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Impedance of Slowly Tapered Structures

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Abstract

The transverse and longitudinal impedances are studied for beam pipe structures whose radius is slowly varying as a function of the longitudinal position. The result is expressed in the form of a simple integral of the radius.

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Contents

1 Introduction 2

2 Basic Equations 3

3 Kernels 5

4 Longitudinal Impedance of the Monopole Mode 8

5 Transverse Impedance 9

6 Summary and Applications 10

7 Acknowledgements 11

1 Introduction

In any colliders, circular or linear, the particle beam has a halo which consists of particles having large transverse amplitudes. They do not contribute to the lumonosity and, moreover, they create background to the experiments. In order to eliminate them we often use scrapers (or masks or collimators). It was pointed out by Bane and Morton[1] that the transverse wakefield excited in the scrapers may damage the beam in the case of SLC because of the required small aperture of the scraper and the small emittance of the beam. This effect is even severer in future linear colliders because of the smaller emittances.

A possible way of relaxing the wakefield effects is to taper the steps of the structure. Bane and Morton pointed out by numerical calculation using TBCI[2] that the transverse wakefield is approximately proportional to square root of the taper angle in the case where the bunch length is comparable or longer than the scraper aperture a. W. Chou[3] computed the wake of tapered cavities also using TBCI and found that the transverse loss factor is proportional to a power of the taper angle, with the power depending on the geometry. In these works it was very hard to confirm the results down to very small taper angles because of the computing time problem. Analytic formulas for very small taper angles are desired.

Cooper-Krinsky-Morton[4] and Chou[5] derived a neat analytic formula, which we call CKMC formula, of the wake function for structures with small but arbitrary variation of the pipe radius, i.e., $\Delta a/a(z) \ll 1$, using the boundary perturbation method. The formula is very useful for various applications but it cannot be applied to the scrapers, in which $\Delta a/a$ has to be very large.

In this note we shall investigate the impedance of structures with a slow but arbitrary variation of the radius $a(z)$ (but the total change can be large).

Consider the exit of a non-tapered scraper shown in Fig.1. If the scraper is long enough, the field in the scraper gap is nearly a plane wave. The wave is diffracted at the exit edge of the scraper and, if the wave length is much shorter than the scraper aperture, i.e.,

$ka \gg 1$ where k is the wave number, the diffracted wave (wavy line in the figure) will be only in the region with the small opening angle $\sim 1/ka$. Therefore, if the taper angle $a' = da/dz$ is larger than $1/ka$, the taper does not help in suppressing the diffraction. Thus, we define the 'small angle' or 'slow variation' by $kaa' \ll 1$. We shall consider only the high frequency regime $ka \gg 1$ because of the mathematical difficulty of the low frequency regime. (If we allow $ka \lesssim 1$ together with an arbirary shape, we have to take into account narrow resonances because the conditon allows a cavity-like structure. The condition $ka \gg 1$ actually means waves above cutoff.)

We assume ultrarelativistic particles and an axisymmetric structure with a perfectly conducting wall throughout this paper.

In the next section we formulate our problem and reduce the Maxwell equations. Sec. 3 is devoted to giving mathematical tools to solve the equations. The longitudinal and the transverse impedances are derived in Sec. 4 and 5, respectively. A summary of the results and some applications are given in the last section.

2 Basic Equations

Let us consider a structure specified by $a(\epsilon z)$. A smaller ϵ means a slower variation of the boundary. We shall expand the Maxwell equations in terms of ϵ and take terms upto order ϵ^2, which is the first non-trivial order. After obtaining results, we set $\epsilon = 1$.

To do so, first we define a new curvilinear coordinate system (ζ, ρ, ϕ) instead of the cylindrical coordinate (z, r, ϕ). The new coordinate (ζ, ρ) is a conformal mapping of (z, r)

$$z + ir = H(\zeta + i\rho)$$

with the condition that $\rho = 0$ and $\rho = 1$ correspond to $r = 0$ and $r = a(z)$, respectively. The solution expanded in terms of ϵ is

$$z(\zeta, \rho) = \Re H(\zeta + i\rho) = \xi(\zeta)/\epsilon - \epsilon(\frac{1}{2}\rho^2 aa' + u) + O(\epsilon^3) \tag{2.1}$$

$$r(\zeta, \rho) = \Im H(\zeta + i\rho) = a\rho[1 - \epsilon^2(\rho^2(aa')'/6 + u') + O(\epsilon^4)]. \tag{2.2}$$

Here, $\xi(\zeta)$ is defined as the inverse of

$$\zeta = \int_0^\xi \frac{dz}{a(z)} \tag{2.3}$$

and

$$u(\zeta) = -\frac{1}{6}aa' + \frac{a}{3}\int \frac{a'^2}{a}dz.$$

Here and hereafter a is considered to be a function of ξ, the latter being a function of ζ only but not of ρ, unless the argument is written explicitly. The prime denotes the derivative with respect to ξ.

