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A (Very) Few Points on the Theory of Gravitational Lenses

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1. Introduction

Gravitational lensing is such a rapidly growing field that it is hopeless to provide a complete review. Our goal here is simply to discuss the theory of gravitational lens statistics in outline, with some commentary on the sources of problems in the standard models. A particularly important question is the completeness of surveys for gravitational lenses, and we discuss a simple test that shows the existing radio surveys are incomplete. Finally we give a brief discussion of how to use gravitational lenses to determine the structure of the lens galaxy and possibly to superresolve the underlying radio source. A full review can be found in the book on lenses by Schneider, Ehlers & Falco (1992), and a review of the current observations is given by Walsh in this volume.

2. Cross Sections

The cross section of a gravitational lens is the area on the sky behind the lens that produces multiple images. For a generic lens it looks like the diagram in Figure 1, where the interior of the astroid produces five images, and the region between the ellipse and the astroid produces three images. The lines separating regions producing differing numbers of images are termed caustics. The characteristic physical scale of the cross section and the typical deflection produced by the lens is

$$b = 4\pi \left(\frac{\sigma}{c}\right)^2 \frac{D^A_{LS}}{D^A_{OS}} = 1''\!.8 \left(\frac{\sigma}{250 \text{ km s}^{-1}}\right)^2 \frac{D^A_{LS}}{D^A_{OS}}$$

where σ is the velocity dispersion of the lens, D^A_{LS} is the angular diameter distance between the lens and the source, and D^A_{OS} is the angular diameter distance to the source (Gott & Gunn 1974). The outer caustic of the lens has a typical radius of b,

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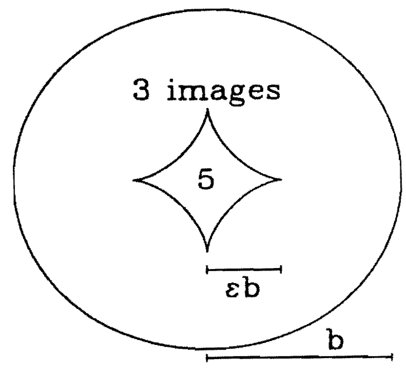

Figure 1. Typical caustic structure of a gravitational lens. The caustics (solid lines) separate regions that produce different numbers of images.

while the inner astroid has size ϵb, where $\epsilon \sim 0.1$ is the dimensionless ellipticity of the lens potential. Naively we expect that only fraction $\epsilon^2 \sim 0.01$ of gravitational lenses are five image lenses.

The major factor that modifies the magnitude of the cross sections is the addition of a core radius s to the lens potential. Core radii become important when the angular size of the core is comparable to the deflection angle b. Since the angular size is $0\overset{''}{.}007(s/100 \text{ pc})(r_H/D^A_{OL})$, where $r_H = c/H_0$ is the Hubble radius and D^A_{OL} is the angular diameter distance to the lens, we find that core radii become important once they are much more than a few hundred parsecs. Evidence from direct observations of galaxy cores (eg Lauer 1985), the number of lenses, and the absence of odd images in the center of lens systems (Wallington & Narayan 1992) all point towards $s \lesssim 300$ pc for a typical L_* galaxy. If we consider an isothermal sphere with a softened core (Blandford & Kochanek 1987) we find that the total cross section, $\sigma_T \simeq \pi b^2(1-(s/b)^{2/3})^3$, depends on the core radius, while the five image cross section, $\sigma_5 = (3/2)\pi b^2 \epsilon^2$, is largely unaffected if $s \lesssim 0.5b$.

3. Optical Depth

The cross section produced by an individual galaxy must be averaged over the distribution of galaxies between us and the source being lensed. A reasonable model is a Schechter (1976) distribution of isothermal spheres with a Tully-Fisher (1977) relation between luminosities and velocity dispersions. This introduces three parameters: the slope of the Schechter function, $\alpha \simeq -1$, the slope of the Tully-Fisher relation, $\gamma \simeq 4$, and the velocity dispersion scale of an L_* galaxy, $\sigma_* \simeq 250$ km s^{-1}. Under these assumptions, the optical depth can be computed analytically using proper motion distances D_{OS} in all standard cosmologies (Kochanek 1992), and it has the particularly simple form

$$\tau = \frac{1}{30}\tau_* D_{OS}^3 \quad \text{where} \quad \tau_* = 16\pi^3 n_* r_H^3 \left(\frac{\sigma_*}{c}\right)^4 \Gamma[1+\alpha+4\gamma^{-1}] \simeq 0.05 - 0.08$$

in flat cosmologies (Turner 1990). The resulting optical depth for a source at a redshift of $z_s = 2$ is about 0.0013 in an $\Omega_M = 1$ Einstein-DeSitter universe, and, at 0.017, it is over ten times higher in a flat universe with $\Omega_\Lambda = 1$. As was first pointed out by Turner (1990), the incidence of gravitational lenses is a powerful means of testing for the presence of a cosmological constant. Now we know that about 1% of bright quasars are lensed, so either we are missing something or $\Omega_\Lambda = 1$!

