particular cases of

$$i\epsilon\frac{d\varphi(x)}{dx} = A(x)\varphi(x) \tag{1.12}$$

where $A(x)$ is a linear operator on $\mathcal{H} = \mathbf{C}^2$ with two distinct real eigenvalues. Our purpose is to study a scattering problem for (1.12) in the "semi-classical" limit $\epsilon$ tends to zero under the hypothesis that $A(x)$ is analytic, has two distinct real eigenvalues for all $x \in \mathbf{R}$ and well-defined limits when $x \to \pm\infty$. It is natural to express the solutions of (1.12) as linear combinations of eigenvectors of $A(x)$:

$$\varphi(x) = \sum_{i=1}^{2} c_j(x)e^{-i/\epsilon\int_0^x e_j(x')dx'}\varphi_j(x) \tag{1.13}$$

where $A(x)\varphi_j(x) = e_j(x)\varphi_j(x)$. Our conditions on the behaviour of $A(x)$ for large $|x|$ will imply that

$$\lim_{x\to\pm\infty} c_j(x) = c_j(\pm\infty) \tag{1.14}$$

exist, so that the following scattering problem is well-defined:

Given $c_j(-\infty), j = 1,2$ find $c_j(+\infty), j = 1,2$, i.e. find the matrix $S$ defined by

$$\begin{pmatrix} c_1(+\infty) \\ c_2(+\infty) \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} c_1(-\infty) \\ c_2(+\infty) \end{pmatrix} \tag{1.15}$$

There is a "canonical" choice of eigenvectors of $A(x)$ specified (up to a global factor) by the condition

$$P_j(x)\frac{d\varphi_j(x)}{dx} = 0 \tag{1.16}$$

where $P_j(x)$ is the eigenprojection corresponding to $e_j(x)$. In particular, it is immediate to verify that for $A(x)$ given by (1.9) or by (1.11) with the identification $\omega^2(x) \equiv E - V(x)$, the eigenvectors associated with $e_j(x) = (-1)^j\omega(x)$

$$\varphi_1(x) = \begin{pmatrix} \dfrac{1}{\sqrt{\omega(x)}} \\ -\sqrt{\omega(x)} \end{pmatrix}, \quad \varphi_2(x) = \begin{pmatrix} \dfrac{1}{\sqrt{\omega(x)}} \\ +\sqrt{\omega(x)} \end{pmatrix} \tag{1.17}$$

satisfy (1.16), so that (1.13) gives the solutions of (1.9) and (1.11) as superpositions of the two well-known WKB functions

$$e^{-i/\epsilon\int_0^x e_j(x')dx'}\varphi_j(x) \tag{1.18}$$

When this choice of eigenvectors is made, a solution $\varphi(x)$ of (1.12) characterized by $c_j(-\infty) = 1$ and $c_k(-\infty) = 0, k \neq j$, satisfies

$$\sup_{x\in\mathbf{R}} |\varphi(x) - e^{-i/\epsilon\int_0^x e_j(x')dx'}\varphi_j(x)| = O(\epsilon) \tag{1.19}$$

Consequently,

$$S = 1 + O(\epsilon) \tag{1.20}$$

The approximations (1.19) and (1.20) are true without assuming analyticity of $A(x)$. On the other hand, if analyticity holds, we can approximate the solutions of (1.12) and thus determine the matrix $S$ up to error terms $O(\exp(-\kappa\epsilon^{-1})), \kappa > 0$ (see corollary 2.5)

$$S_{kj} = s_j(\epsilon)\delta_{kj} + O(\exp(-\kappa\epsilon^{-1})) \tag{1.21}$$



where $|s_j(\epsilon)| = O(1)$. These results are corollaries of the iterative scheme presented in section 2, which will be used in section 3. Actually they are derived for $A(x)$ a $n \times n$ matrix whose eigenvalues are assumed to be real and non degenerate for any $x \in \mathbf{R}$.

The asymptotic formulae (1.21) imply in particular that the non-diagonal terms of $S$ are $O(\exp(-\kappa\epsilon^{-1}))$. These terms are important in applications because they are related, for equation (1.1), to the probability of a quantum transition between the two levels of the system or, in the case of equation (1.2), to the above barrier reflection cœfficient and, in the case of equation (1.4) to the quantity $\Delta J(\epsilon)$. Under further hypotheses on the analytic behaviour of the eigenvalues of $A(x)$ we show that it is possible to find an asymptotic expression for $S_{21}$ or $S_{12}$ accurate up to exponentially small *relative* corrections. The asymptotic formula is expressed by means of the complex degeneracy points of the analytic continuations of eigenvalues $e_j(x)$. If there are $p$ contributing degeneracy points, the asymptotic expression reads (see theorem 3.7 and (2.43), (2.45))

$$S_{21} = \sum_{k=0}^{p} e^{-i\theta^*(k,\epsilon)}e^{-i\gamma^*(k,\epsilon)\epsilon^{-1}} + e^{-\tau\epsilon^{-1}}O\left(e^{-\kappa\epsilon^{-1}}\right), \quad \kappa, \tau > 0 \tag{1.22}$$

where $\theta^*(k,\epsilon)$ is $O(1)$ and $Im\gamma^*(k,\epsilon) = -\tau + O(\epsilon^2)$, $k = 1, \cdots, p$. It should be noted that the error term is smaller by an exponentially decreasing factor than the least significant term in the sum (1.22). This asymptotic formula is proven in section 3, which is the main part of the paper. It is obtained by combination of our iterative scheme with a method due to Fröman, Fröman [1]. We give in section 4 explicit formulae in terms of $A(x)$ for the expressions $\theta^*(k,\epsilon)$ and $\gamma^*(k,\epsilon)$ appearing in (1.22). The consequences of our asymptotic analysis of the matrix $S$ for the applications mentioned above are formulated in section 4 as well.

Before ending this introduction let us make some very brief comments on the vast literature devoted to the exponential decay of non diagonal elements of the matrix $S$. We do not attempt at all to give an exhaustive account of it but we want to set in context our work relative to the main results. We quote these results according to their content and not chronologically. The reader may find further references in the books [2] and [3]. The intermediate result (1.21) is not new, see [2], [3] and references therein, but we nevertheless obtain a new derivation of it in section 2. For recent related results see also

[4]. The asymptotic expression (1.22) generalizes however several rigorous results which were obtained either in the case of equations (1.7) and (1.11) or in the study of $\Delta J(\epsilon)$. When one complex eigenvalue degeneracy only contributes, it is known since the works [1], [5], [6] that

$$S_{21} = e^{-i\theta} e^{-i\gamma\epsilon^{-1}} + O(\epsilon) e^{Im\gamma\epsilon^{-1}}, \quad Im\gamma < 0 \tag{1.23}$$

with $\theta = \pi/2$ for equation (1.11) and, provided $A(x)$ is a real symmetric matrix, for equation (1.7) as well. It was shown recently that when $A(x)$ is a hermitian matrix in (1.7), $\theta$ can take any complex value [7], see also [8]. A corresponding asymptotic expression for $\Delta J(\epsilon)$ in this situation can be found in [9], [10], [11]. See also [12] for more recent related results. The expression (1.23) was then generalized in two ways for equations (1.7) and (1.11). First, when several eigenvalue degeneracy points contribute to the asymptotics of $S_{21}$, it was proven using standard stretching and matching techniques that [5], [13]

$$S_{21} = \sum_{k=0}^{p} e^{-i\theta(k)} e^{-i\gamma(k)\epsilon^{-1}} + O(\epsilon^{\alpha}) e^{Im\gamma\epsilon^{-1}} \tag{1.24}$$

where $0 < \alpha < 1$ and $Im\gamma(k) = Im\gamma < 0$, $\forall k$. The leading term of (1.24) gives rise to the so called "Stückelberg oscillations" as $\epsilon \to 0$, a phenomenon which is illustrated numerically in [13]. Note also that the error term is $O(\epsilon^{\alpha})$ instead of $O(\epsilon)$ which is a common drawback of the method employed to get (1.24). Then, higher order corrections to formula (1.23) were studied systematically in [14], [15] for equation (1.11) and in [16] for equation (1.7):

$$S_{21} = e^{-i\theta^q(\epsilon)} e^{-i\gamma^q(\epsilon)\epsilon^{-1}} + O\left(\epsilon^{q+1}\right) e^{-\tau\epsilon^{-1}}, \quad \forall q \in \mathbf{N}, \quad \tau > 0 \tag{1.25}$$

where $Im\gamma^q(\epsilon) = -\tau + O(\epsilon^2)$ and $\theta^q(\epsilon) = O(1)$. The iterative scheme of section 2 was introduced in [16] to derive this expression in the adiabatic context. Thus the asymptotic expression (1.22) captures all the features of these previous results and it holds for more general situations than those described by the particular matrices in (1.7) and (1.11). Moreover, it yields an expression accurate up to exponentially small corrections for the logarithm of $S_{21}$ since we can write for $p = 1$

$$\ln S_{21} = -i\frac{\gamma^*(\epsilon)}{\epsilon} + \theta^*(\epsilon) + O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{1.26}$$



## 2. APPROXIMATE SOLUTION

We consider a slightly more general problem than in the introduction. Let $\mathcal{H} = \mathbf{C}^n$, with the usual scalar product, and $A(x), x \in \mathbf{R}$, be a linear operator on $\mathcal{H}$. We study the equation $\left(' = \frac{d}{dx}\right)$

$$\begin{aligned} i\epsilon U'(x, x_0) &= A(x)U(x, x_0) \\ U(x_0, x_0) &= \mathbf{1} \end{aligned} \tag{2.1}$$

under the condition that $A(x)$ is analytic in $x$ and for each $x$ the spectrum of $A(x)$ consists of $n$ distinct real eigenvalues $e_1(x) < \ldots < e_n(x)$, with corresponding eigenprojections $P_1(x), \ldots, P_n(x)$. Note that the evolution $U$ is not unitary in general.

In order to find approximate solution of (2.1) we first consider another problem. Let $\psi(x)$ be a solution of

$$i\epsilon\psi'(x) = A(x)\psi(x) \tag{2.2}$$

If $Q(x_0)$ is a projection such that $Q(x_0)\psi(x_0) = \psi(x_0)$, then for any $x$ we have a projection $Q(x)$ such that $Q(x)\psi(x) = \psi(x)$. Indeed, if $U(x, x_0)$ is the matrix solution of (2.2) such that $U(x_0, x_0) = \mathbf{1}$, we take

$$Q(x) = U(x, x_0)Q(x_0)U(x_0, x) \tag{2.3}$$

The projection $Q(x)$ is a solution of

$$i\epsilon Q'(x) = [A(x), Q(x)] \tag{2.4}$$

with the notation $[A, B] \equiv AB - BA$. Let us suppose that at $x_0$ we have a complete set of projections $Q_j(x_0)$, i.e. $Q_j(x_0)Q_k(x_0) = Q_k(x_0)\delta_{jk}$, $\sum_j Q_j(x_0) = \mathbf{1}$. Then the $Q_j(x)$ form a complete set of projections as well and using the fact that for any projection $P(x)$ we have $P(x)P'(x)P(x) = 0$, it follows that

$$Q_j'(x) = \left[\sum_m Q_m'(x)Q_m(x), Q_j(x)\right] \tag{2.5}$$

Therefore we have for all $j$

$$\left[A(x) - i\epsilon \sum_m Q'_m(x)Q_m(x), Q_j(x)\right] = 0 \tag{2.6}$$

We look for approximate solutions of this equation. Since $[A(x), P_j(x)] \equiv 0$, the eigenprojections $P_j(x)$ are approximate solutions of (2.6) up to an error term $O(\epsilon)$. Let

$$A_1(x) := A(x) - i\epsilon K_0(x) \tag{2.7}$$

with

$$K_0(x) := \sum_m P'_m(x)P_m(x) \tag{2.8}$$

By perturbation theory, if $\epsilon$ is small enough, $A_1(x)$ has $n$ distinct eigenvalues $e_{1,j}(x)$ with corresponding eigenprojections $P_{1,j}(x), j = 1, \ldots, n$, such that $e_{1,j}(x) = e_j(x) + O(\epsilon^2)$, and $P_{1,j}(x) = P_j(x) + O(\epsilon)$. Indeed, $e_{1,j}(x) = e_j(x) - i\epsilon tr\,(P_j(x)K_0(x)) + O(\epsilon^2)$ and $P_j(x)K_0(x)P_j(x) = 0$. The projections $P_{1,j}(x)$ are approximate solutions of (2.6) up to an error term $O(\epsilon^2)$ since $[A_1(x), P_{1,j}(x)] = 0$.

