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INTRODUCTION TO UNIFIED THEORIES OF WEAK, ELECTROMAGNETIC AND STRONG INTERACTIONS
– SU(5) –

par

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1. INTRODUCTION

Our understanding of elementary interactions is mainly based on the observation that they obey a number of symmetry laws. Some of these laws seem to be exact. Such are Poincaré (Lorentz + translational) or CPT invariances. Also, in the field theoretical formulation of the interactions, exact local gauge symmetries are realized, such as the $U_{e.m.}(1)$ symmetry which generates electromagnetic phenomena (QED), or as far as we know, the $SU(3)_c$ colour symmetry, responsible for strong interactions (QCD). Such symmetries, when exact, imply the existence of massless, spin 1, gauge bosons. Particles can then be classified according to representations of the corresponding (Lie) groups, their quantum numbers being the eigenvalues of the commuting generators. For example, the QED Lagrangian is invariant under the local transformations $\Psi \to U_{e.m.}\Psi$,

$$U_{e.m.}(x) = \exp\{-i\, e\, a(x) Q\} \quad , \tag{1.1}$$

of the fermion fields. The charge operator Q is the generator of the transformation, a(x) an arbitrary real function of space time. Local invariance implies the existence of a massless gauge boson, the photon. Clearly only the product eQ is physically relevant. If Q is normalised so that its eigenvalue in the electron state Q_e is -1, then e is the usual electron charge ($e^2/(4\pi) = \alpha$ is the fine structure constant). But at this level, nothing guarantees that another particle has, in units of e, a charge equal to a fraction or to a multiple of Q_e, as we think it is the case. It is so because U(1), being abelian, contains only 1-dimensional irreducible representations, so that different particles, belonging to different irreducible representations of U(1), have unrelated quantum numbers (here charges). Here is the first question which any unification aims to answer :

Question I : Why are all charges commensurate ?

In order to go ahead, we need to classify particles in representations of some larger group g, in such a way that non-trivial commutation relations between the generators imply relations between the Q.N. of the particles(even if these particles belong to different representations). $SU(3)_c$ is another, apparently exact, symmetry of nature. The elementary fermions appear either as members of the fundamental representation (quarks q_i, i=1,2,3) or as singlets of $SU(3)_c$ (leptons, e, ν_e, μ, ν_μ etc...). Therefore, $U_{e.m.}(1)$ cannot be identified to a U(1) subgroup of $SU(3)_c$. Rather, we have as a symmetry group the direct product

$$g_o = SU(3)_c \otimes U_{e.m.}(1) \quad . \tag{1.2}$$

From this example, we see that the needed group g should not contain $U_{e.m.}(1)$ as a factor, if we are to explain that charges are commensurate, but of course it should contain g_o as a (non invariant) subgroup in order to accomodate in particular the colour and charge quantum numbers of the known elementary fields.

Weak interactions (W.I.) are also considered as deriving from a gauge symmetry group, namely SU(2). But this symmetry cannot be an exact symmetry : at present energies, W.I. appear as short range interactions, (Fermi type interactions), which cannot be described by zero mass vector boson exchange. Both their smallness and their short range nature may be accounted for at the same time if the gauge fields associated with the SU(2) symmetry have masses M large as compared to the relevant energy scale (the proton mass). Their exchange in a 2 fermion → 2 fermion transition leads to an effective Fermi interaction of strength G_F of order g^2/M^2, where g is the dimensionless SU(2) coupling constant.

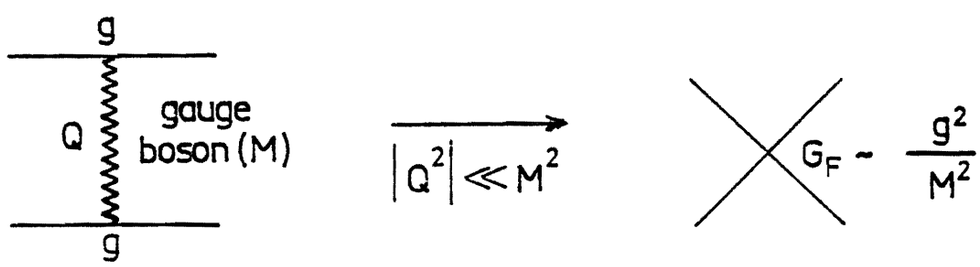

Fig.1.1 - Gauge boson exchange at transfer squared Q^2 and for a boson mass M gives rise to an effective four Fermi interaction of strength $G_F \sim g^2/M^2$ for $|Q^2| \ll M^2$.

Both the range, M^{-1}, and G_F are small for large M. However a mass term cannot be put by hand in the Yang-Mills part $L_{Y.M.}$ of the Lagrangian. The trouble with such a mass term for the gauge boson is that it spoils renormalizibility, the renormalizable theories being the only ones in which we know how to compute (in perturbation). Fortunately, symmetry breaking (mass generation) can be achieved without spoiling renormalizibility (spontaneously broken symmetry - Higgs

mechanism). The historical evolution of the ideas about the application of Y.M. theory to weak interactions may be fruitfully followed by consulting Veltman's lecture notes[1]. In the first part of these lectures, we describe the standard model of electroweak interactions known as the Salam-Weinberg model[2]. We first examine the algebraic structure of the model for an exact SU(2) $\otimes$ U(1) symmetry (Section 2), then the Higgs mechanism in its simplest version (Section 3), and we finally discuss the problem of the so-called Adler anomalies (Section 4).

2. THE ALGEBRAIC STRUCTURE OF THE SU(2) ⊗ U(1) MODEL

We use f as a generic symbol for a Dirac spinor, $\bar{f}$ for its Dirac conjugate and $f_{L,R}$ its left (right) components. Throughout these notes, we adopt the conventions of Bjorken and Drell[3] for the metric and γ matrices

$$f_{L,R} = \left(\frac{1 \mp \gamma_5}{2}\right) f \quad . \tag{2.1}$$

In the Fermi model of neutron β-decay, the 4-fermion interaction is described by

$$\frac{G_F}{\sqrt{2}} H_\mu L^\mu + \text{h.c.} \tag{2.2}$$

H_μ is the hadronic charged weak current, and the leptonic charged weak current is given by

$$L^\mu = \frac{1}{\sqrt{2}} \bar{\nu}_e \gamma^\mu \frac{1-\gamma_5}{2} e^- \quad , \tag{2.2'}$$

which can be written, using 2×2 matrix notations, as

$$L^\mu = (\bar{\nu}_e, e^-)_L \gamma^\mu \frac{\tau^+}{2} \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \tag{2.3}$$

Here τ^+ is the linear combination $\frac{\tau_1 + i\tau_2}{\sqrt{2}}$ of Pauli matrices acting on a two component spinor $\begin{pmatrix} \nu \\ e \end{pmatrix}_L$ formed with the left handed neutrino and electron. The hermitian conjugate of L^μ, which is present in the complete interaction (2.2), contains τ^-. This structure is suggestive of an SU(2) structure of the weak currents where the charged ones behave as the ± components of a weak isospin 1 object $\vec{T}$. Hence one guesses the existence of a neutral component T_3, and thus of a neutral current, represented by $\tau_3/2$ in the space of 2 component (isospin 1/2) spinors. In a Yang-Mills realisation of the SU(2) symmetry, one introduces 3 vector fields W^i_μ , i=1,2,3 also noted $\vec{W}_\mu$, which couple to fermions according to an SU(2) invariant Lagrangian, which can be written in general as

$$-g \bar{F}_L \gamma_\mu \overrightarrow{W^\mu} . \vec{T} F_L + (L \leftrightarrow R) \tag{2.4}$$

F stands for a set of fermions. Its left (right) handed components F_L (F_R) belong to some representation of the SU(2) group. g is the dimensionless weak coupling constant. In the relevant fermion representation, $\vec{T}$ represents the 3 generators of SU(2), whose commutation relations are

$$[T_i, T_j] = i\, \varepsilon_{ijk}\, T_k \tag{2.5}$$

ε_{ijk} is the completely antisymmetric tensor ($\varepsilon_{1\,2\,3} = 1$) . We recover the standard form (2.2') of L_μ if left (right) handed fermions transform like SU(2) doublets (singlets). The representation of $\vec{T}$ is $\vec{\tau}/2$ in the former case (weak isospin 1/2) and 0 in the latter.

The neutral boson W_3^μ couples to the fermion current

$$J_3^\mu = \bar{F}_L\, \gamma^\mu\, \frac{\tau_3}{2}\, F_L \tag{2.6}$$

Written in terms of the charge eigenstates of the $\vec{W}$ bosons,

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\,(W_\mu^1 \mp W_\mu^2),\ W_\mu^3$$

the piece (2.4) of the Lagrangian (with T = 0 for R-fermions) is

$$-\frac{g}{2}\,\{\bar{F}_L\, \gamma^\mu\, W_\mu^+ \tau^+ F_L + \text{h.c.} + \bar{F}_L\, \gamma^\mu\, W_\mu^3\, \tau^3\, F_L\} \quad . \tag{2.7}$$

One could think of identifying this neutral current with the e.m. current. This is not possible for at least two reasons

i) e.m. interactions conserve parity, while the current (2.6) distinguishes between left and right fermions.

ii) the neutrino "charge" defined by the coupling of J_3^μ to $W_{\mu 3}$ does not vanish, but is opposite to the electron charge since

$$\langle\nu|\tau_3|\nu\rangle = -\langle e|\tau_3|e\rangle = 1 \quad .$$

At least one other field B^μ must be added. The simplest way to do so is to consider it as generated by a new U(1) local symmetry, characterized by a coupling g' and a generator Y. The symmetry (here taken to be exact) is thus SU(2) ⊗ U(1). Consider a pair (a,b) of fermions whose left handed components form an SU(2) doublet $\begin{pmatrix} a \\ b \end{pmatrix}_L$. The part $L_{N.C.}^{(a,b)}$ of their Lagrangian corresponding to the total neutral current reads :

$$L^{(a,b)}_{N.C.} = -\frac{1}{2}\Big\{(\bar{a},\bar{b})_L\,\gamma_\mu\,[g\,W^\mu_3\tau_3 + g'B^\mu Y]\begin{pmatrix}a\\b\end{pmatrix}_L + g'\,\bar{a}_R\,\gamma_\mu\,B^\mu\,Y\,a_R + g'\,\bar{b}_R\,\gamma_\mu\,B^\mu\,Y\,b_R\Big\} \tag{2.8}$$

This way of writing underlines the fact that the L and R components are in different representations (isospin 1/2 and 0 respectively) of SU(2). In the first term the generator Y is represented by a multiple of $\mathbf{1}, \begin{pmatrix}Y_L & 0\\ 0 & Y_L\end{pmatrix}$, since a_L, b_L belong to the same representation of SU(2), whereas the eigenvalues Y_{a_R} and Y_{b_R} of Y are different. We now want to identify in $L^{(a,b)}_{N.C.}$ a piece which describes the photon interaction with particles a and b, namely

$$L^{(a,b)}_{e.m.} = -e\,(\bar{a},\bar{b})\,\gamma_\mu\,A^\mu\,Q\begin{pmatrix}a\\b\end{pmatrix}\ , \tag{2.9}$$

with a given charge matrix, diagonal for the physical states,

$$Q^{(a,b)} = \begin{pmatrix}Q_a & 0\\ 0 & Q_b\end{pmatrix} \tag{2.10}$$

This means that Q, the generator of $U_{e.m.}(1)$ must be a linear combination of T_3 and Y. At the same time, we exchange the two fields W^3_μ and B_μ for two new fields, the electromagnetic field A_μ and another one, Z^o_μ, to be identified with the field of the weak neutral boson. This transformation of fields must be orthogonal in order to leave unchanged the purely kinetic part of the Yang-Mills Lagrangian (while preserving the hermiticity of the fields) :

$$K \equiv -\frac{1}{4}\left[\left(K^{W^3}_{\mu\nu}\right)^2 + \left(K^B_{\mu\nu}\right)^2\right] = -\frac{1}{4}\left[\left(K^A_{\mu\nu}\right)^2 + \left(K^{Z_o}_{\mu\nu}\right)^2\right]\ . \tag{2.11}$$

Here for any field G_μ we use the notation

$$K^G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu\ . \tag{2.12}$$

Thus the most general field mixing we can perform is parametrized by a rotation of angle θ (the Weinberg angle)

$$\begin{aligned} W^3_\mu &= \sin\theta\ A_\mu + \cos\theta\ Z^o_\mu \\ B_\mu &= \cos\theta\ A_\mu - \sin\theta\ Z^o_\mu \end{aligned} \tag{2.13}$$

Inserting these expressions into the Lagrangian of Eq. (2.8), we rewrite it as

$$L^{(a,b)}_{N.C.} = L^{(a,b)}_{A} + L^{(a,b)}_{Z^\circ} \tag{2.14}$$

with

$$L^{(a,b)}_{A} = -\left\{(\bar{a},\bar{b})_L\ \gamma_\mu\ [g\ \sin\theta\ T_3 + g'\cos\theta\ \tfrac{Y}{2}]\begin{pmatrix}a\\b\end{pmatrix}_L + (\bar{a},\bar{b})_R\ \gamma_\mu\ [g\ \sin\theta\ T_3 + g'\cos\theta\ \tfrac{Y}{2}]\begin{pmatrix}a\\b\end{pmatrix}_R\right\} A^\mu \tag{2.15}$$

The 2nd term is a short hand notation for the two distinct terms concerning a_R and b_R. It is consistent with Eq. (2.8) if we recall that T_3 is represented by $\tau_3/2$ and 0 respectively in the left and right fermion spaces. Later on we shall come back to the part $L^{(a,b)}_{Z_o}$, proportional to Z^μ_o. For now we express that $L^{(a,b)}_{A}$ is identical with $L^{(a,b)}_{e.m.}$ in Eq. (2.9). This identity gives

$$eQ = \left[g\ \sin\theta\ T_3 + g'\cos\theta\ \frac{Y}{2}\right]_L = \left[g\ \sin\theta\ T_3 + g'\cos\theta\ \frac{Y}{2}\right]_R \tag{2.16}$$

Given the Lagrangian, the second equality determines the Weinberg angle in terms of g and g'. The first one connects the physical charges to the parameters and to the SU(2) ⊗ U(1) quantum numbers. Note that Eq. 2.16 is general whatever representations of SU(2) are used for the left and right fermions. It has to be always understood as a relation between $(p \times p)$ matrices in a p fermion space (a,b,c,...) whose left and righ components are in two different p-dimensional representations of SU(2) ⊗ U(1).

Consequences

i) The second equality in (2.16) has no solution for generic U(1) quantum numbers of the fermions under consideration. Consistency between the several scalar equations contained in this matrix equation implies strong constraints on the weak hypercharges. In the case where a pair $(a,b)_L$ form a doublet while a_R, b_R are singlets, the following relation must hold

$$2\ Y_L = Y_{a_R} + Y_{b_R} \tag{2.17}$$

If there are several such "families" of particles, the corresponding hypercharges ($Y^{(n)}_L$, $Y^{(n)}_{a_R}$, $Y^{(n)}_{b_R}$, n=1,2,...) must satisfy Eq. (2.17) *separately*. There is another constraint for the solution of Eq. (2.16) to exist. One has

$$e(Q_a - Q_b) = g\ \sin\theta = g'\cos\theta\ \frac{(Y_{a_R} - Y_{b_R})}{2} \tag{2.18}$$

so that for any $n=1,2,\ldots$

$$tg\theta = \frac{g'}{g}\,\frac{\left(Y^{(n)}_{a_R} - Y^{(n)}_{b_R}\right)}{2} \tag{2.19}$$

and hence

$$Y^{(1)}_{a_R} - Y^{(1)}_{b_R} = Y^{(2)}_{a_R} - Y^{(2)}_{b_R} = \ldots$$

Remark

At this point any SU(2) transformation on the fields leaves the physics invariant. The direction chosen for the third axis in SU(2), and thus for the electric charge, is still arbitrary. When the symmetry is broken (see next section), the charge generator is by definition that one which is left unbroken.

ii) For a pair (a,b) fulfilling Eq. (2.17), let us consider the charge difference $\Delta_{ab} = Q_a - Q_b$. From Eqs. (2.18,19), Δ_{ab} is universal for all pairs classified in the standard way. Specializing to the particular pair (ν_e, e), we have $\Delta_{\nu e} = 1$ and hence

$$e = g\,\sin\theta = g'\cos\theta\left[\frac{Y_{a_R} - Y_{b_R}}{2}\right] \tag{2.20}$$

We may now use the fact that, as emphasized in Section 1, only the product $g'Y$ is physically meaningful, and choose the normalisation of Y such that $(Y_{a_R} - Y_{b_R})/2 = 1$. Eqs. (2.16) are then replaced by the relations

$$\begin{cases} Q = T_3 + Y/2 \\ e = g\,\sin\theta \\ tg\theta = g'/g \end{cases} \tag{2.21}$$

In usual presentation of this model, the first of Eqs. (2.21) is imposed and the two others are derived from the requirements that QED conserves parity and that the model has the right charge content. Here these two requirements are first expressed through Eqs. (2.16) under their most general form. This approach allows us to emphasize that

a) implementing QED in the SU(2) $\otimes$ U(1) original symmetry implies strong constraints on the charges and weak hypercharges of the fermions(Eqs. (2.17-19)).

b) once these constraints are verified, Eqs. (2.21) result on one hand from the physical fact that $Q_{\nu_e} = 0$, on the other from a proper choice of normalisations of the generators $\vec{T}$ ($\varepsilon_{123} = 1$ in the commutation relation (2.5)) and Y . Of course this choice, which is arbitrary, has no physical consequences.

iii) It happens that fundamental fermions actually seem to appear in nature in pairs with $Q_a - Q_b = 1$, for example the quark pairs (u,d),(c,s),(t,b),... (?) and the lepton pairs $(\nu_e,e),(\nu_\mu,\mu),(\nu_\tau,\tau),\ldots(?)$. One may consider this fact as predicted by the SU(2) ⊗ U(1) model. The prediction is that, in a given n-plet, the charges are Q, Q+1,...,Q+n-1, if Q is the smallest one in this n-plet. But note that it is not a step towards the answer to question I of Section 1, "why are charges commensurate ?". For example, the model can accomodate a fermion pair (A,B) with $Q_A = \pi$ and $Q_B = \pi - 1$, that is charges which are neither commensurate between themselves, nor with the electron charge.

iv) Coming back to the weak neutral current and making use of Eqs. (2.21), we obtain the following expression for $L^{(a,b)}_{Z_o}$ as defined through Eqs. (2.8,13,14):

$$L^{(a,b)}_{Z_o} = \frac{-g\, Z_o^\mu}{2\cos\theta} \left\{ (\bar{a},\bar{b}) [\gamma_\mu (1-\gamma_5) T_3 - 2\sin^2\theta\, \gamma_\mu\, Q] \begin{pmatrix} a \\ b \end{pmatrix} \right\} \tag{2.22}$$

We see that the weak N.C. contains two terms. The first one is the neutral counterpart of the weak charged current, proportional to $1-\gamma_5$ (V-A interaction), the second is proportional to the e.m. current, with a coefficient proportional to $\sin^2\theta$, which is thus measurable for example by comparing the cross-sections for neutral and charge currents. The present experimental value of $\sin^2\theta$ is

$$\sin^2\theta = 0.229 \pm 0.014 \quad [4]$$

From equations (2.21), we know that $\mathrm{tg}\theta = \frac{g'}{g}$, but we have only one relation between g and g', namely

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2} \tag{2.24}$$

Hence the value of the Weinberg angle is not predicted. It is so because the symmetry group is not simple : each of its 2 invariant subgroups SU(2) and U(1) comes with an independent coupling. As a consequence, we may ask a second question which a unified theory has to answer :

Question II : Why is $\sin^2\theta$ what it is ?