4. Magnification Bias

Lenses magnify the source, so that an 18th magnitude lensed quasar came from the more numerous fainter quasars. We approximate the differential source distribution in flux S by a simple power law $dN/dS \propto S^{-\alpha}$. If lenses simply magnified by a fixed amount M, then the probability of finding that an object of flux S_0 is a lens is not just the optical depth τ, but the optical depth multiplied by the ratio of the number of sources at the unlensed flux, S_0/M, to the number of sources at the observed flux, S_0, or τM^α. This extra enhancement of the probability of finding lenses in a flux limited sample is called magnification bias (Gott & Gunn 1974, Turner 1990). Real lenses produce a distribution of magnifications, but the differential probability of a lens magnifying a source by M always has the asymptotic form $dP/dM \propto M^{-3}$ (Schneider *et al* 1992), so the magnification bias has the form $\int dM M^{\alpha-3}$. If the source counts rise more rapidly than the magnification probability drops, then there will be a huge enhancement in the number of lenses. Bright quasars ($m \lesssim 19$) have a differential slope of $\alpha \simeq 3.15$, while faint quasars have a differential slope of $\alpha \simeq 1.7$ so we expect that magnification bias is large for bright quasars and drops off as the magnitude approaches the break at 19. Theoretical calculations of the amount of bias show an enhancement in the probability of lensing bright QSOs by a factor of 20-80. This saves us from $\Omega_\Lambda = 1$ because it increases the probability of finding lenses among bright quasars to about 1% in normal cosmologies.

In real life, you cannot (or at least should not) separate all these effects. For example, five image lenses are more magnified than three image lenses. Therefore the five image systems have more magnification bias, and in a bright sample you find that half of the lenses are five image systems even through the cross section says they should less than a percent of the systems. The presence of five image lenses in the observed samples of lenses is a tell tale sign of magnification bias (Kochanek 1991). Similarly, changing the core radius of the lens changes not only the cross section for producing multiple images, but also the distribution of magnifications. For example, the mean magnification produced by the lens increases as the core radius increases, so the magnification bias increases and partially compensates for the fall in the cross section.

5. Selection Effects

Surveys for gravitational lenses do not find all the lenses present in the sample. Finite resolution and dynamic range limit how many lenses can be found by a survey. The biggest problem, however, is that the costs of confirming that a candidate is a lens are very high. This means that surveys cannot use liberal selection functions when choosing candidates because the survey is then swamped with ambiguous candidates and false positives such as stars, other quasars, secondary radio lobes and so on. Statistical models of surveys should explicitly include a model for the selection effects, and the selection model for the statistics can be chosen to balance survey completeness and contamination by ambiguous candidates. Selection functions lead to significant modifications not only in the cross sections and optical depths but also to the magnification bias.

6. What About Radio Lenses

All these effects are generic to all gravitational lenses even though the discussion was largely phrased in terms of optical quasars. The advantages of quasars for statistical studies is that the redshift and flux distributions of optical quasars are much better characterized than for radio sources. The most important barrier to statistical studies of radio surveys is the paucity of information on the intrinsic properties of the sources.

Radio surveys for lenses also have an additional bias, "size bias", because the sources frequently have extended structure. If the major and minor axes of a source are ℓ_1 and ℓ_2, then the cross section for it being lensed is not just πb^2 but, $\pi(b+\ell_1)(b+\ell_2)$ (Kochanek & Lawrence 1990). This effect doubles the probability of lensing a circular source if $\ell_1 \simeq \ell_2 \simeq 0\farcs2$. The morphology of the lens varies with the size of the source. A source that is much smaller than the deflection scale b, produces lenses like MG0414 that consist of discrete components. When the source size is comparable to the deflection scale, then the resulting lenses will resemble the ring lenses such as MG1131 and B0218. If the source is much larger than the deflection scale, we can only recognize the lens from the pattern of distortions inside the larger structure – no lens has been found in this limit. Size bias can be as important in radio surveys as magnification bias is in optical surveys.