Let

$$K_1(x) := \sum_m P'_{1,m}(x)P_{1,m}(x) \tag{2.9}$$

and

$$A_2(x) := A(x) - i\epsilon K_1(x) \tag{2.10}$$

Again, for $\epsilon$ small enough, $A_2(x)$ has $n$ distinct eigenvalues $e_{2,j}(x)$ with corresponding eigenprojections $P_{2,j}(x)$. Since $A_2(x) = A_1(x) + i\epsilon(K_0(x) - K_1(x))$ and $K_0(x) - K_1(x) = O(\epsilon)$, $P_{2,j}(x)$ is an approximate solution of (2.6) up to an error term $O(\epsilon^3)$. We can iterate this procedure. At the $q^{th}$ iteration we have approximate solutions $P_{q,j}(x)$, up to order term $O(\epsilon^{q+1})$), which are eigenprojections of

$$A_q(x) := A(x) - i\epsilon K_{q-1}(x) \tag{2.11}$$

with

$$K_{q-1}(x) = \sum_m P'_{q-1,m}(x)P_{q-1,m}(x) \tag{2.12}$$



We now construct approximate solutions for (2.1). Let $Q_m(x)$ be a complete smooth family of projections of $\mathcal{H}, Q_m(x)Q_n(x) = \delta_{mn}Q_m(x)$ and $\sum_m Q_m(x) = \mathbf{1}$. We say that an evolution $V(x, x')$, $(V(x', x') = \mathbf{1}, V(x_2, x_1)V(x_1, x_0) = V(x_2, x_0))$, follows the decomposition of $\mathcal{H}$,

$$\mathcal{H} = \bigoplus_m Q_m(x)\mathcal{H}$$

if for all $x, x'$

$$Q_m(x)V(x, x') = V(x, x')Q_m(x') \tag{2.13}$$

It is known (see [17] or [18]) that a smooth evolution with property (2.13) is solution of an equation of the type

$$V'(x, x_0) = (B(x) + \sum_m Q'_m(x)Q_m(x))V(x, x_0), \quad V(x_0, x_0) = \mathbf{1} \tag{2.14}$$

where $B(x)$ is such that

$$[B(x), Q_m(x)] = 0 \quad \forall m \tag{2.15}$$

The idea is to construct approximate solutions of (2.1) by choosing evolutions which follow the decomposition of $\mathcal{H}$ into

$$\mathcal{H} = \bigoplus_m P_{q,m}(x)\mathcal{H} \tag{2.16}$$

Therefore we define $U_q(x, x_0)$ as the solution of

$$i\epsilon U'_q(x, x_0) = (A_q(x) + i\epsilon K_q(x))U_q(x, x_0), \quad U_q(x_0, x_0) = \mathbf{1} \tag{2.17}$$

The next lemma which is taken from [19] gives the main estimate which we need in order to control the error term for the approximate solution $U_q(x, x_0)$. This lemma is also used in section 3.

For any $z \in \mathbf{C}$ and $r > 0$ let $D(z; r) = \{z' \in \mathbf{C} : |z' - z| < r\}$ and $\partial D(z; r) = \{z' \in \mathbf{C} : |z' - z| = r\}$. Given $z_0 \in \mathbf{C}$ and $r_0 > 0$ let $A(z)$ be analytic in $D(z_0; r_0)$ with a spectrum consisting of $n$ distinct eigenvalues $e_j(z)$ with corresponding eigenprojection $P_j(z)$ for all $z \in D(z_0; r_0)$. We define $A_q(z), K_q(z), P_{q,j}(z)$ and $e_{q,j}(z)$ as above by the iteration method based on (2.11) and (2.12). We set $R(z, \lambda) = (A(z) - \lambda\mathbf{1})^{-1}$.

**Lemma 2.1**

*Let $z_0 \in \mathbf{C}, r_0 > 0$ and $A(z)$ be defined on $D(z_0; r_0)$ with the above properties. Let $r_1 > 0$ and $D_j := D(e_j(z_0); 2r_1)$ be $n$ disjoint discs in $\mathbf{C}$, $j = 1, \ldots, n$, such that for all $z \in D(z_0; r_0)$*

$$e_j(z) \in D(e_j(z_0); r_1)$$

*Let*

$$a = a(z_0) := \sup_j \sup_{\lambda \in \partial D_j} \sup_{z \in D(z_0; r_0)} \|R(z, \lambda)\| < \infty$$

*and*

$$b = b(z_0) := \sup_{z \in D(z_0; r_0)} \|K_0(z)\| < \infty$$

*Then there exist $\epsilon^* = \epsilon^*(a, b) > 0$ and $c = c(r_0, r_1, a, b) < \infty$ such that*

$$\|K_q(z) - K_{q-1}(z)\| \leq b\epsilon^q c^q q!$$

*and*

$$\|K_q(z)\| \leq 2b$$

*for all $z \in D(z_0; r_0)$, all $0 < \epsilon \leq \epsilon^*$, and all $q \leq q^*(\epsilon) = \left[\frac{1}{ec\epsilon}\right]$, where $[y]$ is the integer part of $y$ and $e$ is the basis of the neperian logarithm.*

**Remark:** The proof of this lemma is given in [19] for the case $P_1 + P_2 = \mathbf{1}$. However, it is not difficult to extend it for the case $\sum_{j=1}^n P_j = \mathbf{1}$, $n \geq 2$.

**Corollary 2.2**

*Let the hypothesis of lemma 2.1 be satisfied. Then for all $q \leq q^*$*

$$e_{q,j}(z) = e_j(z) + O(b\epsilon^2)$$

**Proof :** Since $P_j(z)K_0(z)P_j(z) = 0$ the statement is true for $q = 1$. For $q \geq 2$ we have

$$\|A_q(z) - A_1(z)\| \leq \epsilon \sum_{m=1}^{q-1} \|K_m(z) - K_{m-1}(z)\| \leq \epsilon b \sum_{m=1}^{q^*} \epsilon^m c^m m! = O(\epsilon^2 b) \tag{2.18}$$



and therefore the statement follows from perturbation theory. □

We now apply lemma 2.1 and its corollary 2.2 in order to control the norm of $U_q(x, x_0)$. It is crucial that $U_q$ follows the decomposition of $\mathcal{H}$ into $\bigoplus_{m \geq 1} P_{q,m}(z)\mathcal{H}$.

**Corollary 2.3**

*Let $r_0 > 0$ be such that for each $x \in \mathbf{R}$ the hypotheses of lemma 2.1 are satisfied on $D(x; r_0)$ with constants $r_1$ and $a$ independent of $x$ and with constants $b(x) \leq b < \infty$. Then for $\epsilon \leq \epsilon^*$ and $q \leq q^*$*

$$\|U_q(x; x_0)\| \leq \exp\left\{O\left(\left|\int_{x_0}^{x} b(x')dx'\right|\right)\right\}$$

**Proof :** We introduce the evolution $W_q(x, x_0)$,

$$W_q'(x, x_0) = K_q(x)W_q(x, x_0), \quad W_q(x_0, x_0) = \mathbf{1} \tag{2.19}$$

From lemma 2.1 we have

$$\|W_q(x, x_0)\| \leq \exp\left(2\left|\int_{x_0}^{x} b(x')dx'\right|\right) \tag{2.20}$$

Let us choose $n$ eigenvectors $\varphi_{q,j}(0)$ of $A_q(0)$ at $x = 0$. The vectors

$$\varphi_{q,j}(x) := W_q(x, 0)\varphi_{q,j}(0), \quad j = 1, \ldots, n \tag{2.21}$$

are eigenvectors of $A_q(x)$ since $W_q(x, 0)$ interpolates between $Q_q(0)$ and $Q_q(x)$ (see (2.13) and (2.14)) and by definition

$$P_{q,j}(x)\varphi_{q,j}'(x) = 0, \quad j = 1, \ldots, n \tag{2.22}$$

Let us write $U_q(x, x_0) := W_q(x, x_0)\Phi_q(x, x_0)$. The unknown operator $\Phi_q(x, x_0)$ is the solution of

$$\begin{aligned} i\epsilon\Phi_q'(x, x_0) &= W_q(x_0, x)A_q(x)W_q(x, x_0)\Phi_q(x, x_0) \\ \Phi_q(x_0, x_0) &= \mathbf{1} \end{aligned} \tag{2.23}$$

The operator $W_q(x_0,x)A_q(x)W_q(x,x_0)$ has eigenvalues $e_{q,j}(x)$ with eigenvectors $\varphi_{q,j}(x_0)$. Therefore

$$\Phi_q(x,x_0)\varphi_{q,j}(x_0) = \exp\left(-i\epsilon^{-1}\int_{x_0}^{x} e_{q,j}(x')dx'\right)\varphi_{q,j}(x_0), \quad j=1,\dots,n \tag{2.24}$$

From corollary 2.2 and the reality of $e_j(z)$,

$$\left|Im\left(\int_{x_0}^{x} e_{q,j}(x')dx'\right)\right| \le O(\epsilon^2)\left|\int_{x_0}^{x} b(x')dx'\right| \tag{2.25}$$

hence

$$\|U_q(x,x_0)\| \le \exp\left\{(2+O(\epsilon))\left|\int_{x_0}^{x} b(x')dx'\right|\right\} \tag{2.26}$$

□

Note that in the above proof we have factorized the evolution $U_q(x,x_0)$ as the product

$$U_q(x,x_0) = W_q(x,x_0)\Phi_q(x,x_0) \tag{2.27}$$

where $\Phi_q$ only is singular in the limit $\epsilon \to 0$ and $\|\Phi_q\| = O(1)$. Since in our simple case $\Phi_q$ is known explicitly, the solution $\psi(x)$ of

$$\begin{aligned} i\epsilon\psi'(x) &= (A_q(x) + i\epsilon K_q(x))\psi(x) \\ \psi(x_0) &= \psi_0 \end{aligned} \tag{2.28}$$

can be written as

$$\begin{aligned} \psi(x) &= U_q(x,x_0)\psi(0) \\ &= \sum_{j\ge1} c_{q,j}(x_0)\exp\left(-i/\epsilon\int_{x_0}^{x} e_{q,j}(x')dx'\right)\varphi_{q,j}(x) \end{aligned} \tag{2.29}$$

where the $c_{q,j}(x_0)$ are defined by the identity

$$\psi_0 = \sum_{j\ge1} c_{q,j}(x_0)\varphi_{q,j}(x_0) \tag{2.30}$$



**Theorem 2.4**

*Let $r>0$ and $g>0$ and let $A(x)$ be analytic in $\Omega_r = \{z = x+iy : x,y \in \mathbf{R}, |y|<r\}$. Let the spectrum of $A(x)$ consist of $n$ real distinct eigenvalues $e_j(x), j=1,\dots,n$, such that for all $x \in \mathbf{R}$*

$$|e_k(x) - e_j(x)| \ge g, \quad k \ne j$$

*Let*

$$\|K_0(x)\| = \|\sum_{j\ge1} P_j'(x)P_j(x)\|$$

*be an integrable function of $x$ which tends to zero as $|x| \to 0$. Then there exist constants $\epsilon^* > 0, C' < \infty, \kappa > 0$ and an approximate solution $V(x,x_0)$ of the equation*

$$\begin{aligned} i\epsilon U'(x,x_0) &= A(x)U(x,x_0) \\ U(x_0,x_0) &= \mathbf{1} \end{aligned}$$

*such that*

$$\sup_{x,x_0\in\mathbf{R}} \|U(x,x_0) - V(x,x_0)\| \le C'\exp(-\kappa\epsilon^{-1})$$

**Remarks :**

i) Neither $U$ nor $V$ are unitary in general, however both their norms are $O(1)$ as $\epsilon \to 0$.

ii) Note that $\lim_{x\to\pm\infty} A(x)$ need not exist, since we only require that $\lim_{x\to\pm\infty} P_j(x) = P_j(\pm\infty)$ exists.

**Proof :** By standard arguments of perturbation theory we can verify the hypothesis of Corollary 2.3 with $b(x)$ integrable on $\mathbf{R}$ (see e.g. section 2 of [20]). We set

$$V(x,x_0) := U_{q^*}(x,x_0) \tag{2.31}$$

with $q^*(\epsilon)$ given in lemma 2.1. The operator $R(x) := V(x_0,x)U(x,x_0)$ is solution of

$$\begin{aligned} i\epsilon R'(x) &= V(x_0,x)(-A_{q^*}(x) - i\epsilon K_{q^*}(x) + A(x))V(x,x_0)R(x) \\ &= i\epsilon V(x_0,x)(K_{q^*-1}(x) - K_{q^*}(x))V(x,x_0)R(x) \end{aligned} \tag{2.32}$$

From the integrability of $b(x)$ and lemma 2.1 we have

$$\begin{aligned}\|R(x)-1\| &\le C''(c\epsilon)^{q^*}q^*! \\ &\le C''(c\epsilon q^*)^{q^*} \\ &\le eC''\exp(-\kappa\epsilon^{-1})\end{aligned} \tag{2.33}$$

where $\kappa = \frac{1}{ec}$. Hence

$$\begin{aligned}\|U(x,x_0)-V(x,x_0)\| &\le \|V(x,x_0)\|\|R(x)-1\| \\ &\le C'\exp(-\kappa\epsilon^{-1})\end{aligned} \tag{2.34}$$

□

We assume that the hypotheses of theorem 2.4 are satisfied and we determine the matrix $S$ up to an error term $O(e^{-\kappa\epsilon^{-1}})$. Since $\|K_0(x)\|$ and thus $\|K_{q-1}(x)\|$ tend to zero at infinity in an integrable way (see lemma 2.1 and corollary 2.3),

$$\lim_{x\to\pm\infty}\|A_q(x)-A(x)\| = 0, \text{ for all } q \le q^* \tag{2.35}$$

and

$$\lim_{x\to\pm\infty} W_q(x,x_0) = W_q(\pm\infty,x_0), \text{ for all } q \le q^* \tag{2.36}$$

Let us choose a point $x_0$ and a set of eigenvectors $\varphi_j(x_0)$ of $A(x_0), j = 1,\dots,n$. Using $W_0(x,x_0)$ we define a set of eigenvectors of $A(x)$ for all $x$,

$$\varphi_j(x) = W_0(x,x_0)\varphi_j(x_0) \tag{2.37}$$

Let $\psi$ be a solution of

$$i\epsilon\psi'(x) = A(x)\psi(x) \tag{2.38}$$

and let us write $\psi$ as

$$\psi(x) = \sum_{j\ge1} c_j(x)e^{-i/\epsilon\int_{x_0}^{x} e_j(x')dx'}\varphi_j(x) \tag{2.39}$$

Since $\|K_0(x)\|$ is integrable, $\lim_{x\to\pm\infty} c_j(x)$ exists (see e.g. lemma 3.2 below).