Let us now summarize the content of this section. The fermion world we know is compatible with a classification of fermions according to the symmetry group SU(2) ⊗ U(1) ; the *algebraic structure* of weak and electromagnetic interactions is well described if all left handed fermions are classified into SU(2) doublets, the right handed fermions being singlets. The couplings characterizing the strength of both interactions are derived from the electron charge e and the Weinberg angle θ whose values are not predicted by the theory. Question I about commensurability of charges remains unsolved.

The model is not yet suitable for weak and electromagnetic *phenomenology*. As announced, SU(2) ⊗ U(1) is strongly broken, at least at present energies. On one hand the gauge vector bosons $W^{\pm}, Z_o$ must acquire masses (large as compared with the present energy scale) in order to reproduce both the weakness and the short range nature of weak interactions (Section 1). On the other hand, fermions (but the neutrinos ?) have non zero masses, which break the symmetry of the Lagrangian. A fermion mass term is

$$- m_a \, [\bar{a}_L a_R + \bar{a}_R \, a_L] \tag{2.25}$$

(We recall that $\bar{a}(1\pm\gamma_5) = \overline{(1\mp\gamma_5)a}$ and that $(1+\gamma_5)(1-\gamma_5) = 0$). Under an SU(2) transformation, a_R and $\bar{a}_R$ remain unchanged whereas a_L and $\bar{a}_L$ are transformed into combinations of a_L, b_L, and $\bar{a}_L, \bar{b}_L$ according to the representations $D^{1/2}$ and $D^{*1/2}$ of SU(2). Thus the mass term of the pair (a,b) is not invariant.

A last ingredient for weak interaction phenomenology is Cabibbo mixing. The quark pairs to be considered in weak doublets are not (u,d),(c,s) etc... but some linear combinations of them (respecting of course charge conservation). Neglecting heavier pairs, one has to replace (u,d),(c,s) by (u,d_c), (c,s_c) with

$$\begin{aligned} d_c &= d \cos\theta_c + s \sin\theta_c \\ s_c &= -d \sin\theta_c + s \cos\theta_c \quad , \end{aligned} \tag{2.26}$$

where θ_c is the Cabibbo angle. Generalisations to more pairs lead to mixing matrices which are no more orthogonal, but unitary, allowing for phases between states and thus "natural" CP violations. This very important subject will be treated in Section (8c) below. In all the rest of these lectures, the

mixing matrix will be taken equal to unity.

The next step to be achieved is to break $SU(2) \otimes U(1)$ symmetry in such a way that

a) the $U_{e.m.}(1)$ symmetry is preserved, that is the photon remains massless,

b) the fermion and weak vector bosons $W^{\pm}, Z_o$ acquire masses,

c) the theory remains renormalizable.

The last requirement is the most constraining. The standard model which fulfills all the above conditions is obtained by spontaneous symmetry breaking, or Higgs mechanism. Its description is the subject of the next section.

3. SPONTANEOUS SYMMETRY BREAKING AND THE HIGGS[5] MECHANISM IN SU(2) ⊗ U(1)

Spontaneous symmetry breaking (SSB), also called "hidden symmetry", refers to the situation where, though the Lagrangian one starts with has some exact symmetry, the corresponding physical world has not. In quantum theory, this feature may occur when there are several (in fact infinitely many) degenerate possible groundstates : this set of degenerate groundstates is symmetric as a whole, but the particular one which is chosen by nature is not. It must be stressed that, once chosen, the vacuum is unique, as it is separated from the other possible ones by infinite potential barriers. The vacuum, however, may preserve part of the original symmetry. Since in general symmetry of the vacuum implies current conservation and symmetry of the physics, the study of the vacuum symmetry properties is of fundamental importance.

a) SSB of a global U(1) symmetry

The simplest field theory example of SSB is the following. Let us consider the Lagrangian for a complex scalar field $\varphi(x) = (\varphi_1(x)+i\varphi_2(x))/\sqrt{2}$, interacting through a potential $V(\varphi)$

$$L(x) = (\partial_\mu\varphi)(\partial^\mu\varphi^*) - V(\varphi) \tag{3.1}$$

$$V(\varphi) = \frac{\lambda}{2}(\varphi^*\varphi)^2 - a\,\varphi^*\varphi \tag{3.2}$$

λ is the coupling constant, a looks like a mass term (with an unspecified sign for the moment). λ must be positive in order for the hamiltonian to be bounded from below. The Lagrangian (3.1) is invariant under the global phase transformation

$$\varphi(x) \rightarrow e^{-i\alpha}\,\varphi(x) \tag{3.3}$$

where α is an arbitrary real constant. Thus, if $\varphi(x)$ is a ground state, $e^{-i\alpha}\,\varphi(x)$ is another one, degenerate with it. We have a continuous set of degenerate ground states, parametrized by α. At the classical level(*), in order for $\varphi(x)$ to be a groundstate, it has to correspond to a minimum of the (classical) hamiltonian. The groundstate $\varphi(x)$ is thus the constant in space time $\frac{\sigma}{\sqrt{2}}e^{i\delta}$ (and therefore

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(*) quantum corrections can be computed in a systematic way using the generating functional of the one particle irreducible graphs[5b] .

of zero kinetic energy), which minimizes the potential $V(\varphi)$. From Eq. (3.2) the potential is extremum for

$$\frac{\partial V}{\partial \sigma} \equiv \frac{\lambda\sigma^3}{2} - a\sigma = 0 \tag{3.4}$$

For $a \leq 0$, the minimum is obtained for $\sigma = 0$: the classical vacuum is $\varphi \equiv 0$; it is U(1) invariant and there is no symmetry breaking. For $a > 0$, the minimum is obtained for $|\varphi| = \frac{\sigma_o}{\sqrt{2}} = \sqrt{\frac{a}{\lambda}}$, $\sigma = 0$ being a maximum of V (the "mass" squared $\frac{\partial^2 V}{\partial\varphi\partial\varphi^*}$ is negative at $\varphi = 0$). As a function of $\mathrm{Re}\varphi$, $\mathrm{Im}\varphi$, the shape of $V(\varphi)$ for $a > 0$ looks like the bottom of a bottle, as indicated on Fig. (3.1).

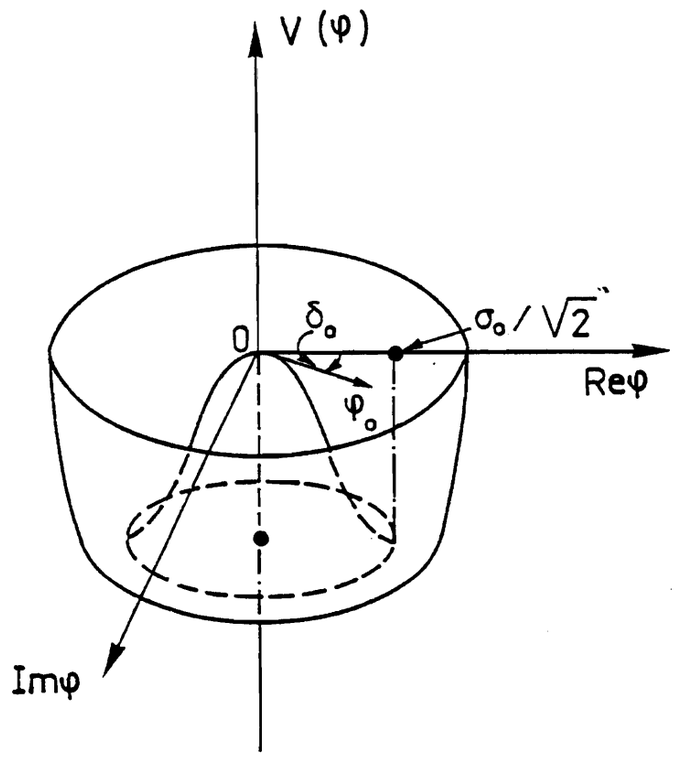

Fig. 3.1 - The minimum of the potential is reached for $|\varphi| = \frac{\sigma_o}{\sqrt{2}} = \sqrt{\frac{a}{\lambda}}$. $\varphi = 0$ *is a maximum. All the* $\varphi = \sigma_o e^{i\delta}$, $\delta \in [0, 2\pi]$ *are degenerate possible vacua.*

Among all the possible degenerate vacua, nature chooses a particular one $\varphi = \frac{\sigma_o}{\sqrt{2}} e^{i\delta_o}$, say, and the U(1) symmetry, represented by the invariance of the figure 3.1 under any rotation around the axis of the bottle, is broken. Now we can always redefine the field φ by the U(1) transformation $\varphi \to e^{-i\delta_o}\varphi$ in such a way that the new field (again noted φ) has the real vacuum expectation value $\sigma_o/\sqrt{2}$.

In order to construct a perturbation theory around the classical solution, we must use a translated field

$$\chi(x) \equiv \frac{\chi_1(x) + i\ \chi_2(x)}{\sqrt{2}} = \varphi(x) - \frac{\sigma_o}{\sqrt{2}} \quad , \tag{3.6}$$

whose vacuum expectation value vanishes. In terms of χ, the Lagrangian (3.1) reads (up to an unimportant constant)

$$\begin{aligned} L = & \frac{1}{2}\ (\partial_\mu \chi_1)^2 - \frac{1}{2}\ m_1^2\ \chi_1^2 \\ & + \frac{1}{2}\ (\partial_\mu \chi_2)^2 \\ & - \frac{\lambda}{8}\ (\chi_1^2 + \chi_2^2)^2 - \frac{\lambda\sigma_o}{2}\ \chi_1\ (\chi_1^2 + \chi_2^2) \quad , \end{aligned} \tag{3.7}$$

where we have set

$$m_1^2 = \lambda\ \sigma_o^2 \tag{3.8}$$

Let us now comment upon this new form of the original Lagrangian, in terms of the two fields χ_1 and χ_2.

i) χ_1 is a scalar *massive* field, with mass proportional to the non vanishing vacuum expectation value $\sigma_o/\sqrt{2}$ of the original field $\varphi(x)$

ii) χ_2 is a scalar *massless* field

iii) as a function of the new fields, the original Lagrangian is no more symmetric. However, the form (3.7) of the Lagrangian is *not* the most general one for two real fields χ_1 and χ_2 with quartic interactions; not only because $m_2 = 0$, but also because the mass m_1 and the 4th and 3rd degree couplings of the fields are related to each other. The use of SSB is thus a way of restricting the number of parameters of a non symmetric theory.

To summarize : starting with a Lagrangian for a complex field $\varphi(x)$, with a *global* U(1) symmetry we end up with a Lagrangian for two interacting scalar fields χ_1 and χ_2. $m_2 = 0$, and m_1 is related to the coupling constants.

The fact that $m_2 = 0$ is a direct consequence of the original symmetry. For small fluctuations of $\chi_2(x)$ around its classical value 0, one may write the original field $\varphi(x)$ as

$$\varphi(x) = \frac{\sigma_o + i\chi_2(x)}{\sqrt{2}} \simeq \frac{\sigma_o}{\sqrt{2}} \exp\,[i\chi_2(x)/\sigma_o] \tag{3.9}$$

Thus to the order χ_2^2, this field configuration leads to the same potential energy as the vacuum $\sigma_o/\sqrt{2}$ does. If moreover we choose χ_2 in (3.9) with no oscillations in $\vec{x}$ (long wave length limit), $\vec{\partial}\chi_2$ also is small (in Fourier space $\vec{k}_2\tilde{\chi}_2 \to 0$ as $\vec{k}_2 \to 0$), i.e. the kinetic energy associated with χ_2 vanishes, and we are left with another ground state. In other words, the energy ω_2 associated with the excitation (3.9) vanishes with $|\vec{k}_2|$, hence $m_2 = 0$.

The existence of a massless scalar field (Goldstone boson), associated with a global symmetry spontaneously broken by an asymmetric vacuum, is generic and is the object of the famous Goldstone theorem. The Goldstone boson survives and remains massless to all orders in perturbation theory. For non abelian global symmetries, one finds as many massless Goldstone bosons as there are group generators which do not leave the vacuum invariant. Neither such particles are observed in nature(*), nor the long range forces they would give rise to. Hence global symmetry breaking is not yet the mechanism we look for to break SU(2) ⊗ U(1) down to $U_{e.m.}(1)$. As we shall see now, in the case of *local* symmetry, the Goldstone degrees of freedom are used to give masses to the gauge fields, and associated massless bosons disappear. This is what is known as the Higgs mechanism.

b) <u>SSB for a *local* SU(2) ⊗ U(1) gauge symmetry. The Higgs mechanism</u>

One introduces a *doublet* Φ of complex scalar fields

$$\Phi = \begin{pmatrix}\Psi\\ \varphi\end{pmatrix} \quad , \qquad \Phi^\dagger = (\Psi^* \;\; \varphi^*) \quad , \tag{3.10}$$

interacting through a potential

$$\begin{aligned} V(\Phi) &= \frac{\lambda}{2}\,(\Phi^\dagger\Phi)^2 - a\,\Phi^\dagger\Phi \quad , \\ &\equiv \frac{\lambda}{2}\,(\Psi^*\Psi + \varphi^*\varphi)^2 - a(\Psi^*\Psi + \varphi^*\varphi) \quad , \end{aligned} \tag{3.11}$$

which is manifestly invariant under both SU(2) and U(1) local transformations

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(*) In a completely different context, the pions are considered as the Goldstone bosons of the spontaneously broken SU(2) chiral invariance $\left(\begin{pmatrix}u\\d\end{pmatrix} \to e^{-i\gamma_5 \frac{\vec{a}.\vec{\tau}}{2}} \begin{pmatrix}u\\d\end{pmatrix}\right)$ of massless QCD. The pion mass is not strictly zero, however, because of the non zero masses of the light quarks.

$$\Phi(x) \to \exp\{-i\,\vec{a}(x).\vec{\tau}/2\}\exp\{-i\,a'(x)\,\frac{Y}{2}\}\,\Phi \quad . \tag{3.12}$$

The kinetic part $(\partial_\mu\Phi^\dagger)(\partial^\mu\Phi)$ is not invariant because $\partial_\mu a(x)$ and $\partial_\mu a'(x)$ are not zero. As usual the symmetry is restored by introducing the gauge fields $\vec{W}_\mu$ and B_μ already considered in the previous chapter, replacing ∂_μ by the covariant derivative D_μ

$$\partial_\mu \to D_\mu = \partial_\mu + \frac{ig}{2}\vec{W}_\mu.\vec{\tau} + \frac{ig'}{2}B_\mu Y \quad . \tag{3.13}$$

The part of the Lagrangian involving Φ then reads

$$L_\Phi = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi) \tag{3.14}$$

L_Φ , as well as the pure Yang-Mills Lagrangian and the part containing fermions (see below), is locally gauge invariant. SSB occurs in a way very similar to that encountered in the previous example. The vacuum is obtained for $\vec{W}_\mu(x) = B_\mu(x) = 0$, and $\Phi(x)$ equal to a constant SU(2) spinor minimizing $V(\Phi)$:

i) $\Phi = 0$ for $a > 0$

ii) $\Phi = \frac{\sigma_o}{\sqrt{2}} \times$(arbitrary constant unit SU(2) spinor) for $a < 0$

The latter case is the one we are interested in. There is now a privileged direction in SU(2) space, which, after a global SU(2) transformation on the fields (a redefinition of the fields), allows one to write the Φ vacuum as

$$\Phi_o = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \sigma_o\end{pmatrix} \tag{3.15}$$

The symmetry associated with the transformations induced by T^+ and T^- is now broken but one combination of T^3 and Y is conserved. This combination coincides with the charge operator $Q = T_3 + Y/2$, provided Y is taken to be $Y_\Phi = 1$ (*). (The field which acquires a non vanishing expectation value has zero charge). With this choice for Y, SU(2) $\otimes$ U(1) correctly breaks down to $U_{e.m.}(1)$. Before investigating the physical content of our theory, we may use the SU(2) invariance to perform in each space time point $\vec{x}$,t a particular SU(2) transformation which

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(*) Had we defined the fields so that $\Phi_o = \frac{1}{\sqrt{2}}\begin{pmatrix}\sigma_o\\ 0\end{pmatrix}$, we would obtain $Y_\Phi = -1$, the hypercharge of the complex conjugate Higgs field.

transforms the $\Phi(x)$ components (not only its vacuum expectation value), into

$$\Phi(x) \equiv \begin{pmatrix} 0 \\ \varphi(x) \end{pmatrix} \quad , \tag{3.16}$$

where $\varphi(x)$ is a *real* field. This prescription for Φ is also a way to fix the gauge of the $W^{\pm}$ and Z fields. In this gauge, 3 out of the 4 real fields contained in Φ, namely the complex Ψ field and the imaginary part of φ have been eliminated. However, their degrees of freedom have not disappeared, because as we shall see very soon, each of the $W^{\pm}, Z_o$ bosons becomes massive and has three degrees of freedom (helicities $\lambda = \pm 1$ and 0) instead of two ($\lambda = \pm 1$). This particular gauge is called the unitary gauge as only the fields corresponding to physical fields explicitely appear in the Lagrangian. We will come back to this question of gauge fixing in Section 4 below.