There is one-to-one correspondence between the two variable ζ and ξ and they have the relation $d\xi = ad\zeta$. In the later sections we often write ξ_1 instead of $\xi(\zeta_1)$ etc. without notice.

The line element is given by

$$ds^2 = g(\zeta,\rho)^2(d\zeta^2 + d\rho^2) + r(\zeta,\rho)^2 d\phi^2 \tag{2.4}$$

with

$$g(\zeta,\rho) = |dH/d\zeta| = a[1 - \epsilon^2(\frac{1}{2}\rho^2 aa'' + u') + O(\epsilon^4)]. \tag{2.5}$$

Formulas of the gradient, rotation, etc. for a scalar V and a vector $\vec{A}$ are

$$\begin{aligned}
\vec{\nabla}V &= \left(\frac{1}{g}\frac{\partial V}{\partial\zeta}, \frac{1}{g}\frac{\partial V}{\partial\rho}, \frac{1}{r}\frac{\partial V}{\partial\phi}\right) \\
\Delta V &= \frac{1}{g^2 r}\left[\frac{\partial}{\partial\zeta}(r\frac{\partial V}{\partial\zeta}) + \frac{\partial}{\partial\rho}(r\frac{\partial V}{\partial\rho})\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} \\
\vec{\nabla}\times\vec{A} &= \left(\frac{1}{gr}[\frac{\partial}{\partial\rho}(rA_\phi) - g\frac{\partial A_\rho}{\partial\phi}], \frac{1}{gr}[g\frac{\partial A_\zeta}{\partial\phi} - \frac{\partial}{\partial\zeta}(rA_\phi)], \frac{1}{g^2}[\frac{\partial}{\partial\zeta}(gA_\rho) - \frac{\partial}{\partial\rho}(gA_\zeta)]\right) \\
\vec{\nabla}\cdot\vec{A} &= \frac{1}{g^2 r}[\frac{\partial}{\partial\zeta}(grA_\zeta) + \frac{\partial}{\partial\rho}(grA_\rho)] + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi}.
\end{aligned}$$

(The components are in the order (ζ,ρ,ϕ).)

We shall work in the frequency domain and omit the factor $e^{-ickt}/2\pi$. We define the source field $(\vec{E}^S, \vec{B}^S)$ as the solution in the vacuum with a unit charge. For the m-th azimuthal mode they are

$$E_z^S = B_z^S = 0, \qquad E_r^S = E_\phi^S = -B_r^S = B_\phi^S = \frac{e^{ikz}}{2\pi r^{m+1}}. \tag{2.6}$$

(The factors $\cos m\phi$ and $\sin m\phi$ are omitted.) Since ζ-axis makes an angle of $\arctan(a')$ with the z-axis on the boundary, the source fields on the boundary $\rho = 1$ are given by

$$E_\zeta^S = -B_\zeta^S = \frac{\epsilon a' V}{\sqrt{1+\epsilon^2 a'^2}}, \qquad E_\rho^S = -B_\rho^S = \frac{V}{\sqrt{1+\epsilon^2 a'^2}}, \qquad E_\phi^S = B_\phi^S = V \tag{2.7}$$

where

$$V(\zeta) = \frac{e^{ikz(\zeta,1)}}{2\pi[a(z(\zeta,1))]^{m+1}} \tag{2.8}$$

The radiated field, which we denote simply as $\vec{E}$ and $\vec{B}$, obeys the sourceless Maxwell equations. We define the transverse Lorentz force by

$$F_\rho \equiv E_\rho - B_\phi, \qquad F_\phi \equiv E_\phi + B_\rho. \tag{2.9}$$

These are not exactly the Lorentz force along the ρ and ϕ direction but they will turn out to be enough later on.