Let us now define a set of eigenvectors of $A_{q^*}(x)$ by choosing

$$\varphi_j^*(-\infty) \equiv \varphi_{q^*,j}(-\infty) := \varphi_j(-\infty) \tag{2.40}$$



and by setting

$$\varphi_j^*(x) = W_{q^*}(x,-\infty)\varphi_j(-\infty) \tag{2.41}$$

We can also write $\psi(x)$ as ($e_j^* \equiv e_{q^*,j}$)

$$\begin{aligned}\psi(x) &= \sum_{j\ge1} c_j^*(x)e^{-i/\epsilon\int_{x_0}^{x} e_j^*(x')dx'}\varphi_j^*(x) \\ &= \sum_{j\ge1} c_j^*(x)e^{-i/\epsilon\int_{x_0}^{x} e_j(x')dx'}e^{-i/\epsilon\int_{x_0}^{x}(e_j^*(x')-e_j(x'))dx'}\varphi_j^*(x)\end{aligned} \tag{2.42}$$

From (2.39), (2.42) and $\lim_{x\to-\infty}\|P_{q^*,j}(x)-P_j(x)\| = 0$ we have

$$\begin{aligned}\lim_{x\to-\infty} e^{+i/\epsilon\int_{x_0}^{x} e_j(x')dx'}P_j(x)\psi(x) &= c_j(-\infty)\varphi_j(-\infty) \\ &= e^{-i/\epsilon\int_{x_0}^{-\infty}(e_j^*(x')-e_j(x'))dx'}c_j^*(-\infty)\varphi_j(-\infty)\end{aligned} \tag{2.43}$$

On the other hand

$$\begin{aligned}\varphi_j^*(\infty) &= W_{q^*}(\infty,-\infty)\varphi_j(-\infty) \\ &= W_{q^*}(\infty,-\infty)W_0(-\infty,x_0)\varphi_j(x_0) \\ &= W_{q^*}(\infty,-\infty)W_0(-\infty,\infty)\varphi_j(+\infty) \\ &\equiv e^{-i\beta_j^*}\varphi_j(+\infty)\end{aligned} \tag{2.44}$$

the last equality defining the factor $e^{-i\beta_j^*}$ where $\beta_j^*$ is in general complex. Thus, similarly,

$$e^{-i/\epsilon\int_{x_0}^{+\infty}(e_j^*(x')-e_j(x'))dx'}e^{-i\beta_j^*}c_j^*(\infty) = c_j(\infty) \tag{2.45}$$

Let $\psi$ be a solution of (2.38) characterized by $c_j(-\infty) = 1$ and $c_k(-\infty) = 0$ for $k \ne j$ which we decompose as in (2.42). From theorem 2.4 and (2.29) an approximate solution of $\psi(x)$ is obtained by replacing $c_j^*(x)$ by $c_j^*(x_0)$ in (2.42), and we have

$$\sup_{x\in\mathbb{R}}|c_j^*(x)-c_j^*(x_0)| = O(e^{-\kappa\epsilon^{-1}}), \quad j = 1,\dots,n \tag{2.46}$$

Therefore

$$c_k(+\infty) = O(e^{-\kappa\epsilon^{-1}}), \quad k \ne j \tag{2.47}$$

and

$$c_j(\infty) = e^{-i\beta_j^*} e^{-i/\epsilon \int_{-\infty}^{+\infty} (e_j^*(x') - e_j(x'))dx'} + O(e^{-\kappa\epsilon^{-1}}) \tag{2.48}$$

The matrix $S$ defined in the introduction is then given by

**Corollary 2.5**

$$S_{kj} = e^{-i\beta_j^*} e^{-i/\epsilon \int_{-\infty}^{+\infty} (e_j^*(x') - e_j(x'))dx'} \delta_{kj} + O(e^{-\kappa\epsilon^{-1}})$$

**Remark:** It should be recalled that we did not write explicitly the $\epsilon$-dependence of $e_j^*$ or $P_{q^*,j}$, but in corollary 2.5 we have $\beta_j^* = \beta_j^*(\epsilon)$ and $e_j^*(x') = e_j^*(x', \epsilon)$.



## 3. ASYMPTOTICS OF THE NON DIAGONAL PART OF THE MATRIX $S$

### 3.1 Stokes lines

From now on, we deal with the case $\mathcal{H} = \mathbf{C}^2$, and we compute in this section an asymptotic expression for $S_{21}$, which, in the simplest case, reads

$$S_{21} = e^{-i\theta^*(\epsilon)} e^{-i\gamma^*(\epsilon)\epsilon^{-1}} \left(1 + O\left(e^{-\kappa\epsilon^{-1}}\right)\right) \tag{3.1}$$

The idea is to combine our iterative scheme (2.11), (2.12) with an analysis in the complex plane by a method due to Fröman, Fröman [1]. In order to perform the analysis we need some precise information about the analytic extension of $A(x)$ into the complex plane. In particular, we must control the Stokes lines of the problem (condition II below). Thus, in this subsection we introduce the notion of Stokes lines and give the conditions which are needed in order to make use of the method of [1] in the next subsection.

Without restricting the generality we impose $trA(x) \equiv 0$. Thus we have $A(x)^2 = \rho(x)\mathbf{1}$, this identity defining the function $\rho(x)$. The eigenvalues of $A(x)$ are then $e_1(x) = -e_2(x)$ and $e_2(x) = \sqrt{\rho(x)}$, with $\sqrt{1} = 1$.

The corresponding eigenprojections are given by

$$P_j(x) = \frac{1}{2}\left(\mathbf{1} + \frac{A(x)}{e_j(x)}\right) \tag{3.2}$$

On $\mathbf{R}$ the eigenvalues are real and distinct and we suppose that there exists $g > 0$ with $\rho(x) > g$, for all $x \in \mathbf{R}$.

Let $\Omega$ be a domain of $\mathbf{C}$, symmetric with respect the real axis, containing $\mathbf{R}$, on which $A$ has an analytic extension. Since $\rho$ is real on $\mathbf{R}$ we have for any $z \in \Omega$, $\rho(\overline{z}) = \overline{\rho(z)}$. The analysis of $S_{21}$ is done by working in the upper half-plane only, whereas the analysis of $S_{12}$ is performed in the lower half-plane, as we shall see below. The eigenvalues and eigenprojections also have analytic extensions in $\Omega$, but it is clear that the zeros of $\rho$ in $\Omega$ are singular points for these objects. Some of these singularities play a dominant role in the determination of $S_{jk}$, $j \neq k$.

As in section 2 we introduce new operators $A_q(z)$ for all $z \in \Omega \backslash \{z' : \rho(z') = 0\}$ by the iteration scheme (2.11) and (2.12). In our case we can write

$$\begin{aligned} K_0(z) &= P_1'(z)P_1(z) + P_2'(z)P_2(z) \\ &= [P_1'(z), P_1(z)] = \frac{1}{4\rho(z)}[A'(z), A(z)] \end{aligned} \tag{3.3}$$

where $' = \frac{d}{dz}$ and we compute for all $q$

$$\begin{aligned} A_q(z) &= A(z) - i\epsilon\left[P_{q-1,1}'(z), P_{q-1,1}(z)\right] \\ &= A(z) - \frac{i\epsilon}{4\rho_{q-1}(z)}\left[A_{q-1}'(z), A_{q-1}(z)\right] \end{aligned} \tag{3.4}$$

Indeed, we have $trA_{q'}(z) \equiv 0$, because the trace of a commutator is zero. Thus $\rho_{q'}(z)$ is defined by $A_{q'}^2(z) = \rho_{q'}(z)\mathbf{1}$. Hence the eigenvalues $e_{q',j}(z) = (-1)^j\sqrt{\rho_{q'}(z)}$ and $P_{q',1}(z)$ is given by an expression similar to (3.2). Equation (3.4) clearly shows that although the eigenvectors and eigenprojections are multivalued in $\Omega$ when we perform the analytic continuation, this is not the case for $A_q(z)$. In the above construction we must avoid the zeros of $\rho_{q'}(z)$ for $q' \leq q-1$.

**Condition I:**

*The set $X = \{z \in \Omega : \rho(z) = 0\}$ is a finite set.*

*Let $r_2 > 0$ such that $D(z_j; r_2) \cap D(z_k; r_2) = \emptyset$ for all $z_j \neq z_k \in X$ and let*

$$\widetilde{\Omega} = \Omega \backslash \bigcup_{z_j \in X} D(z_j; r_2) \tag{3.5}$$

*There exist constants $g' > 0$ and $C' < \infty$ such that uniformly on $\widetilde{\Omega}$*

$$|\rho(z)| \geq g', \ \|P_j(z)\| \leq C' \tag{3.6}$$

**Remark:** As we shall see in condition II and III below, we need satisfy (3.6) on a subset of $\widetilde{\Omega}$ only.



Condition I allows to verify the hypotheses of lemma 2.1 uniformly on $\widetilde{\Omega}$. Moreover the operators $A_q(z)$ are holomorphic on $\widetilde{\Omega}$, provided $\epsilon$ is small enough. Indeed for any $\epsilon \leq \epsilon^*$ and $q \leq q^*$

$$\rho_q(z) = \rho(z) + O(b\epsilon^2) \tag{3.7}$$

(The proof is the same as that of corollary 2.2) We define eigenvectors of $A_{q^*}(z), z \in \widetilde{\Omega}$, by the method of section 2: Let $\varphi_j^*(0)$ be an eigenvector of $A_{q^*}(0)$ for the eigenvalue $e_j^*(0)$, $j = 1, 2$. Let $W_*(z|\alpha)$ be the analytic continuation of $W_*(x, 0)$ along a path $\alpha$ in $\widetilde{\Omega}$, starting at 0 and ending at $z$, where

$$\begin{aligned} &W_*'(x,0) = K_{q^*}(x)W_*(x,0), \quad x \in \mathbf{R} \\ &W_*(0,0) = \mathbf{1} \end{aligned} \tag{3.8}$$

The operator $W_*(z|\alpha)$ is a (local) solution of

$$W_*'(z|\alpha) = K_{q^*}(z)W_*(z|\alpha) \tag{3.9}$$

The main property of $W_*(z|\alpha)$, which follows from (3.9) (see (2.13) and (2.14)), is that the vectors

$$\varphi_j^*(z|\alpha) \equiv W_*(z|\alpha)\varphi_j^*(0), \quad j = 1, 2 \tag{3.10}$$

are two eigenvectors of $A_{q^*}(z)$, which are obtained by analytical continuation of $\varphi_j^*(0)$ along $\alpha$. The vector $\varphi_j^*(z|\alpha)$ is an eigenvector for the eigenvalue $e_j^*(z|\alpha)$, which is the analytic continuation of $e_j^*(0)$ along $\alpha$.