For the time being, we keep $\Phi(x)$ under the form (3.16) and translate the field $\varphi(x)$ by its vacuum value in order to do perturbation theory around small fields. We set

$$\frac{\chi(x)}{\sqrt{2}} = \varphi(x) - \frac{\sigma_o}{\sqrt{2}} \tag{3.17}$$

We next identify the spectrum of the theory by looking at the quadratic part L_2 of L_Φ (Eq. 3.14). Apart from the kinetic term $\frac{1}{2}(\partial_\mu\chi)^2$, L_2 contains mass terms which are the terms proportional to σ_o^2 after the replacement (3.17) in L_Φ. As before $V(\Phi)$ generates a mass term $-\frac{1}{2}(\lambda\sigma_o^2)\chi^2$ which tells us that the neutral Higgs boson χ we are left with is massive, its mass being

$$m_H = \sigma_o \sqrt{\lambda} \tag{3.18}$$

Let us collect the σ_o^2 term in $(D_\mu\varphi)^2$. In order to do this it is convenient to remark that it amounts to calculate a quantity L_{φ^2} of the form

$$L_{\varphi^2} = \frac{1}{2}(0,\varphi)\, M^2 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \frac{1}{4}\varphi^2 \, \mathrm{Tr}[(1-\tau_3)M^2] \tag{3.19}$$

with M^2 being a 2×2 matrix given by

$$M = \left[g\, \vec{W}_\mu \cdot \frac{\vec{\tau}}{2} + g' B_\mu \frac{Y}{2} \right]^2 \tag{3.20}$$

It is then straightforward to find

$$L_{\varphi^2} = \frac{(\sigma_o+\chi(x))^2}{8} \times \left\{ g^2(W_{1\mu}^2 + W_{2\mu}^2) + (gW_{3\mu} - g'YB_\mu)^2 \right\} \tag{3.21}$$

From this expression, we immediatly see that W_1 and W_2, and hence $W^\pm$, the charged weak bosons, have acquired non zero masses, namely

$$m_{W^\pm}^2 = \frac{\sigma_o^2 g^2}{4} \tag{3.22}$$

The second term in Eq. (3.21) can be rewritten as $\frac{(\chi+\sigma_o)^2 g^2}{8\cos^2\theta}\left[\cos\theta W_{3\mu} - \frac{g'}{g}\cos\theta B_\mu Y\right]^2$, and we recognize inside the square bracket the combination

$$Z_\mu^o = \cos\theta W_{3\mu} - \sin\theta B_\mu \tag{3.23}$$

which we had identified in section 2, as the neutral weak vector boson (Eq.(2.13). Eq. (3.23) is true provided $g'Y/g = tg\theta$, which, together with $g'/g = tg\theta$ (Eq. 2.21) requires that the eigenvalue of Y for the Higgs doublet is 1. This is just a check as we had already obtained Y=1 by noting above that the part of the Higgs doublet Φ which acquires a non vanishing vacuum expectation value must correspond to a neutral field, not to destroy exact $U_{e.m}(1)$ invariance. The orthogonal combination, the photon $A_\mu \equiv \sin\theta W_{3\mu} + \cos\theta B_\mu$ does not appear in $L\varphi^2$ and thus remains massless. This has to be so since A_μ is associated with the unbroken symmetry. A second consequence of this SSB scheme is the celebrated relation between the charged and neutral weak boson masses

$$m_{Z_o}^2 = \frac{\sigma_o^2 g^2}{4\cos^2\theta} = \frac{m_W^2}{\cos^2\theta} \tag{3.24}$$

As to the physical Higgs boson χ, we note (Eqs. (3.11,16 and 17)) that it has cubic and quartic self couplings. It also couples to the gauge boson (Eq. 3.21) : one or two Higgs bosons couple to a pair of Z^o's and to a pair W^+W^- with predicted strengths.

A few numbers

By comparison of the Salam-Weinberg model for β decay to the Fermi interaction (see Fig. 1.1), one gets

$$G_F/\sqrt{2} = g^2/(8m_W^2) = \frac{1}{2\sigma_o^2} \quad , \tag{3.25}$$

independent of the gauge coupling g.
From unification of weak and electromagnetic interactions

$$g^2/4\pi = \alpha/\sin^2\theta$$

From low energy weak interaction phenomenology

$$G_F = 1.05 \times 10^{-5}\ m_{proton}^{-2}$$

From high energy weak interaction phenomenology (charged and neutral weak currents, ed polarization experiments etc...)

$$\sin^2\theta = 0.229 \pm 0.014$$

Eqs. (3.24,25) then allow to compute (with errors essentially due to that on $\sin^2\theta$)

$$\begin{cases} g^2/4\pi = 3.11.10^{-2} \\ m_W = 81.1 \text{ GeV} \\ m_{Zo} = 90.1 \text{ GeV} \\ \sigma_o = 260 \text{ GeV} \\ m_{Higgs} = 260\ \sqrt{\lambda}\ \text{ GeV} \end{cases} \tag{3.26}$$

As λ is undetermined, the Higgs mass is unknown.

c) Couplings of Higgs to fermions. The fermion masses.

Fermions have been left aside in the above description of the Higgs SSB mechanism. We recall that in the unbroken theory, all fermions are massless since a mass term of the form

$$-m_f\ (\bar{f}_L\ f_R + \bar{f}_R\ f_L)$$

is not invariant under SU(2) when L and R fermions belong to different representations of SU(2). For the standard model where L-fermions (Resp. R-fermions) are doublets (Resp. singlets), the above term transform under SU(2) like a doublet.

With a Higgs doublet Φ at our disposal we can build a new piece $L_{\Phi F}$, by coupling directly the Φ field to $\bar{f}_L f_R$ in an SU(2) invariant way. In fact, there are two independent singlet fermions associated with each SU(2) doublet, u_R and d_R for example for the (u,d) pair (with the possible exception of the (ℓ,ν) pairs if the ν's are indeed massless - no right neutrino). In the unitary gauge, $\Phi = \begin{pmatrix} 0 \\ \varphi(x) \end{pmatrix}$, and φ may couple to a $T_3 = -1/2$ fermion, while the charge

conjugate field $\Phi^c = i\tau_2 \begin{pmatrix} 0 \\ \varphi(x) \end{pmatrix}^* = \begin{pmatrix} \varphi(x) \\ 0 \end{pmatrix}$ may independently couple to a $T_3 = +1/2$ fermion. As an example, a general Yukawa coupling of Φ to (u,d) quarks is

$$-g_d\ \varphi(x)\bar{d}_L d_R - g_u\ \varphi(x)\bar{u}_L u_R + \text{c.c.}\ , \tag{3.27}$$

with no relation between g_d and g_u (which can be complex). In the case of SSB, $\varphi(x) = \frac{\sigma_o + \chi(x)}{\sqrt{2}}$ and apart from the Yukawa interaction between χ and the fermions, a mass term is generated independently for each fermion f of the theory which has a right component. The mass term for f is

$$-g_f \frac{\sigma_o}{\sqrt{2}} \bar{f}_L f_R - g_f^* \frac{\sigma_o}{\sqrt{2}} \bar{f}_R f_L$$

By a phase transformation on f_L (and/or f_R), are can always make g_f real positive, so that one finally gets

$$m_f = \frac{\sigma_o}{\sqrt{2}} g_f \tag{3.28}$$

For the general problem of fermion masses, see section 8c.

Fermion masses are thus "naturally" generated by the same SSB mechanism which gives the gauge bosons masses. However, there are no predictions for the fermion masses, as they are proportional to arbitrary Yukawa coupling constants. That neutrino are massless in the present model has been put in by hand in deciding that there is no right handed neutrino(*). Nevertheless, Eq. (3.28) is interesting if read as giving g_f as a function of m_f and σ_o which are known experimentally : the χ field is predicted to couple to a fermion proportionally to the fermion mass. Higgs production in particle collisions is thus most probably accompanied by heavy quark production. The phenomenology, and references to the relevant literature, of the Higgs boson production and decay may be found in Ref.[7]. Various bounds on the Higgs and fermion masses have been discussed (see Ref.[8]). They will not be reproduced here. Let us just say that the general strategy for deriving them is to infer from the success of perturbative calculations in electroweak interactions that the various new couplings introduced

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(*) Using a Higgs triplet, the left handed neutrino by itself could be given a so-called Majorana mass [6]. A Majorana particle is a spin 1/2 particle (massive or not) of definite charge conjugation. Such a particle has no electromagnetic interaction at all. In the zero mass case, there is no difference between Majorana and Dirac particles.

in the model, (λ for the self couplings of the Higgs and g_f for its Yukawa coupling to the fermions) are small enough. From Eqs. (3.18) and (3.28), bounds on these couplings imply bounds on the corresponding masses.

d) The number of parameters in the standard SU(2) $\otimes$ U(1) model

Restricting ourselves to the 6 quark model (3 quark pairs (u,d), (c,s), (t,b) and 3 lepton pairs (ν_e,e), (ν_μ,μ), (ν_τ,τ), with massless neutrinos), we have 9 fermion masses (or their 9 couplings to the Higgs boson). Then come the 4 parameters (3 angles and one CP violating phase) of the unitary matrix which allows us to build the hadronic weak charged currents starting with the mass eigenstate quarks (generalisation of Cabibbo mixing (see section 8c)). Associated with the gauge fields are the coupling α and the Weinberg mixing angle θ, with the Higgs doublet the self coupling λ and the vacuum expectation value σ_o , that is 4 more parameters. The minimum scheme then with 1 Higgs doublet and 3 fermion families involves 17 free parameters. This number increases very fast with the number F of families. As we will see in section 8c, the unitary generalized Cabibbo matrix contains $(F-1)^2$ physically relevant parameters. In addition there are 3F fermion masses, so that the total number of parameters depends on F as F(F+1) : 8 more parameters if F changes from 3 to 4 ! Here we might ask a third question.

Question III : how can one reduce the number of free parameters ?

We close this section with a comment about the possibility of introducing more than 1 Higgs doublet or larger SU(2) Higgs multiplets. First we observe that multi-Higgs systems cause some theoretical problems.

(i) in order for charge to be conserved, all vacuum expectation values have to point in the same SU(2) direction, which impose constraints on the parameters of the Lagrangian.

(ii) unless unnatural conditions are fulfilled[9], models with multi-Higgs systems involve flavour changing neutral interactions induced by Higgs exchange. This is in contradiction with experimental data on, say, $K_L \to \mu^+\mu^-$. That a single Higgs doublet does not lead to such a disease is shown in section 8c : the Higgs interaction becomes diagonal in flavour space at the same time as the mass matrix is diagonalized.

Assuming that these difficulties are overcome, let us examine the consequences of introducing an arbitrary number of arbitrary Higgs multiplets. In Eq. (3.13), $\frac{\vec{W}.\vec{\tau}}{2}$ and $\frac{Y}{2}$ B are replaced respectively by

$$\sum_h \vec{W}.\vec{T}_h \quad \text{and} \quad \frac{1}{2}\sum_h Y_h B \quad , \tag{3.29}$$

where $\vec{T}_h$, Y_h are the representations of the SU(2) ⊗ U(1) generators for a Higgs multiplet labelled by h. For each h, the charges inside the multiplet are given by

$$Q_h = T_{3_h} + Y_h/2 \quad ,$$

where T_{3_h} runs from $-I_h$ to $+I_h$, I_h being the weak isospin of h. If Y_h is such that one of the charges Q_h is zero, the vacuum expectation value of the corresponding field φ_h^o can be used to generate masses for the gauge bosons. For this field, $Y_h/2 = -T_{3_h}$ so that its contribution to the mass term is

$$\frac{1}{2} g^2\sigma_h^2 \left[W_1^2 \; T_{1_h}^2 + W_2^2 \; T_{2_h}^2\right] + \frac{g^2\sigma_h^2}{2\cos^2\theta} (\cos\theta \; W_3 - \sin\theta \; B_\mu)^2 \; T_{3_h}^2$$

where T_{ih}^2, i=1,2,3 here represents the matrix element of $T_{i_h}^2$ in the neutral state φ_h^o. The ratio $M_W^2/(\cos^2\theta \; M_{Z_o}^2)$ can be read off directly from this expression :

$$\rho = \frac{m_W^2}{\cos^2\theta \; m_{Z_o}^2} = \frac{\sum_h \sigma_h^2 \; T_{1_h}^2}{\sum_h \sigma_h^2 \; T_{3_h}^2} \tag{3.30}$$

Given T_{3_h} , the eigenvalue of T_3 for ϕ_h^o whose vacuum expectation value is σ_h, one has

$$T_{1_h}^2 = \frac{1}{2}(T_{1_h}^2 + T_{2_h}^2) = \frac{1}{2}(\vec{T}_h^2 - T_{3_h}^2)$$
$$= \frac{1}{2}(I_h(I_h+1) - T_{3_h}^2)$$

The value predicted for ρ then is in general

$$\rho^{th} = \frac{1}{2} \frac{\sum_h \sigma_h^2 \; (I_h(I_h+1) - T_{3_h}^2)}{\sum_h \sigma_h^2 \; T_{3_h}^2} \tag{3.31}$$

The value of ρ can be experimentally measured by comparing the strength of the charged weak current relative to the neutral weak current. Indeed, in the same way as charged current interactions are measured by

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} \quad ,$$

we get from Eq. (2.22) a Fermi like neutral interaction measured by

$$\frac{G_N}{\sqrt{2}} = \frac{g^2}{\cos^2\theta} \frac{1}{8m_{Z_o}^2} \quad ,$$

so that ρ is given as well by

$$\rho = \frac{G_F}{G_N}$$

The present experimental value determined by comparison of $\nu,\bar{\nu}$ induced charged and neutral current interactions[10] is

$$\rho_{exp} = 0.98 \pm 0.05 \tag{3.32}$$

If there are several different values of I_h in the Higgs sector, Eqs. (3.31, 32) are not very constraining. In fact any a priori value of ρ can be reached. On the contrary an arbitrary number of replications of the same Higgs pattern with the same I and T_3 leads to a simpler equation

$$\rho_{th} = \frac{1}{2} \frac{I(I+1) - T_3^2}{T_3^2} \quad , \tag{3.33}$$

independent of the continuous variables σ_h.

If we admit that experiment indicates $\rho=1$, the equation becomes

$$3T_3^2 = I(I+1)$$

A little bit of arithmetics shows that the lowest solution for I,T_3 integers is

$$I = 3 \qquad T_3 = \pm 2$$

(The next one is $I = 48$, $T_3 = \pm 28$!).

It is a quite exotic configuration, not very attractive to accomodate the known fermions. For 1/2-integer isospin, the lowest solution is the standard one

$$I = 1/2 \qquad T_3 = \pm\, 1/2 \quad ,$$

(The next one is $I = 25/2$, $T_3 = \pm\, 15/2$!).
The conclusion is : there is a remarkable agreement between the standard theory with $I = 1/2$ and the experimental value of ρ. What we have learnt here above is that an arbitrary number of Higgs doublets would work equally well, up to the above mentioned problems i) and ii).

4. ANOMALIES.

Unitarity of the S-matrix and renormalization

In the unitary gauge we have used to describe the Higgs mechanism, the particle content of the theory is made apparent at the very beginning since 3 out of the 4 Higgs fields disappear from the Lagrangian, the corresponding degrees of freedom being carried by the helicity zero components of the weak bosons $W^{\pm}, Z_o$. In particular, there are no unwanted Goldstone modes. However the unitary gauge is not suited for the study of the renormalizability of the theory[11]. In this gauge, the propagator of a massive gauge boson W is in momentum space

$$P_{\mu\nu}^{\text{unit.}}(k) = \frac{-1}{k^2-m_W^2} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}\right] \quad , \tag{4.1}$$

which is of order 1 for large k components, leading to badly behaved Feynman graph integrands. This fact makes the renormalization program difficult to pursue. It is better to come back to the original Lagrangian which involves the Goldstone bosons. The renormalization then is simpler to achieve,but in turn one has to show that the Goldstone bosons have no physical effects. This happens to be true only in the absence of the so-called anomalies, as we are to show now.