By summing and subtracting the eight Maxwell equations ($\vec{\nabla}\times\vec{B} = -ik\vec{E}$, $\vec{\nabla}\times\vec{E} = ik\vec{B}$, $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$), we obtain the equations for E_ζ, B_ζ, F_ρ and F_ϕ upto $O(\epsilon^2)$;

$$(\nabla_\zeta + ika)F_\rho = \frac{\partial E_\zeta}{\partial\rho} - \frac{m}{\rho}B_\zeta \tag{2.10}$$

$$(\nabla_\zeta + ika)F_\phi = \frac{\partial B_\zeta}{\partial \rho} - \frac{m}{\rho}E_\zeta \tag{2.11}$$

$$\nabla_\rho F_\rho + \frac{m}{\rho}F_\phi = -(\overline{\nabla}_\zeta - ika)E_\zeta + \epsilon^2 \rho a a'' E_\rho \tag{2.12}$$

$$\nabla_\rho F_\phi + \frac{m}{\rho}F_\rho = -(\overline{\nabla}_\zeta - ika)B_\zeta + \epsilon^2 \rho a a'' B_\rho \tag{2.13}$$

where

$$\nabla_\zeta f = \frac{1}{a}\frac{\partial}{\partial \zeta}(af), \qquad \overline{\nabla}_\zeta f = \frac{1}{a^2}\frac{\partial}{\partial \zeta}(a^2 f), \qquad \nabla_\rho f = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f).$$

E_ρ and B_ρ still appear in these equations but they can be replaced with the solutions for $\epsilon \to 0$, i.e., the solutions in a uniform cylinder

$$E_\rho^{(0)} = B_\rho^{(0)} = \frac{\rho^{m-1}e^{ik\xi}}{2\pi a^{m+1}}(1 - \delta_{m0}) \tag{2.14}$$

because they appear only in the terms of order ϵ^2.

From these equations we get

$$\Box E_\zeta = 2i\epsilon^2 k a^2 a'' \rho E_\rho^{(0)} \tag{2.15}$$

$$\Box B_\zeta = 2i\epsilon^2 k a^2 a'' \rho B_\rho^{(0)} \tag{2.16}$$

where

$$\Box = \Box_\zeta + \Delta_\rho$$

$$\Box_\zeta = (\nabla_\zeta + ika)(\overline{\nabla}_\zeta - ika) = (\frac{\partial^2}{\partial \xi^2} + k^2)a^2$$

$$\Delta_\rho = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2}.$$

Note that $\Box$ is not exactly the d'Alembertian for our coordinate system. It contains terms upto $O(\epsilon)$ of the real d'Alembertian and it is separable upto this order.

The boundary condition for E_ζ on the wall $\rho = 1$ is

$$\left[E_\zeta + E_\zeta^S\right]_{\rho=1} = 0 \tag{2.17}$$

and that for B_ζ is given by evaluating eq.(2.11) on $\rho = 1$ as

$$\left[\frac{\partial B_\zeta}{\partial \rho}\right]_{\rho=1} = \left[-mE_\zeta^S - (\nabla_\zeta + ika)(E_\phi^S + B_\rho^S)\right]_{\rho=1}. \tag{2.18}$$

3 Kernels

In this section we shall find the kernels for the operator $\Box$ to be used in the later sections. The equation

$$\Box F(\zeta, \rho) = 0 \tag{3.1}$$

can be separated as

$$F(\zeta,\rho) = J_m(\mu\rho)Q(\mu,\xi)/a^2(\xi) \qquad (\xi = \xi(\zeta), \qquad \mu \geq 0)$$

where J_m is the Bessel function and Q is a solution to

$$(\Box_\zeta - \mu^2)(Q/a^2) = (\frac{\partial^2}{\partial\xi^2} + k^2 - \frac{\mu^2}{a^2})Q = 0. \tag{3.2}$$

We denote the solution going to the right by Q. Obviously, the solutions for different μ's are orthogonal to each other in the sense

$$\int_{-\infty}^{\infty} Q(\mu_1,\xi)Q^*(\mu_2,\xi)\frac{d\xi}{a^2} = \int_{-\infty}^{\infty} Q(\mu_1,\xi)Q^*(\mu_2,\xi)\frac{d\zeta}{a} = 0 \qquad (\mu_1 \neq \mu_2). \tag{3.3}$$