**Lemma 3.1**

*Let $z_j$ be a simple zero of $\rho$ in $\Omega$ and let $\eta$ be a simple closed path around $D(z_j; r_2)$, counterclockwise oriented and encircling no other disc $D(z_k; r_2)$ with $\rho(z_k) = 0$. Then for $\epsilon$ small enough,*

*1) the total variation of the argument of $\rho_{q^*}$ along $\eta$ is $2\pi$.*

*2) if $\eta$ starts at $z = 0$, then there exist two complex numbers $\theta_{jk}^*$, $j \neq k$ $j, k = 1, 2$, such that*

$$W_*(0|\eta)\varphi_k^*(0) := e^{i\theta_{jk}^*}\varphi_j^*(0), \quad j \neq k$$

*and*

$$e^{i\theta^*_{kj}} e^{i\theta^*_{jk}} = -1, \quad j \neq k$$

**Proof :**

1) We can write

$$\rho_{q^*}(z) = \rho(z) g(z) \tag{3.11}$$

with $|g(z) - 1| < 1$ for all $z \in \eta$. Thus

$$\begin{aligned} 0 = \frac{1}{2\pi i} \int_\eta \frac{g'(z)}{g(z)} d(z) &= \frac{1}{2\pi i} \int_\eta \frac{\rho'_{q^*}(z)}{\rho_{q^*}(z)} dz - \frac{1}{2\pi i} \int_\eta \frac{\rho'(z)}{\rho(z)} dz \\ &= \frac{1}{2\pi i} \int_\eta \frac{\rho'_{q^*}(z)}{\rho_{q^*}(z)} dz - 1 \end{aligned} \tag{3.12}$$

2) $\varphi^*_j(0)$ is an eigenvector of $A_{q^*}(0)$ for the eigenvalue $e^*_j(0)$. After analytical continuation $e^*_j(0|\eta)$ is an eigenvalue of $A_{q^*}(0)$ and by 1) it is equal to $-e^*_j(0) = e^*_k(0), k \neq j$. Thus $\varphi^*_j(0|\eta) \equiv W_*(0|\eta)\varphi^*_j(0)$ is an eigenvector for the eigenvalue $e^*_k(0)$ and therefore proportional to $\varphi^*_k(0)$. Finally the last identity is a consequence of $detW_*(z|\alpha) = 1$ since $trK_{q^*}(z) \equiv 0$. □

Let $\Sigma$ be a simply connected domain in $\widetilde{\Omega}$, which contains the real axis. In $\Sigma$ the analytical continuations of $e^*_j(x)$ and $\varphi^*_j(x)$ are path-independent so that we write $e^*_j(z)$ instead of $e^*_j(z|\alpha)$ and so on. Let $\psi(z)$ be a solution of

$$i\epsilon\psi'(z) = A(z)\psi(z), \quad z \in \Sigma \tag{3.13}$$

We decompose $\psi(z)$ along the eigenfunctions of $A_{q^*}(z)$,

$$\psi(x) = \sum_{j=1}^{2} c^*_j(z) e^{-i/\epsilon \int_0^z e^*_j(z')dz'} \varphi^*_j(z) \tag{3.14}$$

and we derive a differential equation for the unknown cœfficients $c^*_j(z)$ using the identities

$$A(z) = A_{q^*}(z) + i\epsilon K_{q^*-1}(z) \tag{3.15}$$



and

$$\varphi^{*'}_j(z) = K_{q^*}(z)\varphi^*_j(z) \tag{3.16}$$

We obtain

$$c^{*'}_j(z) = \sum_{k=1}^{2} \exp\left(i\epsilon^{-1}\Delta^*_{jk}(z)\right) a_{jk}(z) c^*_k(z) \tag{3.17}$$

where

$$\Delta^*_{jk}(z) = \int_0^z \left(e^*_j(z') - e^*_k(z')\right) dz' \tag{3.18}$$

and

$$a_{jk}(z) = -\frac{< \varphi^*_j(0)|W_*^{-1}(z)\left(K_{q^*}(z) - K_{q^*-1}(z)\right) W_*(z)\varphi^*_k(0) >}{\|\varphi^*_j(0)\|^2} \tag{3.19}$$

We have a good control of $a_{jk}(z)$ using lemma 2.1 but the factor $\exp\left(i\epsilon^{-1}\Delta^*_{jk}(z)\right)$ may cause troubles when we consider the limit $\epsilon \to 0$ because $Im\Delta^*_{jk}(z) \neq 0$. Since

$$e^*_j(z) = e_j(z) + O(\epsilon^2 b) \tag{3.20}$$

we must actually control the factor $\exp\left(i\epsilon^{-1}\Delta_{jk}(z)\right)$ where

$$\Delta_{jk}(z) = \int_0^z \left(e_j(z') - e_k(z')\right) dz' \tag{3.21}$$

The function $\Delta_{jk}$ is equal, up to a factor $\pm 2$, to the function

$$\Phi(z) := \int_0^z \sqrt{\rho(z')} dz' \tag{3.22}$$

which is naturally associated with the quadratic differential $\rho(z)d^2z$.

**Definition :** *A Stokes line $\alpha$ is a curve in $\Omega \backslash \{z : \rho(z) = 0\}$ such that*

*1) $Im\Phi(z)$ is a constant along $\alpha$*

*2) $\alpha$ is maximal with property 1)*

*3) one of the boundary points of $\alpha$ at least is a zero of $\rho(z)$.*

There are different terminologies in the literature. Sometimes our Stokes lines are called anti-Stokes lines and vice-versa (see below). A Stokes line is always a simple curve and in

our case it is contained either in the upper half-plane or in the lower half-plane. Near a simple zero $z_0$ of $\rho(z)$ the level-lines of $Im\Phi(z)$ are homeomorphic to the level-lines

$$Imz^{3/2} = \text{constant} \tag{3.23}$$

around $z = 0$. For any simple zero $z_0$ of $\rho(z)$ there are exactly three Stokes lines which have $z_0$ as boundary point. We call them the Stokes lines of $z_0$. (See figure 1)

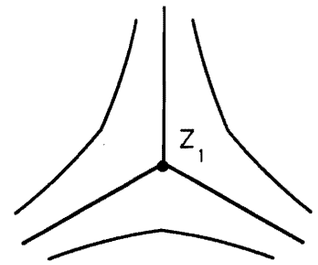


Figure 1: The level lines of $\Phi(z)$ near $z_0$.

**Condition II**

*A) There exists in the upper half-plane a nonempty finite set of simple zeros of $\rho(z)$, $\{z_1, \ldots, z_p\}$ with the properties (see figure 2):*

*1) there exists a Stokes line $l_i$, parametrized by $(t_i, t_{i+1})$, such that*

$\lim_{t\to t_i} l_i(t) = z_i, \quad \lim_{t\to t_{i+1}} l_i(t) = z_{i+1}, \quad i = 1, \ldots, p-1$

*2) there exists a Stokes line $l_0$, parametrized by $(-\infty, t_1)$, such that*

$\lim_{t\to t_1} l_0(t) = z_1, \quad \lim_{t\to -\infty} Rel_0(t) = -\infty, \quad \lim_{t\to -\infty} Iml_0(t) = a^-$

*3) there exists a Stokes line $l_p$, parametrized by $(t_p, \infty)$, such that*

$\lim_{t\to t_p} l_p(t) = z_p, \quad \lim_{t\to\infty} Rel_p(t) = \infty, \quad \lim_{t\to\infty} Iml_p(t) = a^+.$

*B) Along any vertical line $Rez = x$ going from the real axis to $l_0$ or $l_p$, $Im\Phi(z)$ is strictly monotone, provided $|x|$ is large enough.*

**Remark:** Condition II simply describes the situation illustrated in figure 2.



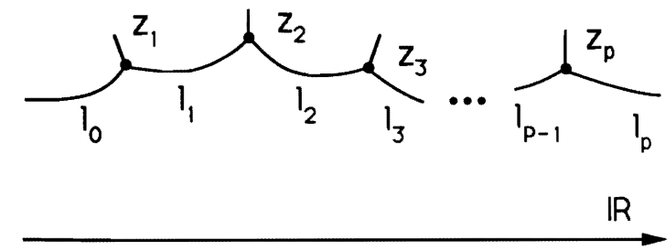


Figure 2: The Stokes lines of condition II.

In our case, if condition II is satisfied then an analogous condition holds in the lower half-plane. It follows from theorem 2.1 in [7] that the region $\Lambda$ in the upper half-plane between the real axis and the closure of the Stokes lines $l_0, \ldots, l_p$ is a simply connected region in $\Omega$ which does not contain zeros of $\rho$ in its interior. In [7], the part B of condition II follows from the existence of limiting matrices when $t$ tends to infinity. As already noticed, such limiting matrices are not supposed to exist here. Let $r > 0$ and let

$$\Sigma_r = \{z \in \mathbf{C} \quad | \quad dist(z, \Lambda) \leq r \quad \text{and} \quad |z - z_i| \geq r, \quad i = 1, \ldots, p\} \tag{3.24}$$

**Condition III**

*There exists $r > r_2$, sufficiently small so that $\Sigma_r$ is a simply connected region in $\Omega$ containing the real axis and such that:*

*For any zero $z_i, i = 1, \ldots, p$, the Stokes lines of $z_i$ in the disc $D(z_i; r)$ intersect the boundary of the disc at a single point, $D(z_i, r) \cap D(z_j, r) = \emptyset$ (See figure 3)*

*The function*

$$b(x) := \sup_{\substack{y: \\ x+iy \in \Sigma_r}} \|K_0(x+iy)\| \tag{3.25}$$

*tends to zero at infinity and is integrable on* $\mathbf{R}$.

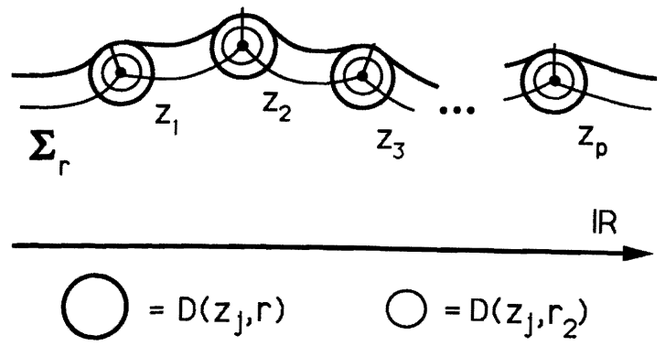


Figure 3: The set $\Sigma_r$ of condition III.

**Remark :** As we already mentioned, we need to verify condition I on $\Sigma_r$ only and not on $\widetilde{\Omega}$ since we shall integrate the differential equation (3.17) along a path in $\Sigma_r$.

### 3.2 Fröman-Fröman's method

We suppose that conditions I to III are satisfied and we study equation (3.17) on $\Sigma_r$. The hypotheses of lemma 2.1 are thus verified uniformly on $\Sigma_r$, so that there exists a $q^\star = q^\star(\epsilon)$ independent of $z \in \Sigma_r$ provided $\epsilon$ is small enough. Let us rewrite equation (3.17) as a Volterra equation

$$c_1^\star(z) = c_1^\star(z_0) + \int_{z_0}^{z} a_{11}(z')c_1^\star(z')dz' + \int_{z_0}^{z} a_{12}(z')e^{i\epsilon^{-1}\Delta_{12}^\star(z')}c_2^\star(z')dz' \tag{3.26}$$

and

$$c_2^\star(z) = c_2^\star(z_0) + \int_{z_0}^{z} a_{22}(z')c_2^\star(z')dz' + \int_{z_0}^{z} a_{21}(z')e^{i\epsilon^{-1}\Delta_{21}^\star(z')}c_1^\star(z')dz' \tag{3.27}$$

**Lemma 3.2**

*If condition I to III hold then* $\lim_{x\to\pm\infty} c_j^\star(x) = c_j^\star(\pm\infty)$ *exist and*

$$\lim_{x\to\pm\infty} \sup_{\substack{y:\\ x+iy\in\Sigma_r}} |c_j^\star(x+iy) - c_j^\star(\pm\infty)| = 0$$



**Proof :** By conditions I to III we get from (3.19) and lemma 2.1, as in section 2,

$$\sup_{\substack{y:\\ x+iy\in\Sigma_r}} |a_{kj}(x+iy)| = b(x)O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{3.28}$$

and for $z = x+iy \in \Sigma_r$

$$\Delta_{jk}^\star(z) = \Delta_{jk}(z) + b(x)O(\epsilon^2) \tag{3.29}$$

Hence the limits $\lim_{x\to\pm\infty} c_j^\star(x)$ exist on the real axis since $\Delta_{jk}$ is real there. Then for all $z = x+iy$ on a vertical segment joining $\mathbf{R}$ and $l_0$ or $l_p$ we can control $|Im\Delta_{jk}(z)|$, provided $|x|$ is large enough, by using part $B$ of condition II. Indeed, for such $z$, $|Im\Delta_{jk}(z)|$ is bounded by twice the value of $|Im\Phi(z)|$ on the Stokes lines. From these estimates and (3.28) we can deduce easily lemma 3.2 using (3.26) and (3.27). □

Instead of integrating (3.17) along the real axis we integrate the equation along the Stokes lines $l_0, \ldots, l_p$, as long as we are at a distance larger than $r$ from a zero of $\rho$. Otherwise we integrate the equation along the boundaries of the discs $D(z_i; r)$, staying always in $\Sigma_r$ (see figure 4).

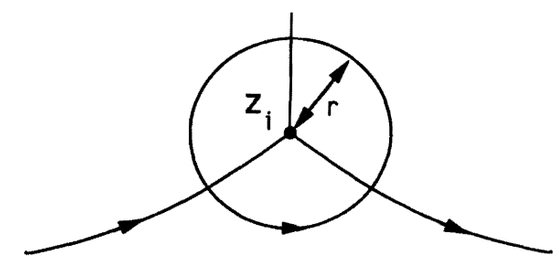


Figure 4: The path of integration close to $z_i$.

Let $z$ and $z_0$ be two points of $\Sigma_r$ and let $T(z, z_0)$ be the matrix-solution of (3.17) with $T(z_0, z_0) = \mathbf{1}$. We can find $T(z, z_0)$ by integrating the equation along any path in $\Sigma_r$ going from $z_0$ to $z$. However because of the factors $\exp\left(i\epsilon^{-1}\Delta_{jk}(z)\right)$ we have a good control of the equation only on particular paths. For instance, the Stokes lines are "good" paths. The main work consists in controlling the equation along the parts of the boundaries of the discs $D(z_i; r)$ when we pass from one Stokes line to the next one.