For simplicity, we consider the case of SSB for a U(1) gauge symmetry, where only one complex Higgs field $\varphi(x)$ is introduced. The non-vanishing vacuum expectation value $\sigma_o/\sqrt{2}$ of φ can be made real by a *global* (i.e. x-independent) phase transformation. So we can write

$$\varphi(x) = \frac{1}{\sqrt{2}} \left(\sigma_o + \chi_1(x) + i\chi_2(x)\right) \quad , \tag{4.2}$$

as in section 3.a, but we *do not set* $\chi_2 \equiv 0$ by a *local* gauge transformation. Proceeding next as in section 3.b, but with $D_\mu = \partial_\mu + igW_\mu$, we get the Lagrangian :

$$\begin{aligned} L = & \frac{1}{2}(\partial_\mu\chi_1)^2 - \frac{1}{2}\lambda\sigma_o^2\chi_1^2 \\ & + \frac{1}{2}(\partial_\mu\chi_2)^2 \\ & - \frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 + \frac{1}{2}g^2\sigma_o^2 W_\mu W^\mu \\ & + g\,\sigma_o\, W_\mu\, \partial^\mu\chi_2 \\ & + \text{terms of degrees 3 and 4 in the fields.} \end{aligned} \tag{4.3}$$

The first two lines are the same as in Eq. (3.7). The third one contains the usual $-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ and the W mass term generated by SSB. Lastly comes a term which is new with respect to the case of the unitary gauge $\chi_2 \equiv 0$, namely $g\,\sigma_o\, W_\mu\, \partial^\mu \chi_2$, coming from $(D_\mu\varphi)^*(D^\mu\varphi)$. This crossed term mixes the W and χ_2 propagations. The action $\int d^4x\, L(x)$ remains unchanged if this term is replaced by $-g\,\sigma_o\left(\partial_\mu W^\mu\right)\chi_2$ (integration by part), which can be eliminated by choosing the (Landau-'t Hooft) gauge

$$\partial_\mu W^\mu = 0$$

The quadratic part of the Lagrangian is diagonalized and the W-propagator now is

$$P_{\mu\nu}^{(\text{Landau})}(k) = \frac{-1}{k^2-m_W^2}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] \quad , \tag{4.4}$$

to be compared with Eq. (4.1) in the preceding case. It behaves like $(k^2)^{-1}$ at large k^2 values, which leads to the dimensional counting of a renormalizable theory.

As a slight disgression, we mention that the Landau gauge is a special case of the general 't Hooft gauge (or R_ξ gauge). In the R_ξ gauge, one adds, to the Lagrangian (4.3), the "gauge fixing" term $-\frac{\xi}{2}\left(\partial_\mu W^\mu - g\,\frac{\sigma_o}{\xi}\,\chi_2\right)^2$, where ξ is the gauge parameter. The Landau gauge is recovered in the limit $\xi \to \infty$. In all cases, the crossed term $-g\,\sigma_o\,\partial_\mu W^\mu\,\chi_2$ is exactly cancelled out. The W-propagator is

$$P_{\mu\nu}^{\xi}(k) = \frac{-1}{k^2-m_W^2}\left[g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{k_\mu k_\nu}{k^2-\frac{m_W^2}{\xi}}\right] \quad ,$$

and the χ_2 field has a ξ dependent mass

$$m_{\chi_2}^2 = \frac{m_W^2}{\xi}$$

It is actually massless in the Landau gauge. In the absence of anomalies, the theory will be gauge invariant in the sense that all ξ-dependent quantities cancel in the S-matrix. Conversely, anomalies are manifested by poles at ξ-dependent positions in the S-matrix.

We close the disgression on R_ξ-gauges, and go on with ξ infinite (Landau).

The χ_2 field, which here is massless, is present in the Lagrangian. Is the unwanted Goldstone boson back ? In fact, the comparison of Eqs. (4.1) and (4.4) leads to

$$P_{\mu\nu}^{(\text{Landau})} = P_{\mu\nu}^{(\text{Unit.})} - \frac{k_\mu k_\nu}{m_W^2\, k^2} \quad , \tag{4.5}$$

which expresses that apart from the physical W pole at $k^2 = m_W^2$, $P^{(\text{Landau})}$ contains a pole at $k^2 = 0$, with furthermore the wrong sign (ghost). There are thus two origins for poles at $k^2 = 0$. It has been shown that, in the absence of fermions, or if the fermions have no axial coupling to the gauge bosons, the poles coming from the χ_2 field and from the gauge field cancel each other, leading to an S-matrix without any Goldstone boson present[12](*) (hence the word unitary used to qualify the particular gauge where χ_2 is absent from the beginning). Let us remark that in the Landau gauge, the Higgs phenomenon is *not* the absence of Goldstone boson - the Goldstone theorem remains valid in the presence of gauge fields, but rather the presence of a ghost pole, which precisely cancels the observable effects of the Goldstone boson.

The anomaly problem

The only exception to the above cancellation theorem is provided by the existence of the so-called triangle anomaly[13]. We present the problem in the form it has been discussed (and solved) by Bouchiat, Iliopoulos, Meyer[14] . One considers a simplified model with a gauge symmetry $U_{e.m.}(1) \otimes U_W(1)$. The model contains one fermion f which has usual E.M. interactions with the photon A_μ , and transforms under the "weak" gauge group U_W as

$$\begin{aligned} f_L &\to e^{-ig\theta(x)}\, f_L \\ f_R &\to e^{ig\theta(x)}\, f_R \end{aligned} \tag{4.6}$$

The f-W interaction is thus described by

$$\begin{aligned} & i\bar{f}_L\, \gamma^\mu(\partial_\mu + ig\, W_\mu) f_L + i\bar{f}_R\, \gamma^\mu(\partial_\mu - ig\, W_\mu) f_R \\ &\quad = i\bar{f}(\partial_\mu + ig\, W_\mu \gamma_5)\gamma^\mu\, f \end{aligned} \tag{4.7}$$

(*) In the case of a non Abelian theory, the quantization procedure introduces additional fields (the Fadeev-Popov ghosts) which also participate in the cancellation of the unwanted Goldstone bosons and finally disappear from the S-matrix.

One next introduces a complex neutral Higgs field φ, coupled to f according to

$$-G\,\bar{f}_R\,f_L\,\varphi^* + \text{h.c.} \tag{4.8}$$

This coupling is U_W invariant provided under $U_W(1)$

$$\varphi \to \varphi\, e^{-2ig\theta(x)} \tag{4.9}$$

$U_W(1)$ is subsequently broken as φ acquires a non vanishing expectation value $\sigma_o/\sqrt{2}$. We set

$$\varphi = \frac{1}{\sqrt{2}}\,(\sigma_o + \chi_1 + i\chi_2) \quad , \tag{4.10}$$

and using the χ fields as usual we get the couplings

$$-\frac{G}{\sqrt{2}}\,\bar{f}\,f\,\chi_1 - i\,\frac{G}{\sqrt{2}}\,\bar{f}\,\gamma_5 f\,\chi_2 \tag{4.11}$$

and the fermion mass is

$$m = \frac{\sigma_o}{\sqrt{2}}\,G\,. \tag{4.12}$$

Finally, according to the φW interaction induced by (4.9) , the W boson has the mass m_W ,

$$m_W^2 = 4g^2\,\sigma_o^2 \tag{4.13}$$

In the Landau gauge, the W propagator is given by Eq. (4.5), its residue at $k^2 = 0$ being $-\frac{k_\mu k_\nu}{m_W^2}$, while the χ_2 propagator $\frac{1}{k^2}$ has residue +1. We are now to show that there is a pole at $k^2 = 0$ in the forward $\gamma\gamma$ elastic amplitude, with a non vanishing residue, the cancellation between the Goldstone pole and that of the W propagator being incomplete. At the lowest order in e and g, these poles occur through the following Feynman graphs (the notations are indicated directly on the drawings, together with the residues at $k^2 = 0$ of the W and χ_2 propagators). The residues corresponding to each of the graphs are

$$R_W = -\frac{e^4 g^2}{m_W^2}\left(k^\alpha T^{\mu\nu}_\alpha\right)\left(k^{\alpha'} T^{\mu'\nu'}_{\alpha'}\right) \tag{4.14}$$

$$R_{\chi_2} = \frac{e^4 G^2}{2}\,U^{\mu\nu}\,U^{\mu'\nu'} \tag{4.15}$$

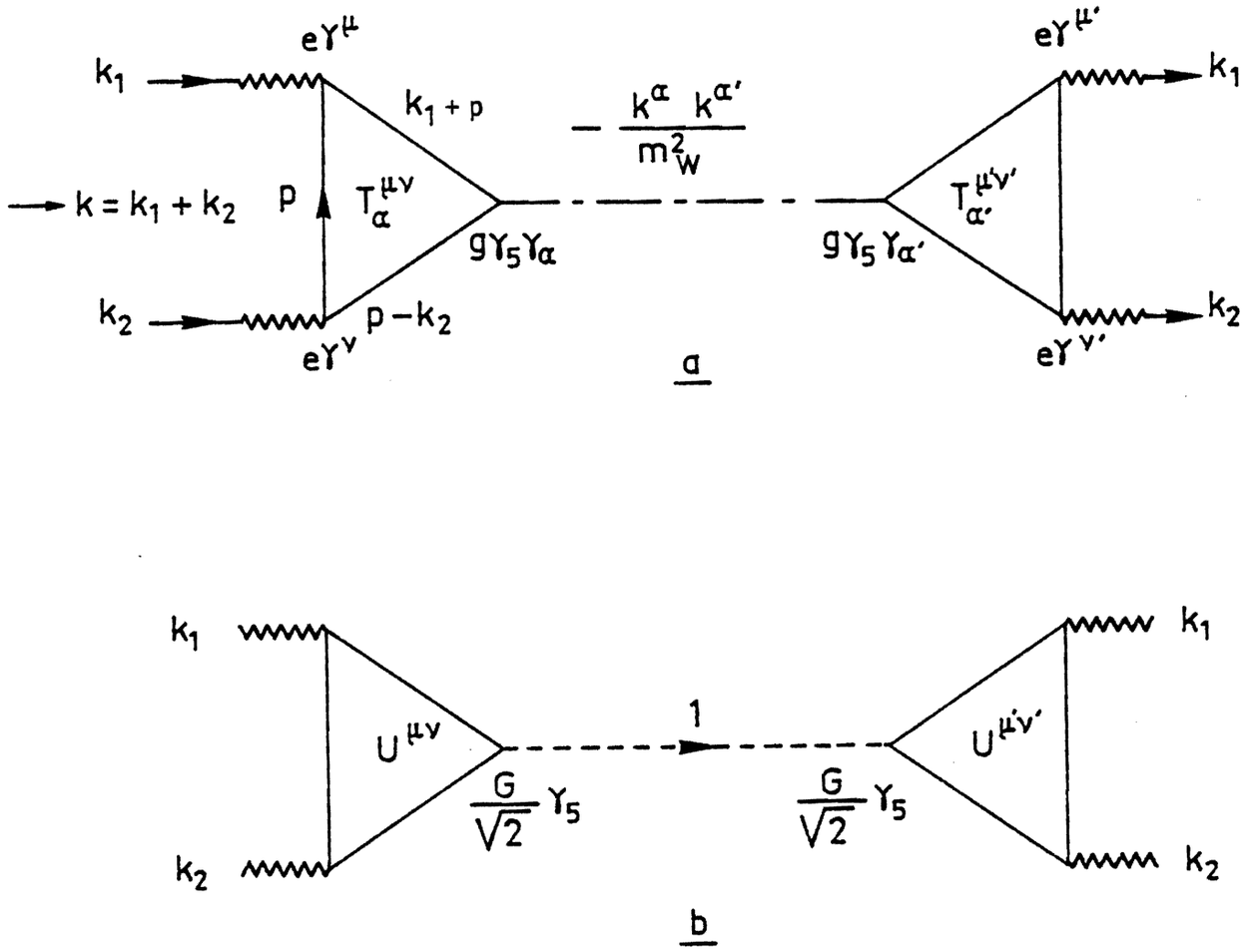


Fig. 4.1 - Contributions to the pole at $k^2=0$ in the forward $\gamma\gamma$ scattering amplitude: W in the direct channel (a) ; Goldstone boson χ_2 in the direct channel (b).

Using the expressions (4.12) and (4.13) of the f and W masses, the total residue can be written as

$$R \equiv R_W + R_{\chi_2} = -\frac{e^4}{4\sigma_o^2}\left\{\left(k^{\alpha}T_{\alpha}^{\mu\nu}\right)\left(k^{\alpha'}T_{\alpha'}^{\mu'\nu'}\right) - 4m^2\ U^{\mu\nu}\ U^{\mu'\nu'}\right\} \quad .(4.16)$$

It is understood in these expressions that $T_{\alpha}^{\mu\nu}$, $U^{\mu\nu}$ represent the sum of the two terms where the two photon lines have been exchanged. Hence they are symmetric in the change ($k_1 \leftrightarrow k_2$, $\mu \leftrightarrow \nu$). It is clear on (4.16) that if $k^{\alpha}T_{\alpha}^{\mu\nu}$, the

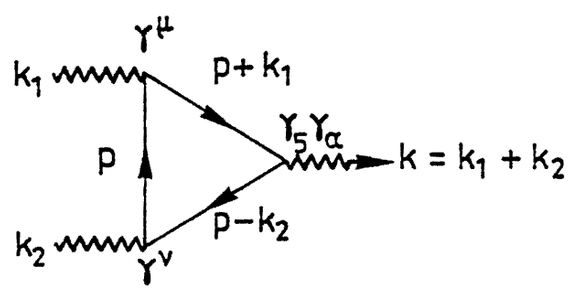
divergence of the axial current carried by W, is equal to $2mU^{\mu\nu}$, then $R=0$. One could expect that it is so : recall that the Noether theorem states that to any exact continuous symmetry of the Lagrangian is associated a conserved current, so that in the limit where the $U_W(1)$ symmetry is restored, $m=0$ and $k^\alpha T_\alpha^{\mu\nu}$ should vanish. In fact, the so-called "normal" Ward identity

$$k^\alpha T_\alpha^{\mu\nu} = 2m\, U^{\mu\nu} \tag{4.17}$$

can be derived, as is the Noether theorem, from the classical equations of motion. It can also be obtained from formal (but illegal) manipulations on Feynman graph integrands. However, Adler has shown[13] by actual calculation of T and U that

$$k^\alpha T_\alpha^{\mu\nu} = 2m\, U^{\mu\nu} + 8\pi^2\, \varepsilon^{\mu\nu\rho\sigma}\, k_{1_\rho}\, k_{2_\sigma} \quad , \tag{4.18}$$

a relation which is known as the *anomalous* Ward identity : in the case of the axial current, the "normal" identity does not survive the one loop correction. We do not want to reproduce the calculation which leads to the result (4.17), but just draw the attention of the reader to a few points which are useful for what follows. The contribution of the graph



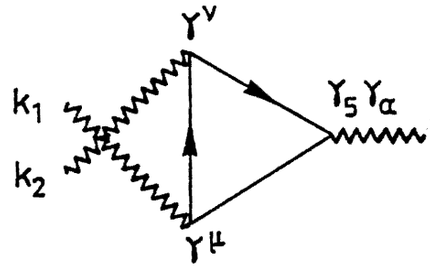
to the tensor $T_\alpha^{\mu\nu}$ is

$$T_{1\alpha}^{\mu\nu} = \int_{\text{Reg.}} d^4p\, \mathrm{Tr}[t_\alpha^{\mu\nu}] \tag{4.19}$$

where the loop integral $\int_{\text{Reg}} d^4p$ means that some regularization is needed, on which we will come back later on. The trace is made on Dirac indices, and

$$t_\alpha^{\mu\nu} = \gamma^\mu \frac{1}{(\not p - m)} \gamma^\nu \frac{1}{(\not p - \not k_2 - m)} \gamma_5\gamma_\alpha \frac{1}{(\not p + \not k_1 - m)} \tag{4.20}$$

The graph with the two photon being interchanged, namely



happens to give the same contribution as $t_\alpha^{\mu\nu}$. In order to show that, it is convenient to use -p as integration variable, so that the second contribution involves Tr $t_{2\alpha}^{\mu\nu}$ with

$$t_{2\alpha}^{\mu\nu} = \gamma^\nu \frac{1}{-\not p-m} \gamma^\mu \frac{1}{-\not p-\not k_1-m} \gamma_5\gamma_\alpha \frac{1}{-\not p+\not k_2-m} \tag{4.21}$$

Using that Tr t_2 is invariant under transposition, and its invariance under cyclic permutations of the factors, its contribution to $T_\alpha^{\mu\nu}$ is equal to that of

$$t_{2\alpha}'^{\mu\nu} = \gamma^{\mu^T} \frac{1}{-\not p^T-m} \gamma^{\nu^T} \frac{1}{-\not p^T+\not k_2^T-m} \gamma_\alpha^T\gamma_5 \frac{1}{-\not p^T-k_1^T-m}$$

$$= -C\, \gamma^\mu \frac{1}{\not p-m} \gamma^\nu \frac{1}{\not p-\not k_2-m} \gamma_\alpha\gamma_5 \frac{1}{\not p+\not k_1-m} C^\dagger \quad ,$$

where C is the charge conjugation matrix, such that

$$C\, \gamma^\beta\, C^\dagger = -\gamma^{\beta^T} \quad , \quad C\, \gamma_5\, C^\dagger = \gamma_5$$

Since $-\gamma_\alpha\gamma_5 = \gamma_5\gamma_\alpha$, we find $t_{2\alpha}'^{\mu\nu} \equiv t_\alpha^{\mu\nu}$

Hence

$$T_\alpha^{\mu\nu} = 2 \int_{\text{Reg}} d^4p\ \text{Tr}[t_\alpha^{\mu\nu}] \tag{4.22}$$

The same doubling occurs for $U^{\mu\nu}$. Note here for future reference that if γ_5 is replaced by 1 (vector coupling), there is one - sign less in the manipulations made for $t_{2\alpha}^{\mu\nu}$ above, so that Tr t and Tr t_2 exactly *cancel* each other (special case of the Furry theorem : a fermion loop with an odd number of photon legs attached gives no contribution).