We normalize Q by the Wronskian as

$$Q^*(\mu,\xi)\frac{\partial Q(\mu,\xi)}{\partial\xi} - Q(\mu,\xi)\frac{\partial Q^*(\mu,\xi)}{\partial\xi} = 2ik. \tag{3.4}$$

In the case $ka \gg 1$, we may use an approximate solution

$$Q(\mu,\xi) \sim \left(1 - \frac{\mu^2}{k^2a^2}\right)^{-1/4} \exp(i\int_0^\xi \sqrt{k^2 - \mu^2/a^2}d\xi) \sim \exp(ik\xi - \frac{i\mu^2}{2k}\int_0^\xi \frac{d\xi}{a^2}). \tag{3.5}$$

Now, let us define the kernel K_1 as the solution to

$$\Box K_1(\zeta,\zeta_1,\rho) = \rho^m\delta(\zeta - \zeta_1), \tag{3.6}$$

which is the form we need in the following, with the boundary condition

(a) right(left)-going wave for $\zeta \to \pm\infty$,

(b) regular at $\rho = 0$

(c) $K_1 = 0$ at $\rho = 1$.

Because of these conditions, K_1 can be expanded as

$$K_1 = \frac{1}{a^2}\sum_{s=1}^{\infty} J_m(j_s\rho)\left\{\begin{array}{ll} A_sQ^*(j_s,\xi) & (\zeta < \zeta_1) \\ B_sQ(j_s,\xi) & (\zeta > \zeta_1) \end{array}\right\} \tag{3.7}$$

where j_s is the s-th zero of $J_m(x)$. The continuity of K_1 at $\zeta = \zeta_1$ and the jump of $\partial K_1/\partial\xi$ determine the coefficients A_s and B_s. With the help of the Wronskian (3.4) and the orthogonality of the Bessel function

$$\int_0^1 J_m(j_s\rho)J_m(j_{s'}\rho)\rho d\rho = \frac{1}{2}[J_m'(j_s)]^2\delta_{ss'}$$

we get

$$K_1(\zeta,\zeta_1,\rho) = \frac{ia(\xi_1)}{ka^2(\xi)}\sum\frac{J_m(j_s\rho)}{j_sJ_m'(j_s)}Q(j_s,\max(\xi,\xi_1))Q^*(j_s,\min(\xi,\xi_1)). \tag{3.8}$$

If the boundary condition is

(c') $\partial K_2/\partial\rho = 0$ at $\rho = 1$

instead of (c), the kernel is

$$K_2(\zeta,\zeta_1,\rho) = \frac{ia(\xi_1)}{ka^2(\xi)}\sum\frac{mJ_m(\mu_s\rho)}{\mu_s^2 J_m''(\mu_s)}Q(\mu_s,\max(\xi,\xi_1))Q^*(\mu_s,\min(\xi,\xi_1)) \tag{3.9}$$

where μ_s is the s-th zero of $J_m'(x)$.

Now, let us find the solution to the equation

$$\Box F(\zeta,\rho) = \rho^m f(\zeta) \tag{3.10}$$

with the boundary condition (a), (b) and

(d) $F(\zeta,1) = g(\zeta)$ at $\rho = 1$.

As one can see easily, $F - \rho^m g$ satisfies an equation of the same form as eq.(3.10) with vanishing boundary condition on $\rho = 1$. It can therefore be represented by the kernel K_1. The solution is

$$F(\zeta,\rho) = \rho^m g(\zeta) + \int K_1(\zeta,\zeta_1,\rho)[f(\zeta_1) - \Box_\zeta g(\zeta_1)]d\zeta_1. \tag{3.11}$$

Similarly, if the boundary condition is

(d') $\partial F/\partial\rho = g(\zeta)$ at $\rho = 1$,

the solution is

$$F(\zeta,\rho) = \rho^m\frac{g(\zeta)}{m} + \int K_2(\zeta,\zeta_1,\rho)[f(\zeta_1) - \Box_\zeta\frac{g(\zeta_1)}{m}]d\zeta_1. \qquad (m > 0) \tag{3.12}$$