**Lemma 3.3**

*Let $z$ and $z_0 \in \Sigma_r$ and let $\alpha$ be a path, parametrized by $[s_0, s_1]$, going from $z_0$ to $z$, and such that $s \longmapsto Im\Delta_{12}(\alpha(s))$ is non decreasing on $[s_0, s_1]$. Then*

$$T(z,z_0) = \begin{pmatrix} 1+O\left(e^{-\kappa\epsilon^{-1}}\right) & e^{-\epsilon^{-1}Im\Delta^*_{12}(z_0)}O\left(e^{-\kappa\epsilon^{-1}}\right) \\ e^{\epsilon^{-1}Im\Delta^*_{12}(z)}O\left(e^{-\kappa\epsilon^{-1}}\right) & 1+O\left(e^{-\kappa\epsilon^{-1}}\right)+ \\ & O\left(e^{-2\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}(Im\Delta^*_{12}(z)-Im\Delta^*_{12}(z_0))} \end{pmatrix}$$

**Proof :** We consider (3.26) and (3.27) along $\alpha$ with $c_1^*(z_0) = 1$ and $c_2^*(z_0) = 0$ and we introduce new variables

$$X_1(s) = c_1^*(\alpha(s)), \quad X_2(s) = e^{i\epsilon^{-1}\Delta^*_{12}(\alpha(s))}c_2^*(\alpha(s)) \tag{3.30}$$

Writing $b_{jk}(s) = a_{jk}(\alpha(s))\frac{d\alpha(s)}{ds}$ we get

$$X_1(s) = 1 + \int_{s_0}^{s} b_{11}(s')X_1(s')ds' + \int_{s_0}^{s} b_{12}(s')X_2(s')ds' \tag{3.31}$$

$$\begin{aligned} X_2(s) = &\int_{s_0}^{s} b_{22}(s')e^{i\epsilon^{-1}(\Delta^*_{12}(s)-\Delta^*_{12}(s'))}X_2(s')ds' \\ &+ \int_{s_0}^{s} b_{21}(s')e^{i\epsilon^{-1}(\Delta^*_{12}(s)+\Delta^*_{21}(s'))}X_1(s')ds' \end{aligned} \tag{3.32}$$

In (3.32) $s' \leq s$ and $\Delta^*_{21}(s') = -\Delta^*_{12}(s')$. Using (3.29) and the hypothesis on the path we have

$$\begin{aligned} &\left|e^{i\epsilon^{-1}(\Delta^*_{12}(s)-\Delta^*_{12}(s'))}\right| = \\ &\exp\left(-\epsilon^{-1}\left(Im\Delta_{12}(s) - Im(\Delta_{12}(s'))\right) + O(\epsilon)\right) = O\left(\exp(O(\epsilon))\right) \end{aligned} \tag{3.33}$$

Let $\|X_i\| = \sup_{s_0 \leq s \leq s_1} |X_i(s)|$. We get from (3.31), (3.32) and (3.33), using (3.28),

$$\begin{aligned} \|X_1\| &\leq 1 + O\left(e^{-\kappa\epsilon^{-1}}\right)(\|X_1\| + \|X_2\|) \\ \|X_2\| &\leq O\left(e^{-\kappa\epsilon^{-1}}\right)(\|X_1\| + \|X_2\|) \end{aligned} \tag{3.34}$$



so that for $\epsilon$ small enough $\|X_1\| + \|X_2\| \leq 2$. Using this a priori estimate in (3.31) and (3.32) we have

$$\sup_{s_0 \leq s \leq s_1} |X_1(s) - 1| = O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{3.35}$$

and

$$\sup_{s_0 \leq s \leq s_1} |X_2(s)| = O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{3.36}$$

Equations (3.35) and (3.36) allow to determine the first column of $T(z, z_0)$,

$$T(z,z_0) = \begin{pmatrix} 1+O\left(e^{-\kappa\epsilon^{-1}}\right) & T_{12}(z,z_0) \\ e^{\epsilon^{-1}Im\Delta^*_{12}(z)}O\left(e^{-\kappa\epsilon^{-1}}\right) & T_{22}(z,z_0) \end{pmatrix} \tag{3.37}$$

Since $|a_{11}(z) + a_{22}(z)| = O\left(e^{-\kappa\epsilon^{-1}}\right)$, we get from the Liouville formula

$$\begin{aligned} detT(z,z_0) &= \exp\left(O\left(e^{-\kappa\epsilon^{-1}}\right)\right) \\ &= 1 + O\left(e^{-\kappa\epsilon^{-1}}\right) \end{aligned} \tag{3.38}$$

Moreover $T^{-1}(z, z_0) = T(z_0, z)$, hence

$$T(z_0,z) = \frac{1}{detT(z,z_0)}\begin{pmatrix} T_{22}(z,z_0) & -T_{12}(z,z_0) \\ -T_{21}(z,z_0) & T_{11}(z,z_0) \end{pmatrix} \tag{3.39}$$

The reverse path $\alpha^{-1}$ from $z$ to $z_0$ is such that $s \longmapsto Im\Delta_{21}(\alpha^{-1}(s))$ is non increasing from $s_1$ to $s_0$. If $c_1^*(z) = 0$ and $c_2^*(z) = 1$ then we can estimate $c_1^*(z_0)$ and $c_2^*(z_0)$ as above, introducing new variables $Y_2(s) = c_2^*(\alpha^{-1}(s))$ and $Y_1(s) = e^{i\epsilon^{-1}\Delta^*_{21}(\alpha^{-1}(s))}c_1^*(\alpha^{-1}(s))$. Thus we can estimate the second column of (3.39). The coefficient $T_{22}(z, z_0)$ is estimated using $detT(z, z_0) = 1 + O\left(e^{-\kappa\epsilon^{-1}}\right)$. □

A Stokes line is a good path because $Im\Delta_{jk}(z)$ remains constant along this line. We have thus the immediate

**Corollary 3.4**

*If there is a Stokes line going from $z_0$ to $z$, then*

$$T(z,z_0) = \begin{pmatrix} 1+O\left(e^{-\kappa\epsilon^{-1}}\right) & O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta^*_{12}(z)} \\ O\left(e^{-\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}Im\Delta^*_{12}(z)} & 1+O\left(e^{-\kappa\epsilon^{-1}}\right) \end{pmatrix}$$

We now come to the difficult part of the method. We must control the matrix solution $T(z, z_0)$ along a portion of $\partial D(z_j, r)$ which is not a good path in the sense that $Im\Delta_{12}(z)$ is not monotone. We must establish two lemmas. The first lemma gives a monodromy matrix around the singularity $z_1$ and easily proven. The second and main lemma is more difficult to establish. Its proof is based on lemmas 3.3 and 3.5 and on a clever use of elementary identities between the coefficients of products of $2 \times 2$ matrices and their inverses [1]. This method has a definite advantage on the use of stretching and matching techniques to compute asymptotics in the sense that it allows to obtain better estimates on the remainders (see (1.19) in the introduction). However it can only be used for simple zeros of the function $\rho(z)$ whereas the stretching and matching method works in more general situations [21].

We consider now the neighborhood of a zero of $\rho(z)$, say $z_1$. Let $\delta$ be the boundary of the disc $D(z_1; r)$ counterclockwise oriented, going from $\zeta_0$ to $\zeta_6$ as in figure 5. On this figure the solid lines are the Stokes lines of $z_1$ and the dashed lines are the anti-Stokes lines of $z_1$, i.e. the lines along which $Re\Delta_{12}(z) \equiv Re\Delta_{12}(z_1)$. The arrows indicate the directions in which $Im\Delta_{12}(z)$ is non decreasing along the boundary of $D(z_1; r)$.

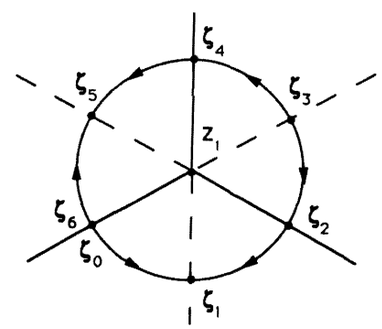


Figure 5: The points $\zeta_j$, $j = 0, \ldots, 6$ on the Stokes and anti-Stokes lines.

We compute the matrix $T(\zeta_6, \zeta_0)$ along $\delta$.



**Lemma 3.5**

$$T(\zeta_6, \zeta_0) = \begin{pmatrix} 0 & e^{i/\epsilon \int_\eta e_1^\star} e^{-i\theta_{21}^\star} \\ e^{i/\epsilon \int_\eta e_2^\star} e^{-i\theta_{12}^\star} & 0 \end{pmatrix}$$

**Proof:** Let us consider $\psi(z)$ at $z = \zeta_0$, the solution which we have obtained by integration along the Stokes line $l_0$ up to $\zeta_0$. We have

$$\psi(\zeta_0) = \sum_{j=1}^{2} c_j^\star(\zeta_0) e^{-i/\epsilon \int_0^{\zeta_0} e_j^\star} \varphi_j^\star(\zeta_0) \tag{3.40}$$

where in (3.40) the integration from 0 to $\zeta_0$ is along $\alpha$ as in figure 6 and, similarly, $\varphi_j^\star(\zeta_0)$ is the analytical continuation of $\varphi_j^\star(0)$ along $\alpha$.

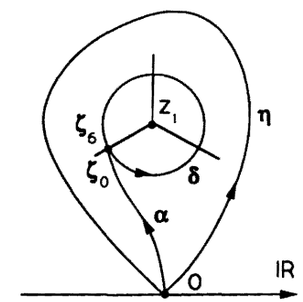


Figure 6: The paths $\alpha$, $\delta$ and $\eta$.

We make the analytical continuation of (3.40) along $\delta$ up to $\zeta_6$. Since $\psi(z)$ is holomorphic at $z_1$ we have $\psi(\zeta_6) = \psi(\zeta_0)$ and we can write

$$\psi(\zeta_0) = \sum_{j=1}^{2} c_j^\star(\zeta_6) e^{-i/\epsilon \int_0^{\zeta_0} e_j^\star} e^{-i/\epsilon \int_\delta e_j^\star} \varphi_j^\star(\zeta_6) \tag{3.41}$$

where now $\varphi_j^\star(\zeta_6)$ is the analytical continuation of $\varphi_j^\star(0)$ along $\alpha$ and then along $\delta$. But this is the same as the analytical continuation of $\varphi_j^\star(0)$ along $\eta$ and then along $\alpha$ as on figure 6. By lemma 3.1 we have therefore

$$\varphi_j^\star(\zeta_6) = e^{i\theta_{kj}^\star} \varphi_k^\star(\zeta_0) \tag{3.42}$$

Similarly we have

$$\int_\alpha e_j^* + \int_\delta e_j^* = \int_\eta e_j^* + \int_\alpha e_k^* \tag{3.43}$$

Hence, by comparing (3.40) and (3.41),

$$c_j^*(\zeta_6) e^{-i/\epsilon \int_\eta e_j^*} e^{i\theta_{kj}^*} = c_k^*(\zeta_0), \quad k \neq j \tag{3.44}$$

□

**Lemma 3.6**

*For $\epsilon$ small enough*

$$T(\zeta_2, \zeta_0) = \begin{pmatrix} 1 + O\left(e^{-\kappa/\epsilon}\right) & O\left(e^{-\kappa/\epsilon}\right) e^{-\epsilon^{-1} Im\Delta(\zeta_0)} \\ e^{-i/\epsilon \int_\eta e_1^*} e^{-i\theta_{12}^*} \left(1 + O\left(e^{-\kappa/\epsilon}\right)\right) & 1 + O\left(e^{-\kappa/\epsilon}\right) \end{pmatrix}$$

**Proof :** The following computations will involve expressions like $\epsilon^{-1} Im\Delta_{12}^*(\zeta_\nu)$ for $\nu = 0, 2, 4, 6$. These expressions are almost equal. Indeed

$$\Delta_{jk}^*(z) = \Delta_{jk}(z) + O(\epsilon^2) \tag{3.45}$$

and for this choice of $\zeta_\nu$ we have

$$Im\Delta_{12}(\zeta_\nu) = Im\Delta_{12}(z_1), \quad \nu = 0, 2, 4, 6 \tag{3.46}$$

since these points are on the Stokes lines of $z_1$. Hence, in particular,

$$e^{\pm\epsilon^{-1} Im\Delta_{12}^*(\zeta_\nu)} = O\left(e^{\pm\epsilon^{-1} Im\Delta_{12}(z_1)}\right), \quad \nu = 0, 2, 4, 6 \tag{3.47}$$

Finally note that

$$\int_\eta e_1^* = \int_\eta e_1 + O(\epsilon^2) = \Delta_{12}(z_1) + O(\epsilon^2) \tag{3.48}$$

Let us denote the coefficient $jk$ of the matrix $T(\zeta_\alpha, \zeta_\beta)$ by $t_{jk}(\alpha, \beta)$ and consider the identity

$$T(\zeta_{\nu+1}, \zeta_\nu) = T(\zeta_{\nu+1}, \zeta_{\nu+2}) T(\zeta_{\nu+2}, \zeta_\nu) \tag{3.49}$$

Using

$$detT(\zeta_\mu, \zeta_\nu) = t_{11}(\mu, \nu) t_{22}(\mu, \nu) - t_{12}(\mu, \nu) t_{21}(\mu, \nu) = 1 + O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{3.50}$$



we obtain for $\nu = 0, 2$ and 4

$$t_{11}(\nu+2, \nu) = \frac{t_{11}(\nu+1, \nu)}{t_{11}(\nu+1, \nu+2)} - \frac{t_{12}(\nu+1, \nu+2)}{t_{11}(\nu+1, \nu+2)} t_{21}(\nu+2, \nu) \tag{3.51}$$

$$\begin{aligned} t_{22}(\nu+2, \nu) = & \frac{t_{11}(\nu+1, \nu+2)}{t_{11}(\nu+1, \nu)} \left(1 + O\left(e^{-\kappa\epsilon^{-1}}\right)\right) \\ & + \frac{t_{12}(\nu+1, \nu)}{t_{11}(\nu+1, \nu)} t_{21}(\nu+2, \nu) \end{aligned} \tag{3.52}$$

$$\begin{aligned} t_{12}(\nu+2, \nu) = & \frac{t_{12}(\nu+1, \nu)}{t_{11}(\nu+1, \nu+2)} - \frac{t_{12}(\nu+1, \nu+2)}{t_{11}(\nu+1, \nu)} \left(1 + O\left(e^{-\kappa\epsilon^{-1}}\right)\right) \\ & - \frac{t_{12}(\nu+1, \nu) t_{12}(\nu+1, \nu+2)}{t_{11}(\nu+1, \nu) t_{11}(\nu+1, \nu+2)} t_{21}(\nu+2, \nu) \end{aligned} \tag{3.53}$$