The integral involved in (4.22) is a priori linearly divergent (of the form $\int \frac{d^4p}{p^3}$ at large p's). As a consequence, given a prescription for computing

(4.22), one might get a different result by making the change of variable

$$p \to p + \lambda_1 k_1 + \lambda_2 k_2 \quad . \tag{4.23}$$

That it is possible to produce an unambiguous contribution from the triangle graph comes from the requirement of gauge invariance with respect to the photon indices μ,ν, which states :

$$k_{1\mu}\, T_\alpha^{\mu\nu} = k_{2\nu}\, T_\alpha^{\mu\nu} = 0 \quad . \tag{4.24}$$

These two relations are used to define two a priori divergent integrals appearing in (4.21) as a combination of the other integrals, which converge. The rest of the calculation is straightforward. One computes $2m\, U^{\mu\nu}$ and compares with $(k_1+k_2)^\alpha T_\alpha^{\mu\nu}$ to find the result announced in Eq. (4.17). The anomalous term $\varepsilon^{\mu\nu\rho\sigma}\, k_{1\rho}\, k_{2\sigma}$ comes from a term proportional to $\varepsilon^{\mu\nu\alpha\beta}(k_{1\beta}-k_{2\beta})$ in $T_\alpha^{\mu\nu}$, the coefficient of which, formally given by a divergent integral, being fixed by the condition (4.24). This condition actually forbids arbitrary translations of the form (4.23), which would generate additional terms linear in the 4-momenta k_1, k_2.

The above results can be understood in the following way. Feynman graphs must be regulated prior to any manipulation. Anomalies happen when there is no regularization which preserves all the Ward identities of the formal theory (the normal ones) for both types of vertices, vector (Eq. (4.24)) and axial (Eq. (4.17)). When it is so, some of the gauge symmetries of the Lagrangian are broken by radiative corrections (in the case of SSB, by "gauge symmetry" we mean independence of the S-matrix on the gauge fixing parameter ξ). The anomalous Ward identity precisely expresses this lack of symmetry of the theory with respect to that of the Lagrangian. The result (4.18) is obtained if the electromagnetic current conservation (Eq. (4.24)) is imposed. One regularizes the theory in a $U_{e.m.}(1)$ invariant way (e.g. Pauli-Villars regulators). Then the regulated T and U obey normal vector and anomalous axial Ward identities. As the regulators are removed, T and U are finite (no renormalization) and thus obey the same identities.

In the discussion of the anomalies, we have emphasized the point of view of the unitary of the S-matrix. The anomaly problem is at the same time intimately connected with the renormalizability of the theory : the counter-terms in the Lagrangian which would cancel the anomalous piece of Eq. (17)

is non renormalizable and gauge dependent.

The solution to the anomaly problem

Suppose now we have not 1, but n fermions f_j in the Lagrangian, with charges e_j, g_j and arbitrary masses m_j (given by arbitrary couplings G_j to Higgs fields φ_j). From the expression (4.14), we see that the residue at $k^2 = 0$ of the pole contained in the W propagator is proportional to

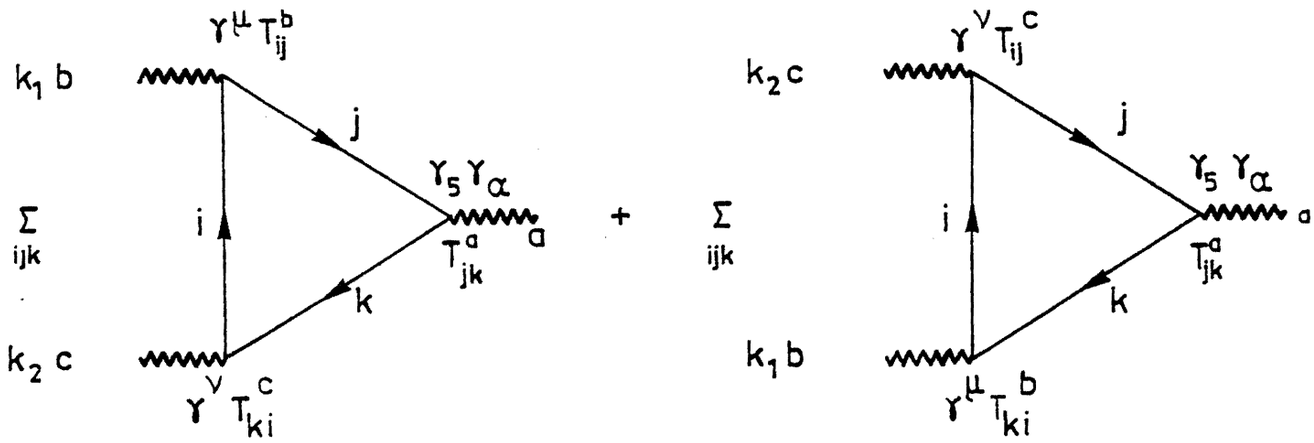
$$a^2 = \left(\sum_{i=1}^{n} e_i^2 \, g_i \right)^2 \quad .$$

Therefore, the condition for the triangle anomaly to disappear is $a = 0$. In the present simplified model, this condition can be realized, e.g., with two fermions of opposite weak charges $g_1 = -g_2$, and electric charges equal or opposite. It has been shown that once the 3 boson vertex has no anomaly at the one loop level, there is no anomaly at all for any diagram at any order in perturbation theory[15]. Hence $a = 0$ is the only condition to be fulfilled for the renormalization program to be pursued.

The condition $a = 0$ clearly is a constraint on the quantum numbers of the fermions. As a consequence, if these quantum numbers are related to each other by symmetry properties induced by some gauge group, the absence of anomaly appears as a condition either on the group itself or on the group representations to which the fermions belong. If this condition is fulfilled the theory is said to be anomaly free. We actually have to consider anomaly free theories only, since otherwise we are faced with all the problems of gauge dependent quantities in the S-matrix or of non renormalizable counterterms in the Lagrangian. Note that these problems are present in any gauge theory, abelian or not, spontaneously broken or not. It follows that the absence of anomaly is a constraint of fundamental importance for grand unified models, as we shall see at length.

We consider the triangle graph in a general case of gauge symmetry, for 3 arbitrary external gauge fields with group indices a,b,c. They couple to the fermion loop through representations T^a, T^b, T^c of the associated generators, with strengths measured by the gauge coupling constants. Note that T^a, T^b, T^c need not be the generators of the same simple Lie group : for example the W^+W^-B vertex involves the generators T^+, T^- of SU(2) and Y of U(1), the coupling constants being respectively g and g'. Finally, the couplings of the gauge fields

to the fermion loop may involve γ_5 or not. In all cases, since $\gamma_5^2 = 1$, any odd (even) number of γ_5's can always be reduced to only one (zero) at a given place. Therefore, as far as the anomaly is concerned, we have to consider



Due to the equality of the two Feynman graph contributions to the anomaly, the absence of anomaly is expressed by

$$D^{abc}(R) \equiv \mathrm{Tr}_R\,[T^a\{T^b,T^c\}] = 0 \qquad . \tag{4.25}$$

The curly bracket means anticommutator, and the trace runs over the indices of the fermion representation R. In order to draw useful consequences from this equation, we first recall a few properties of traces over group representations.

i) $\mathrm{Tr}_R\,[T^a] = 0$ for any representation R (irreducible or not) of any *simple* Lie group. But $\mathrm{Tr}[T^a] \neq 0$ for U(1) which is not simple.

ii) If the group G is a direct product of Lie groups $G^{(1)}$, $G^{(2)}$,..., then any T is the direct sum of its representations in each group $T^{(1)}, T^{(2)}, \ldots$ and

$$\mathrm{Tr}_R[T] = \mathrm{Tr}_{R_1}(T^{(1)}) + \mathrm{Tr}_{R_2}(T^{(2)}) + \ \ldots \tag{4.26}$$

$$\mathrm{Tr}_R[T^{(1)}T^{(2)}\ldots] = \mathrm{Tr}_{R_1}T^{(1)} \times \mathrm{Tr}_{R_2}T^{(2)} \times \ \ldots \tag{4.26'}$$

iii) If the representation R is reducible, $R = R_1 \oplus R_2 + \ldots$

$$\mathrm{Tr}_R[T] = \sum_i \mathrm{Tr}_{R_i}[T] \quad . \tag{4.27}$$

iv) D^{abc} vanishes for any real representation of a Lie group. The proof goes as follows. If T is the matrix representing a generator in a unitary representation R, $-T^T$ represents it in the conjugate representation $\bar{R}$: $T = T^\dagger$ implies

$$\left(e^{-i\vec{a}.\vec{T}}\right)^* = e^{-i(\vec{a}.(-\vec{T}^T))} \quad .$$

If R is real, by definition R and $\bar{R}$ are equivalent, i.e. they differ by a unitary transformation U. Under U, D^{abc} is invariant (trace property). Hence

$$\begin{aligned} D^{abc} &= -\,\mathrm{Tr}\left(T^{a^T}\left\{T^{b^T}\ T^{c^T}\right\}\right) \\ &= -\,\mathrm{Tr}(T^a\{T^b\ T^c\}) = -\,D^{abc} \quad . \end{aligned}$$

v) SU(N), $N \neq 2$ and SO(6) (which is locally identical to SU(4)) are the only simple compact Lie groups which have representations with non zero D^{abc}[16,18]. (Recall that U(1) is not simple. All irreducible representations of U(1) have $D \neq 0$, but the trivial one $Y = 0$). In a first step, the demonstration uses the fact that the only simple compact Lie groups which have non real representations are SU(N) $N \neq 2$, SO(4N+2) $N \geq 1$ and E_6.[17]

vi) It can be shown[18] that for any SU(N) irreducible representation R

$$D^{abc}(R) = d^{abc}\ A(R) \quad , \tag{4.28}$$

where only A(R) depends on the representation. Therefore, if $D^{abc}(R) = 0$ for one particular set a,b,c of indices, then (provided d^{abc} is not zero for this set) $A(R) = 0$ and $D^{abc}(R)$ vanishes for any set of indices. The values of A(R) for all R's of SU(N) are given in Ref.[19].

Coming back to SU(2) $\otimes$ U(1), we examine the anomaly problem for the representation formed by the following 7 fermions :

$$\begin{pmatrix} u \\ d \end{pmatrix}_L , \quad u_R , \quad d_R , \quad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad e^-_R \quad . \tag{4.29}$$

The only triangle diagrams which may cause anomalies are those which contain at least one W, otherwise there is no γ_5 in the loop. If there are 3 W's, then $D^{abc}(R) = 0$ because of the property (v) (all representations of SU(2) are real). If there are one W and two B's, the anomaly vanishes because of properties (i) and (ii). Finally, if there are two W's and one B, one may replace the generator Y/2 associated with B by $(Q-T_3)$. The term with T_3 vanishes (3 SU(2) generators in D^{abc}) and we are left with only two cases corresponding to $W^+W^-\gamma$ and $W_oW_o\gamma$ vertices.

These vertices are proportional respectively to

$$D^{+-\gamma} = Tr(\{T^+T^-\}Q)$$

and $$D^{33\gamma} = Tr(T_3^2\, Q) \quad .$$

$\{T^+T^-\} = 2(T_1^2 + T_2^2)$ gives the same contribution as T_3^2 and therefore the anomalous term is weighted by the quantity

$$A_{SU(2)\otimes U(1)} = Tr(T_3^2\, Q) \quad .$$

The contribution of the quarks comes from the left handed doublet $(u,d)_L$

$$A^{quarks} = \frac{1}{4}\,(Q_u + Q_d) = \frac{1}{12} \quad .$$

The contribution of the leptons is

$$A^{leptons} = \frac{1}{4}\,Q_e = -\frac{1}{4} \quad ,$$

so that the anomaly does not vanish. Including other similar families cannot help. A way out is to postulate the existence of new quarks and leptons with right handed SU(2) couplings in order to cancel separately A^{quarks} and $A^{leptons}$ (vector like theories). There is however no experimental evidence for such new fermions. A far simpler mechanism is the one where the quark and lepton contributions to the anomaly cancel each other. This happens if the quarks appear with 3 colours. Then :

$$A = Tr(T_3^2\, Q) = 3\, A^{quarks} + A^{leptons} = 0 \quad .$$

We have shown that no triangle graph with external W's or γ (or B) gives rise

to any anomaly if we consider a 15 member family with quarks being triplets (3) (antiquarks antitriplets $\bar{3}$) and leptons singlet representations of SU(3). We are thus naturally led to the gauge group SU(3) $\otimes$ SU(2) $\otimes$ U(1), and we have to check that there is no new anomaly associated with triangle graphs with one or several external gluons. That it is so follows from the following considerations. If there is one external gluon, D vanishes from properties (i) and (ii) above. For 2 external gluons, we have to consider D^{ggW^o} which vanishes for the same reason, whereas $D^{gg\gamma}$ does not appear (no γ_5 involved in the corresponding triangle graph). The above arguments work for all similar 15 dimensional families (c_i, s_i, ν_μ, μ), $(t_i, b_i, \nu_\tau, \tau)$ etc..., or linear combinations of them. So the anomaly cancels family by family. Why it is so is a question which has to do with the general problem of flavours : why successive generations, how many of them, etc... .

We conclude that the gauge group SU(3) $\otimes$ SU(2) $\otimes$ U(1) has no anomaly at all for replicated 15 dimensional representations, the colour degree of freedom being essential for the argument. This symmetry group is spontaneously broken through a Higgs mechanism involving a colour singlet, SU(2) doublet of complex Higgs fields as before (section 3). Therefore the symmetry is broken down to SU(3) $\otimes$ $U_{e.m.}(1)$ at low energy, which is exactly the remaining exact symmetry we are looking for.

PART II

GRAND UNIFIED THEORIES-SU(5)

Before going into the main subject of this second part, we summarize the results we have obtained with the standard model of electroweak interactions. This model

- gives rise to a renormalizable theory of weak interactions,
- unifies charged and neutral weak currents,
- leads to the right phenomenology of all electroweak processes which have been measured up to now.

The colour degree of freedom has been useful to prevent from anomalies which otherwise would spoil the renormalizability of the theory. For that purpose, a global $SU(3)_c$ symmetry is sufficient, but we know that when extended into a local gauge symmetry (QCD), SU(3) is a good candidate to be a theory of strong interactions. In particular it has the property of being asymptotically free, allowing perturbative expansions for the so-called hard phenomena. There are also indications that it gives rise to confinement of quarks and gluons.

Therefore the $SU(3) \otimes SU(2) \otimes U(1)$ local gauge group provides us with a very successful model of elementary interactions (but gravity). However the model has some serious defects which we recall here by collecting the various questions we have been led to ask in Part.I.

I. Why is electric charge quantized ? ($Q = T_3 + Y/2$ and Y, the U(1) generator may take any a priori value).

II. How to predict the value of the Weinberg angle ? At the present level, we still have in fact 3 independent couplings α, α_s and $\sin\theta$.

III. How can we reduce the number of free parameters ? In addition to the couplings, we would also like to predict the fermion masses.

Grand unified models answer these questions in a definite (questions I and II) or a partial way (question III). It was outlined in the general introduction (section 1), especially in connection with the problem of the quantization of charges, that a unified model should be built on a symmetry group g containing $g_o = SU(3)_{colour} \otimes U(1)$ as an unbroken, non invariant subgroup. This group also has to contain SU(2), the weak group, in order to reproduce the successful Salam-Weinberg model of electroweak interactions. Finally, the theory should be free of anomalies (section 4) in order to be a renormalizable theory. However the large group g cannot be an exact symmetry group since only g_o is observed as an exact symmetry at low energies. It happens that the above considerations lead to rather stringent restrictions in model building : it is shown in section 5 how SU(5), which fulfills all the conditions required, appears as a very natural (and unique in a sense to be made precise) candidate for g. The SU(5) model is described in section 6, and its predictions on SU(3), SU(2), U(1) couplings shown in section 7. The Higgs mechanism is introduced in section 8 in order to get the breaking of SU(5) into $SU(3) \otimes SU(2) \otimes U(1)$ in a first stage, and into $SU(3) \otimes U_{e.m.}(1)$ in a second one. It is a very nice property of SU(5) that these breakings can actually be achieved in a rather elegant way through the Higgs mechanism. The great weakness of the model however is undoubtely that the coupling constants of the model have to be adjusted with a fantastic precision in order to yield the required low energy physics, since the fundamental scale of the theory (responsible for the first stage breaking), is as large as about 10^{14} GeV (section 9 - Renormalization of couplings and masses). The possibility of matter instability (nucleon decay) being the most spectacular prediction of grand unified models, we finally present the main results obtained in the recent past years on the proton lifetime (section 10).

Apart from the hierarchy problem, another question for unified theories is that of family replications. One would like to be able at least to classify all families in one irreducible representation of the unifying group. In SU(5), the number of families is somewhat restricted by phenomenological implications of the model (value of $\sin\theta$, fermion mass ratios), but there is no theoretical argument in favour of any particular family number. Other groups than SU(5) are of interest, in particular SO(10), especially if neutrinos happen to be massive. We will however limit ourselves to SU(5) in the present lectures.