Another differential equation we need is

$$(\nabla_\zeta + ika)F(\zeta) = f(\zeta) \tag{3.13}$$

where f contains only right(left)-going wave at $\zeta = \pm\infty$. The solution which satisfies the same boundary condition is given by

$$F(\zeta) = -\int_\zeta^\infty d\zeta_1\frac{a(\xi_1)}{a(\xi)}e^{-ik(\xi-\xi_1)}f(\zeta_1). \tag{3.14}$$

However, we shall only encounter the case where $f(\zeta)$ is $\exp(ik\xi)$ times a slowly varying function. In such a case we may approximate as

$$F(\zeta) = \frac{f(\zeta)}{2ika}. \tag{3.15}$$

4 Longitudinal Impedance of the Monopole Mode

We are now ready to solve the equations for E_ζ and B_ζ. In this section we solve E_ζ and evaluate the longitudinal impedance for the mode $m = 0$. (For the modes $m > 0$ we need other field components, too, even for the longitudinal impedance.)

From eq.(2.15), the boundary condition (2.17) and the formula (3.11) we obtain

$$E_\zeta(\zeta,\rho) = -\frac{\epsilon a'\rho^m}{2\pi a^{m+1}} e^{ikz(\zeta,1)} + \frac{ik\epsilon^2}{\pi}\int_{-\infty}^{\infty} d\zeta_1 K_1(\zeta,\zeta_1,\rho)\left[\frac{(2-\delta_{m0})aa'' - (m-1)a'^2}{a^m} e^{ik\xi}\right]_{\zeta=\zeta_1}. \tag{4.1}$$

The first term must be treated carefully because it is $O(\epsilon)$. We can replace $z(\zeta,1)$ with $\xi(\zeta)$ in the argument of a and a' but $z(\zeta,1)$ in the exponent must be retained.

Let us compute the longitudinal impedance for $m = 0$ mode. We may evaluate E_z exactly on $r = 0$, where $E_z = E_\zeta$.

$$\begin{aligned}\frac{Z_L^{(m=0)}}{Z_0} &= -\int_{-\infty}^{\infty}\left[E_z(z,r)e^{-ikz}\right]_{r=0} dz \\ &= -\int_{-\infty}^{\infty} E_\zeta(\zeta,0)e^{-ikz(\zeta,0)}d\xi \times [1+O(\epsilon^2)],\end{aligned}$$

where Z_0 is the impedance of the vacuum. (We define the longitudinal wake to be positive for deceleration.) Substituting eq.(4.1) into this equation and using the explicit form of $Q(\mu,\xi)$ in eq.(3.5), we get

$$\begin{aligned}\frac{Z_L^{(m=0)}}{Z_0} &= \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\frac{a'}{a}e^{ik[z(\zeta,1)-z(\zeta,0)]}d\xi \\ &+ \frac{\epsilon^2}{\pi}\int d\xi_1\int d\xi_2 e^{ik(\xi_1-\xi_2)}\frac{(aa''+a'^2)_{\xi_1}}{a^2(\xi_2)}\sum_s\frac{1}{j_s J_0'(j_s)}\exp\left[ik(\xi_b-\xi_a) - \frac{ij_s^2}{2k}\int_{\xi_a}^{\xi_b}\frac{d\xi}{a^2}\right],\end{aligned}$$

where $\xi_b = \max(\xi_1,\xi_2)$ and $\xi_a = \min(\xi_1,\xi_2)$. The region $\xi_2 < \xi_1$ does not contribute much to the integral because of the rapid oscillation of $\exp(2ik(\xi_1-\xi_2))$, which reflects the causality. (Rigorously speaking, the causality can be 'violated' in the frequency domain within a distance of the order of the wave length.) Then, the integral over ξ_2 from ξ_1 to infinity can be done explicitly and we get

$$\frac{Z_L^{(m=0)}}{Z_0} = \frac{\epsilon}{2\pi}\log\frac{a(\infty)}{a(-\infty)} - \frac{ik\epsilon^2}{4\pi}\int_{-\infty}^{\infty} a'^2 d\xi - \frac{2ik\epsilon^2}{\pi}[aa']_{-\infty}^{\infty}\sum_s\frac{1}{j_s^3 J_0'(j_s)}.$$

Assuming that the initial and final beam pipe radii are the same, we finally obtain from the second term

$$Z_L^{(m=0)}(k) = -\frac{ikZ_0}{4\pi}\int_{-\infty}^{\infty} dz\left[\frac{da}{dz}\right]^2. \tag{4.2}$$

Note that this is the contribution of the phase difference $k[z(\zeta,1) - z(\zeta,0)]$.