These identities express, in particular, the elements of the matrix $T(\zeta_2, \zeta_0)$ as functions of the element $t_{21}(2, 0)$ and other matrix elements which we can control by means of lemma 3.3:

$$t_{11}(2, 0) = 1 + O\left(e^{-\kappa\epsilon^{-1}}\right) + O\left(e^{-\kappa\epsilon^{-1}}\right) e^{-\epsilon^{-1} Im\Delta_{12}(z_1)} t_{21}(2, 0) \tag{3.54}$$

$$t_{22}(2, 0) = 1 + O\left(e^{-\kappa\epsilon^{-1}}\right) + O\left(e^{-\kappa\epsilon^{-1}}\right) e^{-\epsilon^{-1} Im\Delta_{12}(z_1)} t_{21}(2, 0) \tag{3.55}$$

$$t_{12}(2, 0) = O\left(e^{-\kappa\epsilon^{-1}}\right) e^{-\epsilon^{-1} Im\Delta_{12}(z_1)} + \left(O\left(e^{-\kappa\epsilon^{-1}}\right) e^{-\epsilon^{-1} Im\Delta_{12}(z_1)}\right)^2 t_{21}(2, 0) \tag{3.56}$$

We are thus lead to the determination of $t_{21}(2, 0)$. Note that these estimates are true for the elements of $T(\zeta_6, \zeta_4)$ if we replace the arguments $(2, 0)$ by $(6, 4)$. Consider now the identity

$$T(\zeta_3, \zeta_2) T(\zeta_2, \zeta_0) T(\zeta_0, \zeta_6) = T(\zeta_3, \zeta_4) T(\zeta_4, \zeta_6) \tag{3.57}$$

Using lemma 3.1 and $e_1^* \equiv -e_2^*$ to compute $T(\zeta_0, \zeta_6) = T(\zeta_6, \zeta_0)^{-1}$, we obtain for the cœfficient 22 of (3.57)

$$\begin{aligned} & t_{21}(3, 2) t_{11}(2, 0) e^{i\theta_{12}^*} e^{i\epsilon^{-1} \int_\eta e_1^*} + \\ & t_{22}(3, 2) t_{21}(2, 0) e^{i\theta_{12}^*} e^{i\epsilon^{-1} \int_\eta e_1^*} = \\ & t_{21}(3, 4) t_{12}(4, 6) + t_{22}(3, 4) t_{22}(4, 6) \end{aligned} \tag{3.58}$$

and for the cœfficient 21 of (3.57)

$$t_{21}(3,2)t_{12}(2,0)e^{i\theta_{21}^*}e^{i\epsilon^{-1}\int_\eta e_2^*}+$$
$$t_{22}(3,2)t_{22}(2,0)e^{i\theta_{21}^*}e^{i\epsilon^{-1}\int_\eta e_2^*}=$$
$$t_{21}(3,4)t_{11}(4,6)+t_{22}(3,4)t_{21}(4,6) \tag{3.59}$$

Lemma 3.3, (3.39) and (3.47) yield

$$t_{21}(3,2)=-t_{21}(2,3)\left(1+O\left(e^{-\kappa\epsilon^{-1}}\right)\right)=O\left(e^{-\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}Im\Delta_{12}(z_1)} \tag{3.60}$$

$$t_{21}(3,4)=-t_{21}(4,3)\left(1+O\left(e^{-\kappa\epsilon^{-1}}\right)\right)=O\left(e^{-\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}Im\Delta_{12}(z_1)} \tag{3.61}$$

$$t_{22}(3,4)=t_{11}(4,3)\left(1+O\left(e^{-\kappa\epsilon^{-1}}\right)\right)=1+O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{3.62}$$

$$t_{22}(3,2)=t_{11}(2,3)\left(1+O\left(e^{-\kappa\epsilon^{-1}}\right)\right)=1+O\left(e^{-\kappa\epsilon^{-1}}\right) \tag{3.63}$$

whereas from (3.39) and the remark following (3.56) we have

$$t_{12}(4,6)=O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}+\left(O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}\right)^2t_{21}(6,4) \tag{3.64}$$

and

$$t_{22}(4,6)=1+O\left(e^{-\kappa\epsilon^{-1}}\right)+O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}t_{21}(6,4) \tag{3.65}$$

Now we use (3.58) and the above results to get

$$t_{21}(2,0)e^{i\theta_{12}^*}e^{i\epsilon^{-1}\int_\eta e_1^*}=1+O\left(e^{-\kappa\epsilon^{-1}}\right)$$
$$+O\left(e^{-2\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}t_{21}(2,0)+O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}t_{21}(6,4) \tag{3.66}$$

Hence we see that we have to estimate $t_{21}(6,4)$ as well to determine $t_{21}(2,0)$. This is done by performing a similar computation: We estimate $t_{11}(4,6)$ as a function of $t_{21}(6,4)$ as above and we consider equation (3.59). After multiplication by $e^{-i\theta_{21}^*}e^{-i\epsilon^{-1}\int_\eta e_2^*}$ and using

$$Im\int_\eta e_2^*=-Im\int_\eta e_1^* \tag{3.67}$$



we obtain another equation for $t_{21}(6,4)$ and $t_{21}(2,0)$

$$-t_{21}(6,4)e^{-i\theta_{21}^*}e^{-i\epsilon^{-1}\int_\eta e_2^*}=1+O\left(e^{-\kappa\epsilon^{-1}}\right)$$
$$+O\left(e^{-2\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}t_{21}(6,4)+O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}t_{21}(2,0) \tag{3.68}$$

Therefore, from (3.66) and (3.68) we deduce the a priori estimates

$$e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}|t_{21}(2,0)|=O(1) \tag{3.69}$$

$$e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}|t_{21}(6,4)|=O(1) \tag{3.70}$$

which finally yield

$$t_{21}(2,0)=e^{-i\theta_{12}^*}e^{-i\epsilon^{-1}\int_\eta e_1^*}\left(1+O\left(e^{-\kappa\epsilon^{-1}}\right)\right) \tag{3.71}$$

□

This lemma and corollary 3.4 allow to obtain an asymptotic expression for $\ln S_{21}$ beyond all orders by integrating (3.17) from $-\infty$ to $+\infty$ along the paths described above: Let us recall that we have

$$Im\Delta_{12}(z_1)\equiv Im\Delta_{12}(z_i),\quad i=1,\dots,p \tag{3.72}$$

Thus, along the Stokes lines we use the matrices given by corollary 3.4 and which we can write

$$T:=T(z,z_0)=\begin{pmatrix}1+O\left(e^{-\kappa\epsilon^{-1}}\right) & O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}\\ O\left(e^{-\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}Im\Delta_{12}(z_1)} & 1+O\left(e^{-\kappa\epsilon^{-1}}\right)\end{pmatrix} \tag{3.73}$$

On the other hand when we go from one Stokes line $l_{j-1}$ to the next one $l_j$ we use the matrix given by lemma 3.6

$$S_j:=\begin{pmatrix}1+O\left(e^{-\kappa\epsilon^{-1}}\right) & O\left(e^{-\kappa\epsilon^{-1}}\right)e^{-\epsilon^{-1}Im\Delta_{12}(z_1)}\\ e^{-i/\epsilon\int_{\eta_j}e_1^*}e^{-i\theta_{12}^*(j)}\left(1+O\left(e^{-\kappa\epsilon^{-1}}\right)\right) & 1+O\left(e^{-\kappa\epsilon^{-1}}\right)\end{pmatrix} \tag{3.74}$$

where $\int_{\eta_j}e_1^*$ and $\theta_{21}^*(j)$ are the quantities which are associated with the simple zero $z_j$ of $\rho(z)$. Therefore if we start at $-\infty$ with the values $c_1^*(-\infty)=1$ and $c_2^*(0)=0$, then the coefficients $c_1^*(+\infty)$ and $c_2^*(+\infty)$ are obtained by computing

$$\begin{pmatrix}c_1^*(\infty)\\ c_2^*(\infty)\end{pmatrix}=TS_pTS_{p-1}\dots S_1T\begin{pmatrix}1\\0\end{pmatrix} \tag{3.75}$$

which proves the final theorem of this section, (restoring the $\epsilon$ dependence):

**Theorem 3.7**

*Under conditions I to III, the solution of (3.17) such that $c_1^\star(-\infty) = 1$ and $c_2^\star(0) = 0$ is given at $x = +\infty$ by*

$$c_1^\star(\infty) = 1 + O\left(e^{-\kappa\epsilon^{-1}}\right)$$

*and*

$$c_2^\star(\infty) = \sum_{k=1}^{p} e^{-i/\epsilon \int_{\eta_k} e_1^\star(z,\epsilon)dz} e^{-i\theta_{12}^\star(k,\epsilon)} + O\left(e^{-\kappa\epsilon^{-1}}\right) e^{\epsilon^{-1} Im\Delta_{12}(z_1)}$$

*where $Im \int_{\eta_k} e_1^\star(z,\epsilon)dz = Im\Delta_{12}(z_1) + O(\epsilon^2)$ and $\theta_{12}^\star(k,\epsilon) = O(1)$.*



## 4. APPLICATIONS

### 4.1 Explicit Formulae

Let us start by deriving explicit formulae for the eigenvectors $\varphi_j^\star(z)$ of $A_{q^\star}(z)$ defined by (3.10). They will then allow us to give the precise relation between the cœfficients $c_j(z)$ defined by the expansion:

$$\varphi(x) = \sum_{j=1}^{2} c_j(z) e^{-i/\epsilon \int_0^x e_j(x')dx'} \varphi_j(z) \tag{4.1}$$

and the cœfficients $c_j^\star(z)$ defined by

$$\varphi(x) = \sum_{j=1}^{2} c_j^\star(z) e^{-i/\epsilon \int_0^x e_j^\star(x')dx'} \varphi_j^\star(z) \tag{4.2}$$

Note that here we have chosen $x_0 = 0$. Consider the operator $A_{q^\star}(z)$, $z \in \Sigma$, where $\Sigma$ is a simply connected domain of $\tilde{\Omega}$. We can write

$$A_{q^\star}(z) = \begin{pmatrix} ic^\star(z) & a^\star(z) \\ b^\star(z) & -ic^\star(z) \end{pmatrix} \tag{4.3}$$

with

$$\rho_{q^\star}(z) \equiv \rho_\star(z) = a_\star(z)b_\star(z) - (c_\star(z))^2 \tag{4.4}$$

**Lemma 4.1**

*The eigenvectors of $A_{q_\star}(z)$ defined by (3.10) are given by*

$$\varphi_j^\star(z) = \frac{\chi_j^\star(z)}{\|\chi_j^\star(-\infty)\|} e^{-i(-1)^j \sigma_\star(z)}, \quad j = 1, 2$$

*where*

$$\chi_j^\star(z) = \begin{pmatrix} \sqrt{\frac{a_\star(z)}{\sqrt{\rho_\star(z)}}} \\ (-1)^j \sqrt{\frac{\sqrt{\rho_\star(z)}}{a_\star(z)}} - i\frac{c_\star(z)}{\sqrt{\sqrt{\rho_\star(z)}a_\star(z)}} \end{pmatrix}$$

*and*

$$\sigma_\star(z) = \frac{1}{2}\int_{-\infty}^{z} \frac{c_\star(u)a_\star'(u) - c_\star'(u)a_\star(u)}{\sqrt{\rho_\star(u)}a_\star(u)} du$$

*for any* $z \in \Sigma \backslash Y_\star$ *with* $Y_\star = \{z \in \Sigma : a_\star(z) = 0\}$.

**Remarks:**

i) Any traceless matrix can be written under the form given above and the lemma actually requires the existence of distinct eigenvalues only. It is true in particular for the operator $A(z)$ written as in (4.3) without indices $\star$.

ii) The vectors $\varphi_j^\star(z)$ are actually analytic in the whole set $\Sigma$ since the operator $W_\star(z)$ is analytic in $\Sigma$.