We recall that gravity is left aside in all these models. Superunification, based on extended supergravity, with all particles (from spin zero bosons to

the graviton) in the same irreducible multiplet, may give an answer to the problems left opened. The corresponding models are however in their early infancy and will not be considered here.

5. IN SEARCH OF THE UNIFICATION GROUP

In the above introduction, we have given arguments for embedding the $SU(3) \otimes SU(2) \otimes U(1)$ group into a simple Lie group G . This will give us relations between various previously unrelated parameters of the theory. In particular the strong, weak and U(1) coupling constants will be equal (up to well defined numerical factors due to different normalization conventions). As a consequence, $\sin \theta_W$ will no more be a free parameter. Such a theory predicts the existence of exotic(*) bosons. After breakdown of G to $SU(3) \otimes SU(2) \otimes U(1)$, the resultant exotic interactions are suppressed by the extremely high mass acquired by these bosons, so that the $SU(3) \otimes SU(2) \otimes U(1)$ pattern "observed" at present energies (up to a few hundred GeV) is recovered. The relations between coupling constants implied by G are no more exact after breaking, but are still good approximations at very high energies. To be more precise, if the theory is renormalized at a scale very large compared to that where the breaking occurs, its symmetry (with all its consequences) is practically restored. Conversely, in order to allow for perturbative calculations, at present energies, the theory has to be renormalized using a substraction point at a scale μ comparable with these energies. There the symmetry G is strongly broken, but the mass and coupling parameters at the scale μ can be computed from what they are in the symmetric case by the so-called renormalization group equations (RGE).

Since strong and electroweak interactions have very different strengths at present energies, and since couplings vary only logarithmically with the scale μ , one finds a theory containing exotic bosons of fantastically high mass (of the order of 10^{15} GeV). Such high masses are independently required in order to push the rate of proton decay (induced by the exotic interactions) below present experimental bounds (lifetime $\sim 10^{30}$ years). However, one finds that the predicted unification scale is smaller than the Plank mass scale (10^{19} GeV) where gravity can no more be neglected.

In what follows, we assume that there are no other fermions than the presently known (but the top quark t) fermions, grouped into "replicated"

(*) By "exotic", we mean new (with respect to the known γ , gluons and weak bosons) and with unfamiliar charges. Also, they have characteristics of quarks and leptons at the same time (leptoquarks).

families :

$$(u,d,\nu_e,e) \quad , \quad (c,s,\nu_\mu,\mu) \quad , \quad (t,b,\nu_\tau,\tau)$$

which in the simplest unification scheme (the only one to be considered here) are classified in equivalent multiplets. One thus gives up explaining the existence of these families as well as their number. In this respect, the situation will not be better than in the $SU(2) \otimes U(1)$ model. In a maximal unification, one would like to be able to classify all particles in a single irreducible multiplet.

We assume that all neutrinos are strictly massless and for definiteness consider the first family. There are 15 spinors of given chirality. 7 of them have chirality +1 (right handed in the zero mass limit), namely

$$(e^- , u_i , d_i)_R \quad .$$

The 8 other ones have chirality -1 ; they are

$$(e^- , \nu_e , u_i , d_i)_L \quad .$$

$i = 1,2,3$ is the colour index. Since the gauge transformations are γ_5 independent, fermions belonging to the same irreducible representation of the gauge group must have the same chirality. Rather than the above right handed fermions, we thus consider their 7 charge conjugates, which are left handed.

In order to be more precise about our definitions concerning fermions and their representation in Dirac space, we recall a few properties of Dirac spinors. Let f be a fermion spinor. From

$$f_R = \frac{1+\gamma_5}{2} f$$

follows

$$\overline{f_R} = \bar{f} \left(\frac{1-\gamma_5}{2}\right) \quad .$$

The charge conjugate of f is represented by the spinor

$$f^C = C(\bar{f})^T$$

where C is the charge conjugation matrix. It is unitary and verifies

$$C^{\dagger} \gamma_{\mu} C = -\gamma^{\mu^T} \quad .$$

From these definitions, we obtain the following property of the charge conjugate $(f_R)^C$ of a right handed fermion :

$$(f_R)^C = C(\overline{f_R})^T = C \frac{1-\gamma_5}{2} \bar{f}^T = \frac{1-\gamma_5}{2} C \bar{f}^T$$
$$= \frac{1-\gamma_5}{2} f^C = (f^C)_L \quad .$$

One should be careful about the place of the index R,L with respect to the parenthesis : the conjugate of a *right* handed fermion is the corresponding *left* handed antifermion $(f^C)_L$. In the representation used throughout these notes for the γ matrices, C is given by

$$C = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \end{array}\right) \quad .$$

We have

$$C^T = C^{\dagger} = -C$$

The following relation holds for two fermions f and f'

$$\overline{f'^C} f^C = -f'^T \bar{f}^T = +\bar{f} f' \quad .$$

The second equality is due to the fact that fermion fields anticommute. For the same reason

$$\overline{f'^C} \gamma^{\mu} f^C = -\bar{f} \gamma^{\mu} f' \quad , \tag{5.1}$$

and

$$\overline{f'^C} \not{\partial} f^C = \bar{f} \not{\partial} f' \quad . \tag{5.2}$$

The last equation is valid up to a total derivative which does not contribute in the action $\int L(x)\, d^4x$.

When the fermions have group indices, we denote them f_i . If f_i transform according to a representation D

$$f_i \rightarrow D_{ij} f_j \qquad ,$$

then its conjugate $(f_i)^C$ transforms according to D^* . We denote it as f^i and thus have

$$f^i \equiv (f_i)^C \rightarrow (D_{ij})^* f^j \qquad .$$

The place of the indices will then be very important in all subsequent manipulations.

With these properties and conventions, our 15 basic left-handed fields transform according to the following SU(3) ⊗ SU(2) representations

$$\left.\begin{array}{lcl} \nu_L , e_L^- & : & (1,2) \\ u_{i_L}, d_{i_L} & : & (3,2) \\ e_L^+ & : & (1,1) \\ u_L^i & : & (\bar{3},1) \\ d_L^i & : & (\bar{3},1) \end{array}\right\} \qquad (5.3)$$

The bracket (n_3,n_2) means that the corresponding fermions belong to the representation of dimension n_3 of SU(3) and to the representation of dimension n_2 of SU(2). $\bar{n}$ means the conjugate representation of n .

If the unifying group is to be a simple compact Lie group(*)[20], it must be

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(*) The definition of "compact", together with useful properties of Lie groups which are not all explained in the text, can be found in Appendix A. The role of compactness for the commensurability of the charges is underlined there.

i) either one of the classical groups listed below

Name	Definition	Number of generators	Rank
SU(n)	n×n complex unimodular unitary matrices U $U^{\dagger} = U^{-1}$, $\det U = 1$	n^2-1	n - 1
SO(n)	n×n real unimodular orthogonal matrices □ $□^T = □^{-1}$, $\det □ = 1$	$\frac{n(n-1)}{2}$	[n/2] the integer part of n/2
$US_p(2n)$	2n×2n complex unitary symplectic matrices S $S \begin{pmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{pmatrix} S^T = \begin{pmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{pmatrix}$	n(2n+1)	n

The following equivalences between some of these groups hold :

$SO(2) \sim U(1)$

$SO(3) \sim SU(2)$

$SO(4) \sim SU(2) \times SU(2)$ (SO(4) is thus not simple)

$SO(5) \sim US_p(4)$

$SO(6) \sim SU(4)$

ii) or one of the 5 exceptional groups listed below

Name	# Generators	Rank
G_2	14	2
F_4	52	4
E_6	78	6
E_7	133	7
E_8	248	8

In order to choose among these candidates, let us first remark that the massless fermion free Lagrangian

$$L_o = i\left\{\bar{\nu}\,\partial\!\!\!/\,\nu_L + \overline{e^-}\,\partial\!\!\!/\,e_L^- + \sum_i \bar{u}^i\,\partial\!\!\!/\,u_{iL} + \sum_i \bar{d}^i\,\partial\!\!\!/\,d_{iL} + \overline{e^+}\,\partial\!\!\!/\,e_L^+ + \sum_i (\bar{u})_i\,\partial\!\!\!/\,u_L^i + \sum (\bar{d})_i\,\partial\!\!\!/\,d_L^i\right\}$$

has global $U(15) = SU(15) \otimes U(1)$ symmetry. Hence provided one admits (it will always be the case in practice) that G is a symmetry separately of the free and of the interaction Lagrangians, G must be a subgroup of SU(15), or SU(15) itself.

The choice is further strongly reduced by the following considerations. The 3 and $\bar{3}$ representations of SU(3) being not equivalent to each other, the basic 15-plet representation, which has the decomposition

$$(1,2) + (3,2) + (1,1) + 2(\bar{3},1)$$

under $SU(3) \otimes SU(2)$ (Eq.(5.3)), is complex (not equivalent to its conjugate). We recall that the only simple Lie groups which admit complex representations are[17]

$$SU(N) \quad , \quad N \neq 2$$

$$SO(4N+2) \quad , \quad N \neq 1$$

and $$E_6 \quad .$$

Next we remark that $SU(3) \otimes SU(2) \otimes U(1)$ has rank 4 (4 commuting generators). G thus has at least rank 4 . From the above tables, the only admissible rank 4 group is SU(5) (*).

Let us now show that, irrespective of the rank (≥ 4), SU(5) is in fact the only admissible group which provides us with a 15-dimensional representation where the 15 particle family can be arranged. From the above tables

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(*) If one allows G to be not simple, but a power of simple groups, all of them associated with the same coupling constant, $SU(3) \otimes SU(3)$ is another rank 4 candidate. One SU(3) must be identified with SU(3) colour, so that the other one should be the electroweak group. In such a case, the charge operator Q is traceless in any representation. Since the quarks ($SU(3)_c$ triplets) and the leptons ($SU(3)_c$ singlets) must be in separate representations of the second SU(3), one predicts that the sum of the quark charges should be zero, which is wrong.

and previous comments, we are left with

(*i*) SU(N) , $5 \leq N \leq 15$: the lowest dimension *irreducible* representations are :

- the fundamental (or vector) representation, with dimension $D = N$. It has anomalies, according to the following argument. N-1 generators can be diagonalized at the same time (rank = N-1). One of these generators, Z, can be chosen to have all diagonal elements equal to 1 except one equal to 1-N (trace = 0). $\mathrm{Tr}\, Z^3$ is proportional to $N(N-1)(N-2)$, $\neq 0$ for $N > 2$, and thus the fundamental representation has an anomaly (see Section 4).

- the antisymmetric tensor, with $D = N(N-1)/2$. $D = 15$ for $N = 6$, but this representation must be rejected because its reduction to SU(3) is not real.

- the symmetric tensor, with $D = N(N+1)/2$. $D = 15$ for $N = 5$. SU(5) will be discussed at length below. This representation will be seen to be unacceptable.

- the other irreducible representations all have too large dimensions.

Finally, the only reducible 15-dimensional representation is the sum of the fundamental and antisymmetric tensor representations of SU(5). It will happen to be the correct one.

(*ii*) E_6 : the smallest representation has 27 dimensions.

(*iii*) SO(4N+2) , $N \geq 2$. The lowest dimension representations are :

- the vector representation, with $D = 4N+2$. D cannot be 15, furthermore, this representation is real.

- the spinor representation, with $D = 2^{2N}$ (≥ 16) . Hence there is no 15-dimensional representation. Here one should notice that if neutrinos are massive, then the right handed ν has to fit into the considered representation of the unifying group, enlarging the family to 16 members. This fact makes the study of SO(10) very important, although it is not made here. Also of course one should underline the fundamental interest of all experiments about neutrino masses and oscillation phenomena.

6. DESCRIPTION OF THE SU(5) MODEL[21]

We have just shown that all the simple compact Lie groups other than SU(5) contain no suitable 15-dimensional representation. It remains to be shown that among the 15-dimensional representations (irreducible or not) of SU(5) there is at least one with the required properties.

All SU(N) representations can be obtained from properly symmetrized tensorial powers of the fundamental (N dimensions) representation. Let ψ_α be this fundamental (or vector) representation, $(\psi_\alpha)^\dagger \equiv \psi^\alpha$ its complex conjugate. ψ_α transforms under SU(N) infinitesimal transformations according to

$$\psi_\alpha \to \psi_\alpha - i\ \delta\alpha_\alpha^\beta\ \psi_\beta \tag{6.1}$$

or in matrix notations,

$$\psi \to \psi - i\ \delta\alpha\ \psi$$

and

$$\psi^\dagger \to \psi^\dagger + i\ \psi^\dagger\ \delta\alpha \qquad \delta\alpha = \delta\alpha^\dagger$$

By tensorial product of two vectors ψ and φ, one obtains two irreducible representations : the antisymmetric $\left(D = \frac{N(N-1)}{2}\right)$ and the symmetric $\left(D = \frac{N(N+1)}{2}\right)$ tensor representations. Explicitely, they are

$$M_{\alpha\beta}^{A,S} = \frac{1}{\sqrt{2}}\ (\psi_\alpha\varphi_\beta \mp \psi_\beta\varphi_\alpha)$$

which both transform according to

$$M \to M - i[\delta\alpha M + M\delta\alpha^T] \quad . \tag{6.2}$$

The two conjugate representations are obtained by the similar products of $\psi^\dagger$ and $\varphi^\dagger$.

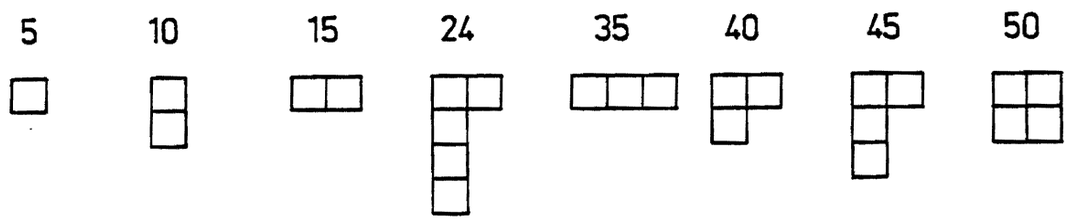
The traceless part of $\psi \otimes \varphi^\dagger$ forms the adjoint representation :

$$M_\alpha^{adj\,\beta} = \psi_\alpha\varphi^\beta - \frac{1}{N}\ \delta_\alpha^\beta \sum_\gamma \psi_\gamma\varphi^\gamma$$

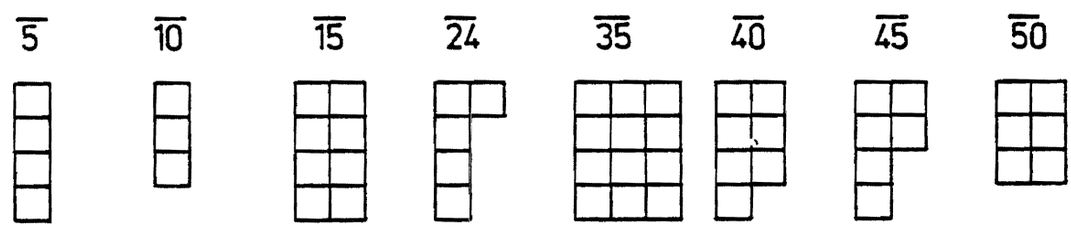
which transforms according to

$$\begin{aligned} M &\to M - i(\delta\alpha M - M\delta\alpha^\dagger) \\ &= M - i[\delta\alpha, M] \quad . \end{aligned} \tag{6.3}$$

For completeness, we give the list of all SU(5) representations of dimension less or equal to 50, together with their Young tableaux (see appendix B).



Their conjugate representations are



Note the identity of the 24 and $\overline{24}$ tableaux, which reflects that the 24 is equivalent to the $\overline{24}$: it is real. In order to assign the 15-family particles to a representation of SU(5), we need the SU(3) $\otimes$ SU(2) decomposition of the representations of interest. The SU(3) and SU(2) subalgebras of SU(5) can be realized as

- the 8 matrices $\left(\begin{array}{c|c} \lambda_i & \\ \hline & \end{array}\right)$

where the λ_i's (i=1,8) are the 8 SU(3) 3×3 matrices.

- the 3 matrices $\left(\begin{array}{c|c} & \\ \hline & \tau_i \end{array}\right)$

where the τ_i's are the 2×2 Pauli matrices.

Accordingly, the representation 5 can be decomposed into $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi_4 \\ \psi_5 \end{pmatrix}$

which transform like (3,1) and (1,2) respectively. In the same way,

$$\overline{5} = (\overline{3},1) + (1,\overline{2}) \quad .$$

The 10 (resp. 15) representation is obtained from the antisymmetric (resp. symmetric) part of the product

$$[(3,1) + (1,2)] \otimes [(3,1) + (1,2)] \tag{6.4}$$

We have

$$(3,1) \otimes (3,1) = (\bar{3},1) + (6,1)$$
$$(1,2) \otimes (1,2) = (1,1) + (1,3) \quad .$$

In the right hand side of these two equations, the first reps. $(\bar{3},1)$ and $(1,1)$ are symmetric, the other ones antisymmetric. Next we write the sum of the 2 crossed term $(3,1) \otimes (1,2)$, as the sum of the symmetric and antisymmetric products, to be respectively attributed to the 15 and 10 representations.