There is strong evidence that the radio surveys for galactic lenses are incomplete. A simple but very powerful means to demonstrate the incompleteness is to take a known lens, invert it to determine the structure of the lens and the source, and then see how often the observed image morphology is generated when we randomly lens the same source with the same lens. For example we can create a very accurate model of a ring

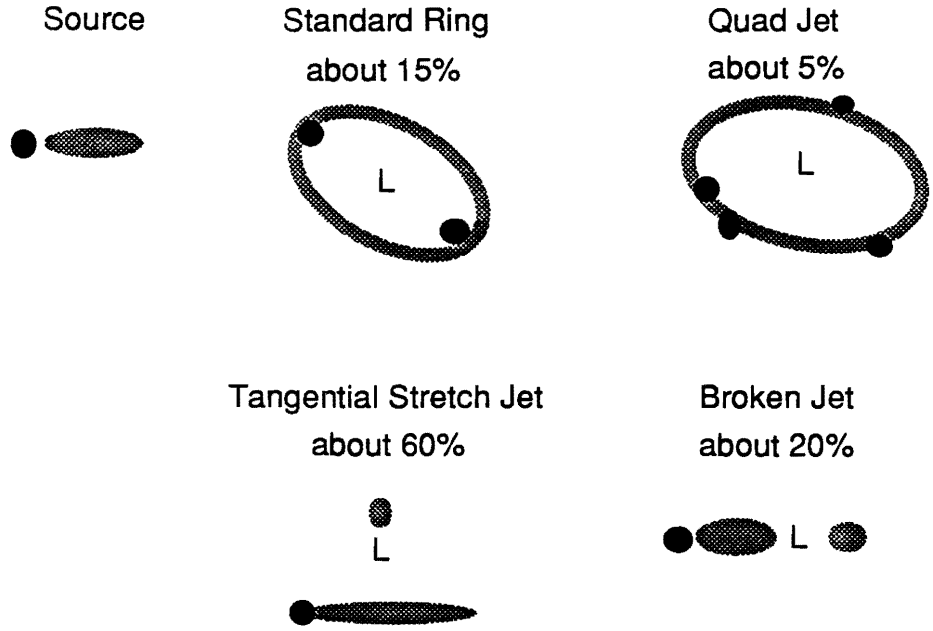

Figure 2. Schematic diagrams of images produced by lensing a core-jet source. The core is solid black, and the jet is shaded. An L marks the center of the lens.

like PKS1830-211 (Kochanek & Narayan 1992) in which the lens image consists of two bright compact cores with a ring, produced by lensing a simple core-jet source (see Figure 2). The readily identifiable normal ring geometry represents only about 15% of the ways in which the fixed lens model can lens the fixed source model. Three times rarer is the "quad-jet" morphology in which we see a low lying ring combined with four images of the point source. The two most common geometries are a "tangentially stretched jet" (60% of the time) in which the core-jet source is stretched out by the lens and a second compact image of the source is formed at right angles to the jet, and a "broken jet" (20% of the time) in which the core jet source is roughly unlensed and a second compact image is formed along the jet axis. It is possible to interpret the quad lenses like MG0414 as being the "quad-jet" sources in disguise, although the surface brightness of the jet would be high enough to see the surrounding ring if the PKS1830 source was lensed into the MG0414 geometry. The dominant geometries are not seen simply because of confusion – there are too many other ways in the sky to have another small blob of radio emission near a core-jet source for the survey groups to pursue these candidates.

7. LensClean

Once you have an extended radio lens, you have a huge number of constraints on the mass distribution in the lens galaxy. The key to making use of these constraints is to have a formal fit of a lens model to the data, including considerations of the noise level and statistical limitations. A "chi by eye" approach to modeling gravitational lenses was justified when we were mainly interested in whether it was possible to produce the observed image morphologies using gravitational lenses, but a more formal approach is required to really make use of the data.

We developed an algorithm (Kochanek & Narayan 1992) based on the Clean algorithm of radio astronomy to invert the extended radio lenses. The presence of multiple images in the lens allows us to simultaneously determine the unlensed structure of the source and the properties of the lens galaxies. This gives us a completely independent way from normal dynamical techniques of estimating the mass distribution in galaxies, and it is probably the only technique that can provide accurate measurements of the asymmetries in the mass distribution. The method automatically performs the calculations required to take into account both the resolution of the radio maps and the noise level. The accuracy with which we can constrain the model is simply a function of the dynamic range in the maps and the number of different maps. Each frequency and polarization emphasizes different parts of the image, which helps to break the degeneracies present in any single map. The advantage of a full statistical method like LensClean is that it makes use of every scrap of information available to it rather than focusing on a few visually obvious (and possibly deceptive) features.

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