**Proof:** A direct verification shows that the vectors $\chi_j^\star(z)$ are eigenvectors of $A_{q^\star}(z)$ for the eigenvalues $e_j^\star(z) = (-1)^j\sqrt{\rho_\star(z)}$. We set the notation

$$p_\star(z) = \sqrt{\rho_\star(z)} \tag{4.5}$$

and we introduce the eigenprojectors (see (3.2))

$$P_{q^\star,j}(z) \equiv P_j^\star(z) = \frac{1}{2}\begin{pmatrix} 1+(-1)^j\frac{ic_\star(z)}{p_\star(z)} & (-1)^j\frac{a_\star(z)}{p_\star(z)} \\ (-1)^j\frac{b_\star(z)}{p_\star(z)} & 1-(-1)^j\frac{ic_\star(z)}{p_\star(z)} \end{pmatrix} \tag{4.6}$$

The vectors $\varphi_j^\star(z)$ must satisfy $P_j^\star(z)\varphi_j^{\star\prime}(z) \equiv 0$ (see (2.22)). We compute, dropping the arguments,

$$\chi_j^{\star\prime} = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{p_\star}{a_\star}}\left(\frac{a_\star}{p_\star}\right)' \\ \frac{(-1)^j}{2}\sqrt{\frac{a_\star}{p_\star}}\left(\frac{p_\star}{a_\star}\right)' - i\frac{c_\star'}{\sqrt{p_\star a_\star}} + \frac{i}{2}\frac{c_\star(p_\star a_\star)'}{(p_\star a_\star)^{3/2}} \end{pmatrix} \tag{4.7}$$

and

$$P_j^\star\chi_j^{\star\prime} = i\frac{(-1)^j}{2}\frac{c_\star a_\star' - c_\star' a_\star}{p_\star a_\star}\chi_j^\star \tag{4.8}$$

Consequently, the vectors

$$\varphi_j^\star = \frac{e^{-i\frac{(-1)^j}{2}\int_{-\infty}^{z}\frac{c_\star a_\star' - c_\star' a_\star}{p_\star a_\star}dz}}{\|\chi_j^\star(-\infty)\|}\chi_j^\star \tag{4.9}$$

normalized to 1 at $z = -\infty$, satisfy condition (2.22). □



**Corollary 4.2**

*Let* $z_k \in X$ *and let* $\eta_k$ *be a counterclockwise oriented loop based at the origin which encircles the disc* $D(z_k, r)$ *only and passes through no point of* $Y_\star$. *Then the quantity* $e^{i\theta_{12}^\star(k)}$ *defined in lemma (3.1) is given by*

$$e^{i\theta_{12}^\star(k)} = -ie^{i\pi n_k^\star}\frac{\|\chi_1^\star(-\infty)\|}{\|\chi_2^\star(-\infty)\|}e^{-\frac{i}{2}\int_{\eta_k}\frac{c_\star a_\star' - c_\star' a_\star}{\sqrt{\rho_\star a_\star}}dz}e^{-2i\sigma_\star(0)}$$

*where* $n_k^\star \in \mathbf{Z}$ *depends on* $a_\star$ *and* $\eta_k$.

**Proof:** It is always possible to choose a loop $\eta_k$ as described. By lemma (3.1) we have

$$\sqrt{\rho_\star(0|\eta_k)} = e^{i\pi}\sqrt{\rho_\star(0)} \tag{4.10}$$

and

$$a_\star(0|\eta_k) = e^{i2\pi n_k^\star}a_\star(0) \tag{4.11}$$

with $n_k^\star \in \mathbf{Z}$ since $a_\star(z)$ is single valued in $\widetilde{\Omega}$. As a consequence

$$\chi_2^\star(0|\eta_k) = -ie^{i\pi n_k^\star}\chi_1^\star(0) \tag{4.12}$$

Finally,

$$\sigma_\star(0|\eta_k) = \frac{1}{2}\int_{\eta_k}\frac{c_\star a_\star' - c_\star' a_\star}{\sqrt{\rho_\star}a_\star}dz + \sigma_\star(0) \tag{4.13}$$

so that

$$\varphi_2^\star(0|\eta_k) = \varphi_1^\star(0)(-i)e^{i\pi n_k^\star}\frac{\|\chi_1^\star(-\infty)\|}{\|\chi_2^\star(-\infty)\|}e^{-\frac{i}{2}\int_{\eta_k}\frac{c_\star a_\star' - c_\star' a_\star}{\sqrt{\rho_\star a_\star}}dz}e^{-2i\sigma_\star(0)} \tag{4.14}$$

□

Consider now the two decompositions (4.1) and (4.2). The relation between the coefficients associated with the choice of eigenvectors made in lemma 4.1 is given by the following

**Corollary 4.3**

*The coefficients* $c_j^\star(\pm\infty)$ *and* $c_j(\pm\infty)$ *defined by (4.1), (4.2) and lemma 4.1 are such that*

$$c_j(-\infty) = c_j^\star(-\infty)e^{-i/\epsilon\int_0^{-\infty}e_j^\star(z)-e_j(z)dz}$$

$$c_j(+\infty) = c_j^*(+\infty)e^{-i(-1)^j(\sigma_*(+\infty)-\sigma(+\infty))}e^{-i/\epsilon\int_0^{+\infty} e_j^*(x)-e_j(x)dx}$$

*for* $j = 1, 2$.

**Proof:** We write the operator $A$ under the form

$$A(z) = \begin{pmatrix} ic(z) & a(z) \\ b(z) & -ic(z) \end{pmatrix} \tag{4.15}$$

where we assume, without loss of generality, that

$$\lim_{x\to\pm\infty} a(x) = a(\pm\infty) \neq 0 \tag{4.16}$$

The corresponding eigenvectors $\varphi_j(z)$ are given by the expressions of lemma 4.1, where the indices $\star$ are dropped. As the operators $A(x)$ and $A_{q^*}(x)$ coincide at $|x| = \infty$, we have

$$\chi_j^*(\pm\infty) = \chi_j(\pm\infty) \tag{4.17}$$

and

$$\varphi_j^*(-\infty) = \varphi_j(-\infty) \tag{4.18}$$

Hence

$$\varphi_j^*(+\infty) = \varphi_j(+\infty)e^{-i(-1)^j(\sigma_*(+\infty)-\sigma(+\infty))} \equiv e^{-i\beta_j^*}\varphi_j(+\infty) \tag{4.19}$$

so that formulae (2.43) and (2.45) apply. □

### 4.2 Invariants

Let us consider now the following three classes of operators $A(x)$:

1)

$$A(x) = A(x)^\dagger, \quad x \in \mathbf{R} \tag{4.20}$$

where $\dagger$ denotes the adjoint.

2)

$$A(x) = \begin{pmatrix} ic(x) & a(x) \\ b(x) & -ic(x) \end{pmatrix}, \quad a(x), b(x), c(x) \in \mathbf{R}, \quad x \in \mathbf{R} \tag{4.21}$$

3)

$$A(x) = i\begin{pmatrix} c(x) & \alpha(x) \\ \beta(x) & -c(x) \end{pmatrix}, \quad \alpha(x), \beta(x), c(x) \in \mathbf{R}, \quad x \in \mathbf{R} \tag{4.22}$$



Note in particular that the operator $H(x)$ in equation (1.7) belongs to the first class whereas the operators in equations (1.9) and (1.11) belong to the second class. For these classes of operators there exist expressions involving the coefficients $c_j(x)$ and $c_j^*(x)$ which are constant for all $x \in \mathbf{R}$.

**Lemma 4.4**

*If $A(x)$ belongs the class 1,2 or 3, then the operators $A_q(x)$ constructed by means of the iterative scheme (2.11), (2.12) belong to the same class, for any $q \leq q^*$.*

The proof of this lemma is obtained by a straightforward induction and will therefore be omitted.

**Lemma 4.5**

*i) If $A(x)$ belongs to the class 1, then*

$$|c_1(x)|^2 + |c_2(x)|^2 = |c_1^*(x)|^2 + |c_2^*(x)|^2 \equiv I \quad x \in \mathbf{R}$$

*where $I$ is constant.*

*ii) If $A(x)$ belongs to the class 2 or 3, then*

$$|c_1(x)|^2 - |c_2(x)|^2 = |c_1^*(x)|^2 - |c_2^*(x)|^2 \equiv I \quad x \in \mathbf{R}$$

*where $I$ is constant.*

**Proof:** The first assertion is a direct consequence of the fact that $U(x, x_0)$, $W(x, x_0)$ and $W_{q^*}(x, x_0)$ are unitary if $A(x)$ and $A_{q^*}(x)$ are self-adjoint. Assume now that $A(x)$ belongs to the second class and let

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.23}$$

If $\varphi(x)$ is solution of equation (1.12)

$$i\epsilon\varphi(x)' = A(x)\varphi(x) \tag{4.24}$$

then $\overline{G\varphi(x)}$ is another solution of this equation. Indeed, $G^2 = 1$ so that we can write

$$i\epsilon\overline{G\varphi(x)}' = \overline{-Gi\epsilon\varphi(x)'} = \overline{-GA(x)GG\varphi(x)} \tag{4.25}$$

and we compute

$$\overline{-GA(x)G} = A(x), \quad x \in \mathbf{R} \tag{4.26}$$

Therefore, as $trA(x) \equiv 0$, the following determinant is constant for any real $x$

$$det\left(\varphi(x), \overline{G\varphi(x)}\right) = \text{constant} \tag{4.27}$$

Observe that the eigenvectors constructed in lemma 4.1 satisfy the identity

$$\overline{G\varphi_j(x)} = \varphi_k(x), \quad j \neq k \tag{4.28}$$

since $\sigma(x)$ is real and $\|\chi_j(x)\|$ is independent of $j = 1, 2$ for real $a(x)$, $b(x)$ and $c(x)$. Then we obtain from the reality of $e_j(x)$ and $e_1(x) = -e_2(x)$

$$\overline{G\varphi(x)} = \overline{c_1(x)}e^{-i/\epsilon\int_0^x e_2(x')dx'}\varphi_2(x) + \overline{c_2(x)}e^{-i/\epsilon\int_0^x e_1(x')dx'}\varphi_1(x) \tag{4.29}$$

It remains to use the the multilinearity of the determinant to get

$$det\left(\varphi(x), \overline{G\varphi(x)}\right) = \left(|c_1(x)|^2 - |c_2(x)|^2\right) det\left(\varphi_1(x), \varphi_2(x)\right) \tag{4.30}$$

and we compute

$$det\left(\varphi_1(x), \varphi_2(x)\right) = 2\frac{\sqrt{\rho(-\infty)}}{a(-\infty) + b(-\infty)} \tag{4.31}$$

using $\rho(x) = a(x)b(x) - (c(x))^2$. The identities (4.28) and (4.29) are also true for the eigenvectors $\varphi_j^\star(x)$ due to lemma 4.4. Hence the same argument and (4.17) show that

$$det\left(\varphi(x), \overline{G\varphi(x)}\right) = \left(|c_1^\star(x)|^2 - |c_2^\star(x)|^2\right) 2\frac{\sqrt{\rho(-\infty)}}{a(-\infty) + b(-\infty)} = \text{constant} \tag{4.32}$$

If $A(x)$ belongs to the third class, we proceed in a similar way. In this case, if $\varphi(x)$ is a solution of (1.12), $\overline{\varphi(x)}$ is another solution and we obtain from the explicit formulae of lemma 4.1 (with the choice $\sqrt{i} = e^{i\pi/4}$)

$$\overline{\varphi_j(x)} = -i\varphi_k(x) \tag{4.33}$$

Finally we compute

$$\begin{aligned} det\left(\varphi(x), \overline{\varphi(x)}\right) &= \left(|c_1(x)|^2 - |c_2(x)|^2\right) 2\frac{\sqrt{\rho(-\infty)}}{\beta(-\infty) - \alpha(-\infty)} \\ &= \left(|c_1^\star(x)|^2 - |c_2^\star(x)|^2\right) 2\frac{\sqrt{\rho(-\infty)}}{\beta(-\infty) - \alpha(-\infty)} = \text{constant} \end{aligned} \tag{4.34}$$



□

**Remark:** It follows from (4.29) in the proof that if $(c_1(x), c_2(x))$ are solutions of (3.17), then $(\overline{c_2(x)}, \overline{c_1(x)})$ provide another solution of (3.17) when $A(x)$ belongs to the class 2 or 3. The corresponding symmetry property when $A(x)$ belongs to class 1 is that if $(c_1(x), c_2(x))$ satisfy (3.17), then $(\overline{c_2(x)}, -\overline{c_1(x)})$ satisfy (3.17) as well. This property can be derived from (3.17) directly by using the anti self-adjointness of $K_q(x)$, $q \leq q^\star$ in this case [13].

### 4.1 Main applications

a) Let $A(x)$ be a $2 \times 2$ hermitian matrix, $x \in \mathbf{R}$, as in equation (1.7). The equation

$$i\epsilon\frac{d\varphi(x)}{dx} = A(x)\varphi(x), \quad \epsilon \to 0 \tag{4.35}$$

describes the adiabatic limit of the dynamics of a two level quantum mechanical system. The squared modulus of the element $S_{21}$ gives the probability $\mathcal{P}(\epsilon)$ of a quantum transition over infinite time between the two eigenstates of the system.