To summarize, we get the following decompositions :

$$5 = (3,1) + (1,2) \tag{6.5}$$
$$\bar{5} = (\bar{3},1) + (1,\bar{2})$$
$$10 = (1,1) + (3,2)_A + (\bar{3},1) \tag{6.6}$$
$$15 = (1,3) + (3,2)_S + (6,1) \quad . \tag{6.7}$$

By computing the decomposition of $5 \otimes \bar{5}$, one gets for the adjoint representation

$$24 = (8,1) + (1,3) + (1,1) + (\bar{3},2) + (3,2) \quad . \tag{6.8}$$

We see from Eq. (6.7) that the 15 has not the right $SU(3) \otimes SU(2)$ content (it contains an SU(2) triplet and a colour sextet), whereas the reducible representation $\bar{5} + 10$ is exactly suited to accomodate the 15 left-handed fermions of Eq. (5.3), higher representations being too large. So, we are led to arrange ν_L and e_L^- and the 3 anti-down quarks d_L^i into a $\bar{5}$ representation. That we put d^i and not u^i in $\bar{5}$ follows from $Tr_{\bar{5}} Q = 0$ since Q is a traceless generator in any representation (SU(5) is a simple Lie group). Turning the argument the other way around, we see that $Q_{di} = \frac{-1}{3} Q_{e^-}$, i.e. the fact that quark and electron charges are commensurate (answer to question I) is required by the postulate of symmetry under the simple group SU(5).

The $\bar{5}$ is thus the vector

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^- \\ -\nu \end{pmatrix}_L$$

The 3 first components transform like the complex conjugate of the SU(3) fondamental representation. In writing the last two fermions, we have used the standard form $\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L$ for the left handed spinor of the representation 2 of SU(2). Since $\bar{2}$ (which is equivalent to 2) appears in the $\bar{5}$ rep., we used

$$\bar{2} \equiv i\tau_2 \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L = \begin{pmatrix} e^- \\ -\nu \end{pmatrix}_L \quad .$$

Put in a different way, if $f_j \in$ rep.2, $f^i = \varepsilon^{ij} f_j$, where ε^{ij} is the antisymmetric tensor with two indices, transforms according to $\bar{2}$. By charge conjugation of the $\bar{5}$ representation, we obtain the 5 rep. which contains the right-handed d_i quarks, positron and antineutrino,

$$\psi = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\nu^c \end{pmatrix}_R \tag{6.9}$$

so that the charge operator Q in this representation is

$$Q_5 = \begin{pmatrix} -1/3 & & & & 0 \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1 & \\ 0 & & & & 0 \end{pmatrix} \quad . \tag{6.10}$$

It generates the $U_{e.m.}(1)$ transformations. Let us explicitely construct the 10 as the antisymmetric product of two 5's, ψ and ψ'. Denoting Ψ as

$$\psi = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \omega_1 \\ \omega_2 \end{pmatrix} \quad , \tag{6.11}$$

where c stands for colour and ω for weak isospin, the 10

$$M_{ij} = \frac{1}{\sqrt{2}} \left(\psi_i \psi'_j - \psi_j \psi'_i\right) \quad ,$$

is represented by

$$M = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} 0 & c_1c_2'-c_2c_1' & c_1c_3'-c_3c_1' & c_1\omega_1'-\omega_1c_1' & c_1\omega_2'-\omega_2c_1' \\ & 0 & c_2c_3'-c_3c_2' & c_2\omega_1'-\omega_1c_2' & c_2\omega_2'-\omega_2c_2' \\ & & 0 & c_3\omega_1'-\omega_1c_3' & c_3\omega_2'-\omega_2c_3' \\ \hline & & & 0 & \omega_1\omega_2'-\omega_2\omega_1' \\ & & & & 0 \end{array} \right)$$

M is antisymmetric and only the elements above the diagonal are written.

Using the fact that in SU(3), the $\bar{3}$ representation c^i transforms as $\varepsilon^{ijk}c_jc_k$, which expresses that the $\bar{3}$ is the antisymmetric part of $3 \otimes 3$, we see

(i) that the upper left 3×3 matrix of M is of the form

$$\begin{pmatrix} 0 & c^3 & -c^2 \\ & 0 & c^1 \\ & & 0 \end{pmatrix} \quad .$$

It is invariant under SU(2), and hence corresponds to SU(2) singlets.

(ii) that, Q_5 being a multiple of unity in this subspace (Eq. (6.10)), all c^i's have the same charge, which is twice the c_i charge. Hence

$$Q(c^i) = -\frac{2}{3} \quad . \tag{6.12}$$

More generally, if $Q_{5\otimes 5}(ij)$ is the charge of the element ij of the product $5\otimes 5$, $Q_{5\otimes 5}(ij) = (Q_5)_{ii} + (Q_5)_{jj}$, Q_5 being given by (6.10). So, the three states c^i, i=1,2,3 are to be identified with the three left handed anti-u-quarks u^i, i=1,2,3.

The lower right 2×2 matrix of M is SU(3) singlet. Its non-vanishing element, $\omega_1\omega_2'-\omega_2\omega_1'$ is the antisymmetric product of 2 SU(2) doublets and thus is SU(2) singlet. Its charge is $Q(e^+)+Q(\nu^c)=+1$. Hence it has to be identified with the positron.

Coming to the elements M_{ij}, i=1,2,3 and j=4,5, they obviously transform

according to the representation 3 of SU(3) at fixed j, and the representation 2 of SU(2) at fixed i : they form the part (3,2) of the representation 10 of SU(5). Their charges are those of the states $(d_i e^+)$ and $(d_i \nu^c)$ respectively for columns 4 and 5, that is 2/3 and -1/3. Hence the 4th column is u_{i_L} and the 5th is d_{i_L}.

In terms of particles, we may thus finally write

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u^3 & -u^2 & u_1 & d_1 \\ -u^3 & 0 & u^1 & u_2 & d_2 \\ u^2 & -u^1 & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{pmatrix} \tag{6.13}$$

In writing this matrix we implicitely made some phase conventions (without any physical consequences) for the various states appearing. For example, we could have written $-u_i$, $-d_i$ instead of u_i, d_i. The $1/\sqrt{2}$ factor is also for convenience.

Remark

That it is possible to arrange the known quarks in the fundamental + antisymmetric tensor representations of SU(5) is not at all automatic. In order to see that, let us assume that there are N_c colours, and that we are to build an $SU(N_c+2)$ unified theory of $SU(N_c) \otimes SU(2)$, where SU(2) is the standard one. Then in the fundamental we set N quarks d , the e^+ and the $\bar{\nu}_e$. From $\mathrm{Tr}\, Q = 0$ we get $Q_d = -\frac{1}{N_c} Q_{e^+}$. In the representation generalizing the expression (6.11) of the antisymmetric tensor, the charges of the anti-up quarks would now be $-2/N_c$, whereas the up quarks appearing in the N_c+1 column would have charge $1-1/N_c$. We see that only for $N_c = 3$, $Q_{u^i} = -Q_{u_i}$. By the way $(N_c \otimes N_c)_A = \bar{N}_c$ only for $N_c = 3$.

The 24 gauge fields, belonging to the adjoint representation of SU(5), can be classified according to the decomposition given in Eq. (6.8). We easily recognize the 8 gluons (rep. (8,1)), the 3 W's (rep.(1,3)) and the B boson (rep.(1,1)). In addition to these bosons of $SU(3) \otimes SU(2) \otimes U(1)$, we next have 12 "exotic" gauge bosons. Six of them, X_i, Y_i (i=1,2,3), belong to the (3,2)

representation, and their charge conjugates X^i, Y^i to the $(\bar{3},2)$ representation. These particles are coloured and thus presumably confined. Each of the pairs (X_i,Y_i) or (X^i,Y^i) form an SU(2) doublet. The charge of any of the 24 gauge bosons is easily derived by building the 24 rep. from the product $5 \otimes \bar{5}$, so that

$$Q_{24}(ij) = (Q_5)_{ii} + (Q_{\bar{5}})_{jj} = (Q_5)_{ii} - (Q_5)_{jj} \quad , \tag{6.14}$$

with Q_5 given by Eq. (6.10). We verify that the gluons (i,j=1,2,3) have no charge. The X_i, defined as the j=4, i=1,2,3 objects, have charge $-\frac{1}{3}-1=-4/3$, and the Y_i (j=5;i=1,2,3) have $Q=-1/3$. Of course the anti X,Y have the opposite charges.

Anomalies

We have seen (section 4) that the 15 member family has no anomaly in the context of $SU(3) \otimes SU(2) \otimes U(1)$, that is when the external legs of the triangle diagram are gluons, $W^{\pm}$, Z_o or γ bosons. The absence of anomaly was guaranted by the relation

$$\mathrm{Tr}[Q\, T_3^{\,2}] = 0 \quad , \tag{6.15}$$

where the trace runs over the 15 fermions. If we restrict the trace to the fermions belonging to the $\bar{5}$ or to the 10, we find

$$\mathrm{Tr}_{\bar{5}}\,[Q\, T_3^{\,2}] = \frac{1}{4}\, Q_e = -\frac{1}{4}$$

$$\mathrm{Tr}_{10}[Q\, T_3^{\,2}] = 3 \times \frac{1}{4} \times (\frac{2}{3} - \frac{1}{3}) = +\frac{1}{4} \quad :$$

the anomaly disappears by cancellation of anomalies in the $\bar{5}$ and the 10. It turns out that this is sufficient to deduce that the $\bar{5} + 10$ representation has no anomaly at all, as shown by application of property vi) of $D^{abc}=\mathrm{Tr}[T^a\{T^bT^c\}]$ (section 4) : since $D^{abc}=0$ when a,b,c correspond to W or γ bosons, it vanishes for any set of indices.

7. RELATIONS BETWEEN THE SU(3), SU(2), U(1) COUPLING CONSTANTS

Let us explicitly build the generators in the representation 5. As it was the case for the charge operator Q, this allows to build them in any representation.

The SU(3) ⊗ SU(2) ⊗ U(1) generators are very simple. The eight gluons are coupled to the SU(3) generators

$$T_G^a = \left(\begin{array}{c|c} \lambda^a/2 & 0 \\ \hline 0 & 0 \end{array}\right) , \qquad a = 1,\ldots,8 \tag{7.1}$$

where the λ^a's are the usual Gell-Man matrices normalized according to $Tr(\lambda_a\lambda_b) = 2\,\delta_{a,b}$. The W bosons are coupled to the SU(2) generators

$$T_W^i = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \tau^i/2 \end{array}\right) , \qquad i = 1,2,3 \tag{7.2}$$

where the τ^i's are the Pauli matrices.
The B boson is coupled to the U(1) generator :

$$T_B = \sqrt{\frac{3}{5}} \left(\begin{array}{c|c} -1/3 & 0 \\ \hline 0 & 1/2 \end{array}\right) = \sqrt{\frac{3}{5}}\,\frac{Y}{2} \qquad , \tag{7.3}$$

where Y is the conventional hypercharge. These 12 generators are (ortho) normalized according to

$$Tr_5\left(T^iT^j\right) = \frac{1}{2}\,\delta^{ij} \qquad .$$

The 12 other hermitian generators are constructed from the requirement of orthonormality. They are similar to the Pauli matrices $\tau_1/2$ and $\tau_2/2$, which couple to W_1 and W_2 in SU(2). We have

$$T^1_{X_j} = \frac{1}{2}\left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right) \begin{array}{l} \left\{\begin{array}{l}\text{line } j\\ \text{column } 4\end{array}\right. \\ \left\{\begin{array}{l}\text{line } 4\\ \text{column } j\end{array}\right. \end{array} \qquad \text{all other elements vanish} \tag{7.4}$$

$$T^2_{X_j} = \frac{1}{2}\left(\begin{array}{c|c} 0 & -i \\ \hline i & 0 \end{array}\right) \begin{array}{l} \left\{\begin{array}{l}\text{line } j\\ \text{column } 4\end{array}\right. \\ \left\{\begin{array}{l}\text{column } j\\ \text{line } 4\end{array}\right. \end{array} \qquad \text{all other elements vanish} \tag{7.5}$$

and similar matrices $T_{Y_j^1}$, $T_{Y_j^2}$ with line or column 4 being replaced by line or column 5. $X_j^{1,2}$ and $Y_j^{1,2}$ are the corresponding 12 gauge bosons. The 24 SU(5) generators of Eqs.(7.2 - 5) are hermitian and they are associated with 24 real gauge fields. As stated in Section 2, one often uses complex combinations of fields with definite electric charges, such as the W^+ and W^- weak vector bosons

$$W^{\pm} = \frac{1}{\sqrt{2}} (W^1 \mp iW^2) \quad , \quad (W^+)^{\dagger} = W^- \quad ,$$

coupled with

$$\tau^{\pm} = \frac{1}{\sqrt{2}} (\tau_1 \pm i\tau_2)$$

according to

$$W_1\tau^1 + W_2\tau^2 = W^+\tau^+ + h.c \quad .$$

In the same way the charged X's

$$X_j \equiv \frac{1}{\sqrt{2}} (X_j^1 - iX_j^2) \quad , \quad X^j = (X_j)^{\dagger}$$

are coupled to the complex combinations of generators

$$(T_X)^j = \frac{1}{\sqrt{2}} (T_{X^1} + i\, T_{X^2}) \quad , \quad (T_X)_j = (T_X{}^j)^{\dagger} \tag{7.6}$$

according to

$$T_{X_j^1} X_j^1 + T_{X_j^2} X_j^2 = (T_X)^j X_j + h.c \quad .$$

With similar notations one also has

$$T_{Y_j^1} Y_j^1 + T_{Y_j^2} Y_j^2 = (T_Y)^j Y_j + h.c \quad .$$

We will use the following notation

$$(T_X)^j = \left(\begin{array}{c|c} 0 & Z_X^j \\ \hline 0 & 0 \end{array} \right) \tag{7.7}$$

$$(T_Y)^j = \left(\begin{array}{c|c} 0 & Z_Y^j \\ \hline 0 & 0 \end{array}\right) \tag{7.8}$$

where all the elements of Z_X^j (Z_Y^j) are zero but the one in the line j, column 4(5) which is equal to $1/\sqrt{2}$.

It is sometimes convenient to use a matrix A_μ which represents the 24 gauge bosons as a whole. By definition (see appendix A)

$$A^\mu = \sqrt{2} \sum_{a=1}^{24} A_a^\mu T^a \quad . \tag{7.9}$$

The factor $\sqrt{2}$ is chosen in such a way that

$$\mathrm{Tr}(A^\mu A^\nu) = 2 \sum_{a,b} A_a^\mu A_b^\nu \ \mathrm{Tr}(T^a T^b)$$

$$= \sum_a A_a^\mu A_a^\nu \quad .$$

Under a global infinitesimal SU(5) transformation defined by the 5 × 5 matrix $\delta\alpha$, A^μ transforms as

$$A^\mu \to A^\mu - i[\delta\alpha, A^\mu] \quad .$$

It has the following expression in terms of the gauge fields

$$A^\mu = \sqrt{2} \left\{ B^\mu T_B + \left(\begin{array}{c|c} \sum_1^8 G_a^\mu \frac{\lambda^a}{2} & \vec{X}.\vec{Z}_X + \vec{Y}.\vec{Z}_Y \\ \hline (\vec{X}.\vec{Z}_X + \vec{Y}.\vec{Z}_Y)^\dagger & \vec{W}.\frac{\vec{\tau}}{2} \end{array}\right) \right\} \tag{7.10}$$

where use has been made of the complex X and Y for convenience.

In terms of the representations Ψ and M , the free Lagrangian is

$$L_{free} = i \sum_k \overline{(\psi_k)} \not{\partial} \Psi_k + i \sum_{k,\ell} \overline{(M_{k\ell})} \not{\partial} M_{k\ell} \quad .$$

We now define by convention, for $k,\ell = 1,\ldots,5$:

$$(\bar{\psi})_k \equiv \overline{(\psi_k)}$$

and

$$(\overline{M})_{k\ell} \equiv \overline{(M_{\ell k})} \quad ,$$

so that $\overline{\psi}$ and $\overline{M}$ exist as respectively a vector and a (antisymmetric) matrix, whose elements are conjugate Dirac spinors. Note the index transposition in the last equation, which is convenient in order to rewrite the free Lagrangian as

$$L_{free} = i\,\overline{\psi}\,\partial\!\!\!/\,\psi + i\,\mathrm{Tr}\,\overline{M}\,\partial\!\!\!/\,M \quad .$$

The couplings of the fermion to the gauge fields are the most easily deduced from this equation by replacing the derivative $\partial\!\!\!/$ by the covariant derivative $D\!\!\!\!/ = \partial\!\!\!/ + ig\,A\!\!\!/$ (see Appendix A). In the 5 representation, one directly obtains

$$L = -g\,A^{\mu}_{a}\;\overline{\psi}\,T^{a}\,\gamma_{\mu}\,\Psi \quad . \tag{7.11}$$

Recalling that M transforms like the product $\psi \otimes \varphi$ of two fundamentals one gets the following 5×5 matrix representation of $A^{\mu}M$ (see Eq.(6.2)) :

$$\begin{aligned}(A^{\mu}M)_{k\ell} &= \sum_a A^{\mu}_a \left(\sum_i T^a_{ki}\,M_{i\ell} + \sum_j M_{kj}\,T^a_{\ell j}\right)\\ &= \sum_a A^{\mu}_a \left(T^a M + M\,T^{a\,T}\right)_{k\ell} \quad .\end{aligned}$$

Hence $\overline{M}\,D\!\!\!\!/\,M$ generates the coupling term

$$-2g\sum_{a=1}^{24} A^{\mu}_a\,\mathrm{Tr}(\overline{M}\,T^a\,\gamma_{\mu}\,M) \quad . \tag{7.12}$$

The factor 2 cancels that which occurs from the $1/\sqrt{2}$ in the definition of M in terms of the individual spinor fields. Eqs.(7.11) and (7.12) contain all the information about any particular coupling. In particular, the gluon, W and B couplings of the SU(3) $\otimes$ SU(2) $\otimes$ U(1) model are recovered under the form

$$\begin{aligned}&-g\,G^{\mu}_a \sum_{\text{quarks}} \bar{q}\,\gamma_{\mu}\,\frac{\lambda^a}{2}\,q\\ &-g\,\vec{W}^{\mu} \sum_{\text{fermions}} \bar{f}_L\gamma_{\mu}\,\frac{\vec{\tau}}{2}\,f_L - g\sqrt{\frac{3}{5}}\;B^{\mu} \sum_{\text{fermions}} \bar{f}\,\gamma_{\mu}\,\frac{Y}{2}\,f\end{aligned} \tag{7.13}$$

By comparing the above expression with Eq.(2.8) we see that the SU(5) model implies

$$g' = \sqrt{\frac{3}{5}}\ g \tag{7.14}$$

As a consequence, we get a prediction for the Weinberg angle defined by Eq.(2.21)

$$\mathrm{tg}\ \theta = \frac{g'}{g} = \sqrt{\frac{3}{5}} \tag{7.15}$$

so that

$$\sin^2\theta = \frac{3}{8} \quad . \tag{7.16}$$

As a second consequence, the model also predicts that not only weak and electromagnetic interactions are governed by the same coupling constant g, but also that g is the strong interaction coupling constant

$$\frac{\alpha}{\alpha_s} = \frac{e^2}{g^2} = \sin^2\theta = \frac{3}{8} \quad . \tag{7.17}$$

We will show later on how these predictions, which are exact in the exact SU(5) symmetry limit, are modified in a predicted way at present energies where SU(5) is strongly broken.