5 Transverse Impedance

The transverse impedance for any mode $m > 0$ can be computed from the longitudinal impedance using the Panofsky-Wentzel theorem. To do so, we have to integrate $E_z = \cos\alpha E_\zeta - \sin\alpha E_\rho$ where α is the angle between the z- and ζ-axes. Near the beam axis α is given by $a'\rho$ and E_ζ and E_ρ are proportional to ρ^m and ρ^{m-1}, respectively, so that the two terms are both proportional to ρ^m if $m > 0$. Thus we also need E_ρ for computing the longitudinal impedance for $m > 0$. Therefore, we shall compute the transverse impedance directly from F_ϕ.

Let us first compute B_ζ. From the equation (2.16), the boundary condition (2.18) and the general formula (3.12) we get

$$B_\zeta = -\frac{\epsilon}{2\pi}\frac{a'\rho^m}{a^{m+1}}e^{ikz(\zeta,1)} - \frac{ik\epsilon^2}{2\pi m}\frac{a'^2\rho^m}{a^m}e^{ik\xi} + \frac{ik\epsilon^2}{\pi}\int d\zeta_1 K_2(\zeta,\zeta_1,\rho)[aa'' - (m-1)a'^2]_{\xi_1}e^{ik\xi_1}. \tag{5.1}$$

In the case $ka \gg 1$, B_ζ is dominated by the oscillating factor $e^{ik\xi}$. We may therefore use eq.(3.15) to solve eq.(2.11). Thus, we get

$$\begin{aligned} F_\phi &= \frac{1}{2ika}\left(\frac{\partial B_\zeta}{\partial\rho} - \frac{m}{\rho}E_\zeta\right) \\ &= -\frac{\epsilon^2}{4\pi}\frac{a'^2\rho^{m-1}}{a^{m+1}}e^{ik\xi} + \frac{\epsilon^2}{2\pi a}\int_{-\infty}^{\infty} d\zeta_1\left(\frac{\partial K_2}{\partial\rho} - \frac{m}{\rho}K_1\right)\left[\frac{2aa'' - (m-1)a'^2}{a^m}\right]_{\zeta_1} e^{ik\xi_1} \end{aligned} \tag{5.2}$$

We evaluate the azimuthal Lorentz force near the beam axis.

$$(\vec{E} + \vec{v}\times\vec{B})_\phi = E_\phi + B_r = E_\phi + B_\rho\cos\alpha + B_\zeta\sin\alpha.$$

Since $E_\phi = O(\rho^{m-1})$, $B_\rho = O(\rho^{m-1})$, $B_\zeta = O(\rho^m)$ and $\alpha = O(\rho)$ near the axis, we find

$$\lim_{r\to 0}(\vec{E} + \vec{v}\times\vec{B})_\phi/r^{m-1} = \lim_{r\to 0} F_\phi/r^{m-1}.$$

Thus, F_ϕ suffices for computing the transverse impedance:

$$\frac{Z_T^{(m)}(k)}{Z_0} = \lim_{r\to 0}\frac{i}{r^{m-1}}\int_{-\infty}^{\infty} F_\phi(r = \text{const.})e^{-ikz}dz. \tag{5.3}$$

Note that the integration is along r=const. but not along ρ=const.

From eqs.(5.3) and (5.2) and the explicit forms of the kernels we get

$$\begin{aligned} \frac{Z_T^{(m)}(k)}{Z_0} &= -\frac{i\epsilon^2}{4\pi}\int_{-\infty}^{\infty} d\xi\frac{a'^2}{a^{2m}} + \frac{\epsilon^2}{2\pi k}\int_{-\infty}^{\infty} d\xi_1\int_{\xi_1}^{\infty} d\xi_2\frac{[2aa'' - (m-1)a'^2]_{\xi_1}}{a^{m+2}(\xi_2)a^m(\xi_1)} \\ &\quad\times\sum_s\frac{m}{2^m m!}\left\{-\frac{m\mu_s^{m-2}}{J_m''(\mu_s)}\exp\left[-\frac{i\mu_s^2}{2k}\int_{\xi_1}^{\xi_2}\frac{d\xi}{a^2}\right] + \frac{j_s^{m-1}}{J_m'(j_s)}\exp\left[-\frac{ij_s^2}{2k}\int_{\xi_1}^{\xi_2}\frac{d\xi}{a^2}\right]\right\}. \end{aligned}$$