**Corollary 4.6**

*If $A(x)$ is hermitian and satisfies conditions I to III,*

$$\mathcal{P}(\epsilon) = |S_{21}|^2 = \left|\sum_{k=1}^{p} e^{-i/\epsilon\int_{q_k} e_1^\star(x,\epsilon)dx} e^{-i\theta_{12}^\star(k,\epsilon)}\right|^2 + O\left(e^{-\kappa\epsilon^{-1}}\right) e^{\epsilon^{-1}2Im\Delta_{12}(x_1)}$$

b) Let $A(x)$ be the matrix (1.11)

$$A(x) = \begin{pmatrix} 0 & 1 \\ E - V(x) & 0 \end{pmatrix} \tag{4.36}$$

associated with the semiclassical regime of Schrödinger equation:

$$-\epsilon^2\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x), \quad \epsilon \to 0 \tag{4.37}$$

where $\inf_{x\in\mathbf{R}} E - V(x) > 0$. A solution $\varphi(x)$ of (1.11) characterized by the asymptotic conditions $c_1(-\infty) = 0, c_2(-\infty) = 1$ describes a particle coming from the right whose energy is strictly above the potential barrier $V(x)$. The reflection coefficient $\mathcal{R}(\epsilon)$ for this

scattering process is then defined by $\mathcal{R}(\epsilon) = \left|\frac{c_1(+\infty)}{c_2(+\infty)}\right|^2$. As it stands here, it cannot be computed from the knowledge of $S_{21}$. However, as a consequence of lemma 4.5 and the remark following it, we can write

$$\mathcal{R}(\epsilon) = \frac{|\tilde{c}_2(+\infty)|^2}{1+|\tilde{c}_2(+\infty)|^2} \tag{4.38}$$

where $\tilde{c}_1(-\infty) = 1$ and $\tilde{c}_2(-\infty) = 0$. Hence the

**Corollary 4.7**

*If $A(x)$ given by (4.46) satisfies conditions I to III,*

$$\mathcal{R}(\epsilon) = \frac{|S_{21}|^2}{1+|S_{21}|^2} = \left|\sum_{k=1}^{p} e^{-i/\epsilon \int_{\eta_k} c_1^*(x,\epsilon)dx} e^{-i\theta_{12}^*(k,\epsilon)}\right|^2 + O\left(e^{-\kappa\epsilon^{-1}}\right) e^{\epsilon^{-1}2Im\Delta_{12}(x_1)}$$

c) Let $A(x)$ be the matrix

$$A(x) = \begin{pmatrix} 0 & 1 \\ \omega^2(x) & 0 \end{pmatrix} \tag{4.39}$$

associated with the equation of motion (1.9) of a classical oscillator whose frequency varies slowly with time

$$\epsilon^2 \frac{d^2u(x)}{dx^2} = -\omega^2(x)u(x), \quad u(0) = u_0, \quad \epsilon\frac{du(0)}{dx} = u_1, \quad \epsilon \to 0 \tag{4.40}$$

We assume that the initial values $u_0$ and $u_1$ are independent of $\epsilon$. In terms of the variable $u(x)$, the adiabatic invariant (1.6) reads (keeping the same notation $J$)

$$J(x,\epsilon) = \frac{\epsilon^2|u'(x)|^2 + \omega^2(x)|u(x)|^2}{\omega(x)} \tag{4.41}$$

Note that we do not require the initial values $u_0$ and $u_1$ to be real. Let us express $\Delta J(\epsilon)$ in terms of the elements of the matrix $S$. We set

$$\Omega(x) = \begin{pmatrix} \omega(x) & 0 \\ 0 & \frac{1}{\omega(x)} \end{pmatrix} \tag{4.42}$$

so that we have with $\varphi(x)$ defined by (1.8)

$$J(x,\epsilon) = <\varphi(x)|\Omega(x)\varphi(x)> \tag{4.43}$$



Writing

$$\varphi(x) = \sum_{j=1}^{2} d_j(x) e^{-i/\epsilon \int_0^x (-1)^j \omega(x')dx'} \varphi_j(x) \tag{4.44}$$

where

$$\varphi_j(x) = \begin{pmatrix} \frac{1}{\sqrt{\omega(x)}} \\ (-1)^j\sqrt{\omega(x)} \end{pmatrix} \sqrt{\frac{\omega(-\infty)}{1+\omega^2(-\infty)}} \tag{4.45}$$

we compute

$$J(x,\epsilon) = 2\frac{\omega(-\infty)}{1+\omega^2(-\infty)}\left(|d_1(x)|^2 + |d_2(x)|^2\right) \tag{4.46}$$

Let us introduce the coefficients $d_j^*(x)$ by

$$\varphi(x) = \sum_{j=1}^{2} d_j^*(x) e^{-i/\epsilon \int_0^x c_j^*(x')dx'} \varphi_j^*(x) \tag{4.47}$$

satisfying the initial condition

$$\varphi(0) = \begin{pmatrix} u_0 \\ iu_1 \end{pmatrix} = d_1^*(0)\varphi_1^*(0) + d_2^*(0)\varphi_2^*(0) \tag{4.48}$$

This last equation and lemma 4.1 allow to express the $d_j^*(0)$ as functions of $u_0$ and $u_1$ and we have in particular $d_j^*(0) = O(1)$. As a consequence of corollary 4.3 we have $|d_j(\pm\infty)| = |d_j^*(\pm\infty)|$, $j = 1,2$, so that

$$\Delta J(\epsilon) = 2\frac{\omega(-\infty)}{1+\omega^2(-\infty)}\left(|d_1^*(+\infty)|^2 + |d_2^*(+\infty)|^2 - |d_1^*(-\infty)|^2 - |d_2^*(-\infty)|^2\right) \tag{4.49}$$

Then it results from the linearity of equation (3.17) and from the remark following the proof of lemma 4.5 that we can write

$$\begin{pmatrix} d_1^*(x) \\ d_2^*(x) \end{pmatrix} = \alpha(\epsilon)\begin{pmatrix} c_1^*(x) \\ c_2^*(x) \end{pmatrix} + \beta(\epsilon)\begin{pmatrix} \overline{c_2^*(x)} \\ \overline{c_1^*(x)} \end{pmatrix} \tag{4.50}$$

where the $c_j^*(x)$ satisfy (3.17) as well with boundary conditions $c_1^*(-\infty) = 1$, $c_2^*(-\infty) = 0$. These boundary conditions together with equation (2.46) allow to express the constants $\alpha(\epsilon)$ and $\beta(\epsilon)$ as functions of the $d_j^*(0)$ which are defined by the initial condition (4.48):

$$\begin{pmatrix} d_1^*(-\infty) \\ d_2^*(-\infty) \end{pmatrix} = \begin{pmatrix} \alpha(\epsilon) \\ \beta(\epsilon) \end{pmatrix} = \begin{pmatrix} d_1^*(0) + O\left(e^{-\kappa\epsilon^{-1}}\right) \\ d_2^*(0) + O\left(e^{-\kappa\epsilon^{-1}}\right) \end{pmatrix} \tag{4.51}$$

We can now express the total variation of the adiabatic invariant as a function of the matrix $S$ and the initial conditions using (4.49) and lemma 4.5:

$$\Delta J(\epsilon) = 2\frac{\omega(-\infty)}{1+\omega^2(-\infty)}\left[4Re\left\{\alpha(\epsilon)\overline{\beta(\epsilon)}c_1^*(+\infty)c_2^*(+\infty)\right\} + 2|c_2^*(+\infty)|^2\left(|\alpha(\epsilon)|^2 + |\beta(\epsilon)|^2\right)\right] \tag{4.52}$$

Hence, by (4.51) and corollary 4.3

**Corollary 4.8**

*If $A(x)$ given by (4.39) satisfies conditions I to III,*

$$\Delta J(\epsilon) = 2\frac{\omega(-\infty)}{1+\omega^2(-\infty)}\left[4Re\left\{d_1(-\infty)\overline{d_2(-\infty)}e^{-2i/\epsilon\int_0^{-\infty}(e_1^*(x,\epsilon)-e_1(x))dx}S_{11}S_{21}\right\}\right.$$
$$\left.+2|S_{21}|^2\left(|d_1(-\infty)|^2 + |d_2(-\infty)|^2\right)\right]$$

*If $d_1(-\infty)\overline{d_2(-\infty)} = 0$*

$$\Delta J(\epsilon) = 4\frac{\omega(-\infty)}{1+\omega^2(-\infty)}\left|\sum_{k=1}^{p} e^{-i/\epsilon\int_{\eta_k} e_1^*(x,\epsilon)dx} e^{-i\theta_{12}^*(k,\epsilon)}\right|^2\left(|d_1(-\infty)|^2 + |d_2(-\infty)|^2\right)$$
$$+O\left(e^{-\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}2Im\Delta_{12}(z_1)}$$

*If $d_1(-\infty)\overline{d_2(-\infty)} \neq 0$*

$$\Delta J(\epsilon) = 8\frac{\omega(-\infty)}{1+\omega^2(-\infty)}Re\left\{d_1^*(0)\overline{d_2^*(0)}\sum_{k=1}^{p} e^{-i/\epsilon\int_{\eta_k} e_1^*(x,\epsilon)dx} e^{-i\theta_{12}^*(k,\epsilon)}\right\}$$
$$+O\left(e^{-\kappa\epsilon^{-1}}\right)e^{\epsilon^{-1}Im\Delta_{12}(z_1)}$$

*where the quantities $d_j^*(0) = O(1)$ are determined by the initial condition (4.48)*

**Remarks:**

i) The coefficients $d_j$ are $O(1)$ since the initial conditions $u_0$ and $u_1$ are independent of $\epsilon$.

ii) The condition $d_1(-\infty)\overline{d_2(-\infty)} \neq 0$ is equivalent to $d_1(0)\overline{d_2(0)} \neq 0$. From (4.45) and (4.48) we compute

$$\begin{aligned} d_1(0) &= \frac{1}{2}\sqrt{\frac{1+\omega^2(-\infty)}{\omega(-\infty)}}\left(u_0\sqrt{\omega(0)} - \frac{i}{\sqrt{\omega(0)}}u_1\right) \\ d_2(0) &= \frac{1}{2}\sqrt{\frac{1+\omega^2(-\infty)}{\omega(-\infty)}}\left(u_0\sqrt{\omega(0)} + \frac{i}{\sqrt{\omega(0)}}u_1\right) \end{aligned} \tag{4.53}$$

so that $d_1(-\infty)\overline{d_2(-\infty)} \neq 0$ is equivalent to $u_1 \neq \pm i\omega(0)u_0$. This condition is always true for real initial values $u_0$ and $u_1$.



**References**


[1] N.Fröman, P.O.Fröman, *JWKB Approximation, Contributions to the Theory*, North Holland 1965.

[2] W.Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, John Wiley-Intersciences, New York 1965.

[3] M.V.Fedoriuk, *Méthodes Asymptotiques pour les Equations Différentielles Ordinaires Linéaires*, Mir Moscou 1987.

[4] Ph.A.Martin, G.Nenciu, *Semi-Classical Inelastic S-Matrix for One-Dimensional N-State Systems*, EPFL Preprint (1992).

[5] J.Davis, P.Pechukas, *Nonadiabatic Transitions Induced by a Time-Dependent Hamiltonian in the Semiclassical/Adiabatic Limit: The Two-State Case*, J. Chem. Phys. **64** (1976) pp. 3129-3137.

[6] J.-T.Hwang, P.Pechukas, *The Adiabatic Theorem in the Complex Plane and the Semi-Classical Calculation of Non-Adiabatic Transition Amplitudes*, J. Chem. Phys. **67** (1977) pp. 4640-4653.

[7] A. Joye, H. Kunz, C.-E. Pfister, *Exponential Decay and Geometric Aspect of Transition Probabilities in the Adiabatic Limit*, Ann. Phys. **208** (1991) pp. 299-332.

[8] M.V. Berry, *Geometric Amplitude Factors in Adiabatic Quantum Transitions*, Proc. Roy. Soc. London A **430** (1990) pp. 405-411.

[9] R.E.Meyer, *Adiabatic variation, Part II. Action change for the simple oscillator*, J. Appl. Math. Phys.**24** (1973) pp. 517-524.

[10] W.Wasow, *Calculation of an adiabatic invariant by turning point theory*, SIAM J. Math. Anal.**5** (1974) pp. 673-700.

[11] M.V.Fedoriuk, *An Adiabatic Invariant for a System of Linear Oscillators and Scattering Theory*, Differential Equations **12** (1976) pp. 713-718.

[12] J.B.Keller, Y.Mu, *Changes in Adiabatic Invariants*, Ann. Phys. **205** (1991) pp. 219-227.

[13] A.Joye, G.Mileti, C.-E.Pfister, *Interferences in Adiabatic Transition Probabilities mediated by Stokes Lines*, Phys. Rev. A **44** (1991) 4280-4295.

[14] N.Fröman *Connection formulas for Certain Higher Order Phase-Integral Approximations*, Ann. Phys. **61** (1970) pp. 451-464.

[15] B.Lundborg *Phase-Integral Treatment of Wave Reflection by real Potentials*, Math. Proc. Camb. Phil. Soc. **85** (1979) pp. 493-522.

[16] A.Joye, C.-E.Pfister, *Full Asymptotic Expansion of Transition Probabilities in the Adiabatic Limit*, J. Phys. A **24** (1991) pp. 753-766.

[17] S.G.Krein, *Linear Differential Equations in Banach Spaces*,American Mathematical Society Providence 1971.

[18] T.Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag Berlin Heidelberg New York 1980.

[19] A.Joye, C.-E Pfister, *Superadiabatic Evolution and Adiabatic Transition Probability between Two Non-degenerate Levels Isolated in the Spectrum*, J. Math. Phys. **34** (1993) pp. 454-479.

[20] A.Joye, C.-E Pfister, *Exponentially Small Adiabatic Invariant for the Schrödinger Equation*, Commun. Math. Phys. **140** (1991) pp. 15-41.

[21] A.Joye, *Non-trivial Prefactors in Adiabatic Transition Probabilities Induced by High Order Complex Degeneracies*, CPT CNRS Luminy Preprint (1993).