As one can see from the exponent, only the region $0 < \xi_2 - \xi_1 \lesssim ka^2$ can contribute to the integral. That is, the wake field can reach the distance only upto $\sim ka^2$. The change of

the aperture over this distance is $\Delta a/a \sim kaa' \ll 1$. Therefore, the factor $a^{m+2}(\xi_2)a^m(\xi_1)$ can be replaced with $a^2(\xi_2)a^{2m}(\xi_1)$ because it is in the term already of order ϵ^2. Then, we can carry out the integration over ξ_2 explicitly and, summing the series involving the zeroes of the Bessel function, we finally obtain

$$Z_T^{(m)}(k) = -\frac{imZ_0}{(m+1)\pi}\int_{-\infty}^{\infty} dz \left[\frac{a'}{a^m}\right]^2. \tag{5.4}$$

6 Summary and Applications

Our main results are eqs.(4.2) and (5.4). They are valid under the conditions of ultrarelativistic particles, perfectly conducting wall, $ka \gg 1$ and $kaa' \ll 1$.

The wake function for a point charge is then given by

$$W_L^{(m=0)}(s) = \frac{cZ_0}{4\pi}\delta'(s)\int_{-\infty}^{\infty} dz a'^2(z) \tag{6.1}$$

$$W_T^{(m)}(s) = \frac{m}{m+1}\frac{cZ_0}{\pi}\delta(s)\int_{-\infty}^{\infty} dz \left[\frac{a'(z)}{a^m(z)}\right]^2. \tag{6.2}$$

The CKMC formulas[4,5] of the wake function are valid for small variation of a but arbitrary ka. They are intended for periodic structures but it is easy to take the long period limit to get the impedance for non-periodic structures. By expressing the results in terms of impedance, we have, in our notation,

$$Z_L^{(m=0)}(k) = ikZ_0\int_{-\infty}^{\infty} dq|A(q)|^2 \sum_{s=1}^{\infty}\frac{q}{k-\frac{1}{2}(q+j_{0s}^2/qa_0^2)} \tag{6.3}$$

$$Z_T^{(m=1)}(k) = \frac{2iZ_0}{a_0^2}\int_{-\infty}^{\infty} dq|A(q)|^2 \times\sum_{s=1}^{\infty}\left\{\frac{q}{(\mu_{1s}^2-1)[k-\frac{1}{2}(q+\mu_{1s}^2/qa_0^2)]}+\frac{q}{k-\frac{1}{2}(q+j_{1s}^2/qa_0^2)}\right\}, \tag{6.4}$$

or, summing the series,

$$Z_L^{(m=0)}(k) = ikZ_0\int_{-\infty}^{\infty} dq|A(q)|^2 q^2 a_0^2\left[\frac{J_0'(x)}{xJ_0(x)}\right]_{x=a_0\sqrt{k^2-(k-q)^2}} \tag{6.5}$$

$$Z_T^{(m=1)}(k) = 2iZ_0\int_{-\infty}^{\infty} dq|A(q)|^2 q^2\left[\frac{1}{x^2}\left(\frac{xJ_1'(x)}{J_1(x)}-\frac{J_1(x)}{xJ_1'(x)}\right)\right]_{x=a_0\sqrt{k^2-(k-q)^2}}, \tag{6.6}$$

where $a_0 = a(\pm\infty)$, $A(q)$ is the Fourier transform of the boundary shape:

$$\frac{a(z)-a_0}{a_0} = \int_{-\infty}^{\infty} dq A(q)e^{iqz},$$

and j_{ms} and μ_{ms} are zeroes of $J_m(x)$ and $J_m'(x)$. When $kaa' \ll 1$, the terms with j_{ms} and μ_{ms} dominate in the denominators in eqs.(6.3) and (6.4), or $x \sim 0$ dominates in eqs.(6.5)

and (6.6). Therefore, the integrand becomes proportional to $|qA(q)|^2$. The expression for $Z_L^{(m=0)}$ obtained in this way is the same as ours (4.2), which means the CKMC formula is valid even when the variation of $a(z)$ is large (but slow). The result for $Z_T^{(m=1)}$ also agrees with our formula (5.4) for $m = 1$ except that $a^2(z)$ in the denominator of our formula is replaced with a_0^2.

There are three scaling parameters, ka, kaa' and $\Delta a/a$. During the reduction of the CKMC formulas for small kaa', we did not need the assumption of large ka. Therefore, it turned out that our formula for small kaa' is not only valid for large ka and any $\Delta a/a$ but also for any ka if $\Delta a/a$ is small. It is unlikely that our formula is valid for any ka and any $\Delta a/a$ but it is interesting to see down to how small ka it can be used in practice.

Let us apply our formula for the transverse dipole wake for a scraper depicted in Fig.2 by the solid line. The point charge wake function is

$$W_T(s) = \frac{cZ_0}{2\pi}\delta(s)2I$$

with

$$I \equiv \int_0^\infty dz \frac{a'^2(z)}{a^2(z)} = \theta(\frac{1}{a} - \frac{1}{b}).$$

Thus, for given scraper gap a and the beam pipe radius b, the wake is linearly proportional to the taper angle if the taper angle is small $\theta \ll 1/ka$.

For given length of the taper $l = (b-a)/\theta$, we can reduce the wake by shaping the wall properly (the dashed line in Fig.2). The best shape (for the dipole wake) is the exponential function

$$a(z) = a\left[\frac{b}{a}\right]^{(z/l)}$$

for which $I = [\log(b/a)]^2/l$. The reduction factor is

$$\frac{I_{\text{exponential}}}{I_{\text{linear}}} = \frac{R(\log R)^2}{(R-1)^2} \qquad (R = b/a).$$

The exponential shaping is effective when R is large. (A reduction by half for $R = 20$.) It is interesting that the linear taper is the best for the monopole wake.

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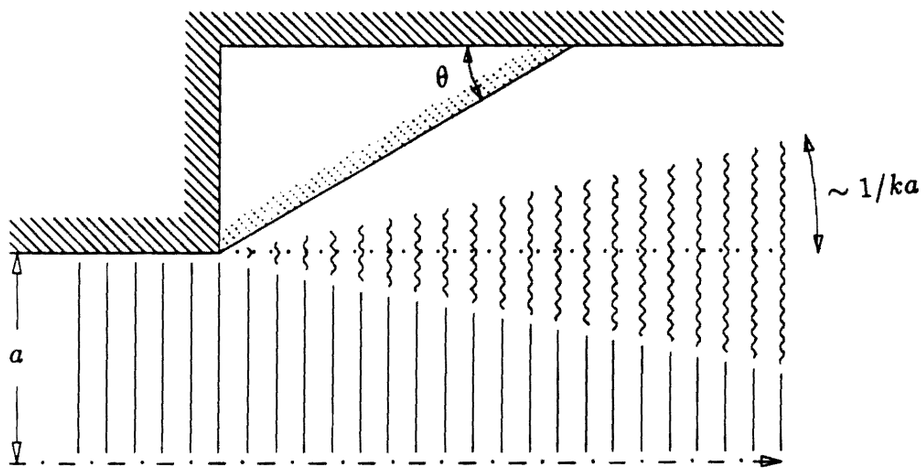

Fig.1. Diffraction of a high frequency wave at the exit of a scraper.

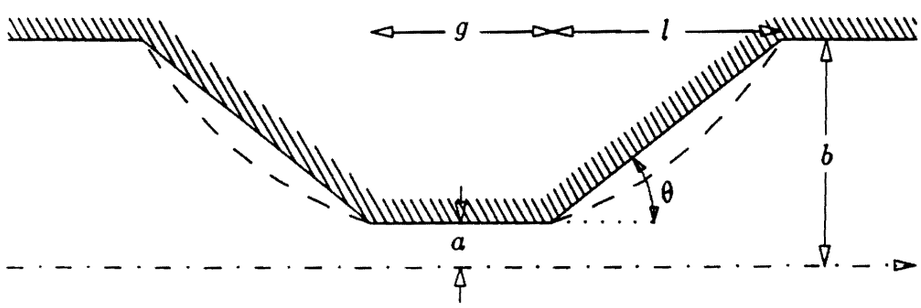

Fig.2. Linearly and exponentially tapered scrapers.