Before going on it is useful to notice that the predictions we have obtained in SU(5) for $\sin\theta$ and α/α_s are valid in much more general situations. Indeed they only rely on the existence of a multiplet, irreducible or not, made of one or several families ($\{u,d,e,\nu_e\}$, $\{c,s,\mu,\nu_\mu\}$, ...) , and on the fact that the unifying group is a simple group. Let T_3 and Y/2 be the matrix representations, for the quoted multiplet, of the third component of the weak isospin and of the weak hypercharge (with their usual eigenvalues). In general, they are not normalized in the standard way, that is one has

$$\mathrm{Tr}\ T_3^2 \neq \mathrm{Tr}\ \frac{Y^2}{4} \quad .$$

The hypercharge generator T_B is suitably normalized if written

$$T_B = C\ \frac{Y}{2}$$

with

$$C = \frac{4\ \mathrm{Tr}\ T_3^2}{\mathrm{Tr}\ Y^2} \quad .$$

T_3 and T_B are then coupled to the corresponding gauge fields with the same strength g . Thus it is quite general to have

$$tg^2\theta = \left(\frac{g'}{g}\right)^2 = C^2 = \frac{4 \text{ Tr } T_3^2}{\text{Tr } Y^2}$$
$$= \frac{3}{5}$$

in all cases where the same 15 family (or several of them) forms a representation of a simple group. From the same argument

$$\frac{\alpha}{\alpha_s} = \frac{\text{Tr}(\lambda_a/2)^2}{\text{Tr } Q^2} = \frac{3}{8}$$

is general under the same conditions.

Apart from the SU(3) ⊗ SU(2) ⊗ U(1) interactions, which are now related with each other, there are new ones mediated by the X and Y gauge bosons. From the general expressions of Eqs.(7.11 and 12), using the explicit contents of the ψ and M representations, and the matrices T_X , T_Y of Eqs.(7.7 and 8), the new interaction terms are for the first family

$$- \frac{g}{\sqrt{2}} X_i^\mu \left(\bar{d}^i \gamma_\mu e^+ + \varepsilon^{ijk} \bar{u}_j \gamma_\mu u_{k,L}\right)$$
$$- \frac{g}{\sqrt{2}} Y_i^\mu \left(\bar{\nu} \gamma_\mu d_L^i - \bar{u}^i \gamma_\mu e_L^+ + \varepsilon^{ijk} \bar{u}_j \gamma_\mu d_{k,L}\right) + \text{h.c} \quad . \qquad (7.18)$$

The experimental implications (proton decay) of these exotic interactions will be discussed later on. Here we only remark that these new interactions violate baryon (B) and lepton (L) number conservation. This is an unavoidable consequence of any unification scheme where quarks and leptons appear in the same irreducible representations. As we will see however (Section 8d), there is a global U(1) invariance left which implies that B-L is conserved.

8. HIGGS BOSONS, SU(5) BREAKING, MASSES

SU(5) has to be broken ; if it was not, we would not have time to discuss it before all our protons have decayed. This means that the breaking of SU(5) down to $SU(2) \otimes SU(2) \otimes U(1)$ must occur at an extremely high energy scale, in such a way that the gauge bosons responsible for the exotic interactions acquire masses of the same order of magnitude and lead to essentially no observable effect (with the eventual exception of proton decay at an extremely low rate). After such a (first step) breaking, we come back to the situation of the $SU(3) \otimes SU(2) \otimes U(1)$ model, with however the bonus of relations between the couplings. The latter symmetry gets itself broken down to $SU(3) \otimes U_{e.m.}(1)$ (second step), like in the Salam-Weinberg model. This breaking occurs at a considerably lower energy than the first one.

In the SU(5) model, symmetry breaking is supposed to be spontaneous, through the Higgs mechanism. We recall the main lines of this scheme. The Higgs boson potential is supposed to be minimum for a non zero vacuum expectation value of some of the Higgs fields. Through their minimal couplings to the gauge bosons, they give some of these gauge field masses, and, through Yukawa couplings, they also give fermion masses. There are as many generators broken by the non vanishing expectation values as there are real Higgs bosons which disappear, transfering their degrees of freedom to the gauge bosons. If the symmetry of the Higgs potential is *not* larger than the symmetry of the whole Lagrangian, the other Higgs bosons survive as massive physical particles. If this symmetry is larger, some of these Higgs bosons remain massless (pseudo-Goldstone bosons[22]), at the lowest order in perturbation.

As the two stages of breaking occur at very different energy scales, it is convenient to treat them successively.

a) The first stage of spontaneous symmetry breaking (SSB) $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)$

Our first problem is to determine the nature of the Higgs multiplet required. As one wants to give masses to the 12 exotic bosons X and Y, one needs at least 12 Higgs bosons. On an other hand, given a Higgs representation, there are only a few possible patterns of unbroken subgroups. The remaining

symmetries after SSB of SU(N) have been studied in the literature for the Higgs representations of low dimensions. The results are just listed below, illustrations being given in the forthcoming detailed discussions about SU(5).

Higgs boson representation	Remaining symmetry after spontaneous breaking
Fundamental $D = N$	$SU(N-1)$
Antisymmetric 2nd rank tensor $D = \frac{N(N-1)}{2}$	$SU(N-2)$ or $SO(2\ell+1)$
Symmetric 2nd rank tensor $D = \frac{N(N+1)}{2}$	$SU(N-1)$ or $SO(N)$
Adjoint $D = N^2-1$	$SU(\ell) \otimes SU(N-\ell) \otimes U(1)$ or $SU(N-1) \otimes U(1)$

Table 8.1

Symmetry pattern after SSB by a given Higgs representation. In the second column, ℓ stands for the integer part of N/2 . "Or" means that one of the two possibilities or the other is chosen according to the values of the parameters in the Higgs potential.

This table shows that 24 Higgs fields in the adjoint representation offer the simplest possibility(*) for obtaining the required first stage breaking of SU(5). As an illustration of how these results on SU(N) breaking are obtained, we examine the two cases of the fundamental and the adjoint Higgs representations, the former one being of interest for the second stage breaking of SU(5).

i) Higgs fields in the fundamental representation of SU(N)

The Higgs fields form a vector of dimension N, $\{h_\alpha\}$, which transforms

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(*) One can show that only the real representations of SU(5) may contain a singlet of $SU(3) \otimes SU(2) \otimes U(1)$, and thus have to be considered. The next real irreducible representations beyond the adjoint have dimensions 75 and 200 !

according to Eq.(6.1)

$$h \rightarrow (1 - i\delta\alpha)\, h$$

The most general SU(N) invariant renormalizable Higgs potential can be written as

$$V(h) = -\mu^2\, h^\alpha h_\alpha + \frac{\lambda}{2}\,(h^\alpha h_\alpha)^2 \quad . \tag{8.1}$$

μ^2 and λ are real numbers. λ must be >0 in order for V to be bounded from below. The ground state is obtained for

$$\frac{\partial V}{\partial h_\alpha} = 0 \quad , \quad \alpha = 1,\ldots,N$$

that is for

$$(-\mu^2 + \lambda\, h^\alpha h_\alpha)\; h^\alpha = 0 \tag{8.2}$$

One solution of (8.2) is $\{h\} = 0$, for which $V = 0$. This is the minimum for negative μ^2 . For positive μ^2 , the minimum of V is reached for the other solution

$$h_o^\alpha h_{o\alpha} = \frac{\mu^2}{\lambda} \tag{8.3}$$

of energy $\qquad V = -\dfrac{\mu^4}{2\lambda}$.

Thus for $\mu^2 > 0$, there is spontaneous symmetry breaking. $\{h_o\}$ is a vector of fixed length and of arbitrary orientation in the representation space. One can always transform the h_o fields into a new vector (which we relabel h_o), by a global SU(N) transformation, in such a way that the new h_o points along the Nth coordinate axis. Then

$$h_o = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sqrt{\mu^2/\lambda} \end{pmatrix} \quad .$$

This vector is invariant under the SU(N-1) subgroup acting upon the (N-1) first coordinates. This is the result announced in the above table. There are $(N^2-1) - ((N-1)^2 - 1) = 2N-1$ broken generators. According to the general

theorem, 2N-1 out of the 2N real Higgs bosons of {h} disappear, so that there is only one real physical Higgs particle left.

ii) Higgs fields in the adjoint representation of SU(N)

They can be represented as a second rank traceless tensor $H_i^{\ j}$, which transforms under SU(N) according to Eq.(6.3)

$$H \to H - i[\delta\alpha , H] \quad .$$

The most general invariant potential is (up to a cubic term, which is absent if we impose the symmetry $H \to -H$)

$$V = -\frac{\mu^2}{2} \mathrm{Tr}\, H^2 + \frac{\lambda_1}{4} (\mathrm{Tr}\, H^2)^2 + \frac{\lambda_2}{4} \mathrm{Tr}\, H^4 \quad . \tag{8.4}$$

There is SSB if μ^2 is positive. We want to find out the possible ground states for this potential[23]. Due to the SU(N) invariance of V, the various possible ground states differ from each other by SU(N) transformations. We may thus look for the minimum of V after a transformation of the fields which diagonalize the (hermitian) matrix H . The new fields (relabelled H) form the matrix

$$H_i^{\ j} = \delta_i^{\ j} a_i$$

$$\sum_{i=1}^{N} a_i = 0$$

so that the potential reads

$$V = -\frac{\mu^2}{2} \sum a_i^2 + \frac{\lambda_1}{4} \left(\sum a_i^2\right)^2 + \frac{\lambda_2}{4} \sum a_i^4 \quad . \tag{8.5}$$

We first look for a minimum of V at $\sum a_i^2 = R^2$ fixed. One has to minimize $\lambda_2 \sum a_i^4$ at fixed R^2 and for $\sum a_i = 0$. Using 2 Lagrange multipliers ρ_2 and ρ_1 for these constraints, the a_i's must verify

$$\lambda_2 a_i^3 + 2\rho_2 a_i + \rho_1 = 0 \quad .$$

So, all the a_i's are solutions of the same 3rd degree equation, and thus take at most 3 different values. If they are all equal, they all vanish due

to $\sum a_i = 0$ (V=0). We now specialize to SU(5) and enumerate all possible configurations with at least 2 different values for the 5 a_i's .

(A) 2 different values. The solution is

(1) either $\frac{R}{\sqrt{20}}(1,1,1,1,-4)$

(2) or $\sqrt{\frac{2}{15}}\, R\,(-1,-1,-1,\frac{3}{2},\frac{3}{2})$.

(B) 3 different values. Their sum is 0 as there is no second degree term in the 3rd degree equation they verify. One finds

(3) either $\frac{R}{\sqrt{2}}(0,0,0,1,-1)$

(4) or $\frac{R}{2}(1,1,-1,-1,0)$.

For these 4 solutions, the values of V are of the form

$$V_n = -\frac{\mu^2}{2}R^2 + \frac{R^4}{4}(\lambda_1+\lambda_2 s_n)$$

with $s_n = \left(\frac{13}{20}, \frac{7}{30}, \frac{1}{2}, \frac{1}{4}\right)$ respectively. There is a minimum with respect to R^2 for $\lambda_1+\lambda_2 s_n > 0$ (otherwise V is not bounded from below). The minimum, obtained for

$$R^2 = \frac{\mu^2}{\lambda_1+\lambda_2 s_n} ,$$

is

$$V_n = \frac{-\mu^4}{4(\lambda_1+\lambda_2 s_n)} .$$

For a given point in the λ_1,λ_2 plane, the absolute minimum is the smallest of the 4 values V_n .

For $\lambda_2 < 0$, the minimum solution is solution (1).

$$H_o = \sigma_o \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \\ & & & & -4 \end{pmatrix} , \quad \sigma_o^2 = \frac{\mu^2}{20\lambda_1 + 13\lambda_2}$$

For $\lambda_2 > 0$, the minimum solution is solution (2).

$$H_o = \sigma_o \begin{pmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & & & 3/2 & \\ 0 & & & & 3/2 \end{pmatrix} \quad , \quad \sigma_o^2 = \frac{\mu^2}{\frac{15}{2}\lambda_1 + \frac{7}{4}\lambda_2} \tag{8.6}$$

In order to determine the remaining symmetry, we remember that under a transformation of SU(5) represented by $\delta\alpha$, H_o transforms according to

$$H_o \to H_o - i[\delta\alpha , H_o] \quad ,$$

so that H_o is invariant for all $\delta\alpha$'s such that

$$[\delta\alpha , H_o] = 0 \quad .$$

For $\lambda_2 > 0$ and $30\lambda_1 + 7\lambda_2 > 0$, $\delta\alpha$ is any matrix of $SU(3) \otimes SU(2) \otimes U(1)$, while for $\lambda_2 < 0$ and $20\lambda_1 + 13\lambda_2 > 0$, $\delta\alpha$ is any matrix of $SU(4) \otimes U(1)$. This property illustrates the last result of Table 8.1. The allowed regions in the λ_1,λ_2 plane and the remaining symmetry pattern after SSB are sketched in Fig.8.1.

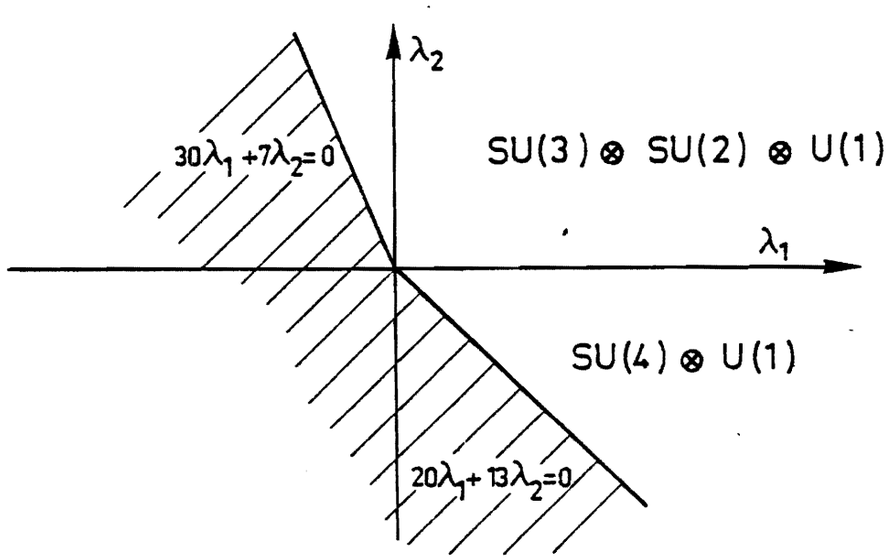


Fig.8.1 - SU(5) *breaking by Higgs bosons in the adjoint representation. In the plane of the two couplings* λ_1 *and* λ_2 *of the Higgs potential (Eq.(8.5)), the forbidden domain (unbounded potential) is shaded. In the allowed domain, the regions where the symmetries* $SU(3) \otimes SU(2) \otimes U(1)$ *and* $SU(4) \otimes U(1)$ *respectively are preserved are separated by* $\lambda_2 = 0$.

From now on, we assume that we are indeed in the situation where λ_2 is positive and larger than $-30\lambda_1/7$. This situation is stable against radiative corrections to the Higgs potential[24].

In the same way as the gauge fields may be considered as members of the algebra, and represented in the representation 5 by Eq.(7.9), we write the Higgs fields in the adjoint representation as the 5×5 matrix,

$$H = \sqrt{2} \sum_{a=1}^{24} H_a T^a \quad ,$$

which satisfies

$$\mathrm{Tr}(H\,H) = \sum_{a=1}^{24} H_a^2 \quad .$$

Using the explicit form of the T_a's given in Eqs.(7.1,2,3,7,8), we write

$$H = (\sqrt{15}\,\sigma_o + \sqrt{2}\,H_B)\,T_B + \sqrt{2}\left(\begin{array}{c|c} \sum_1^8 H_G^a \frac{\lambda_a}{2} & \vec{H}_X.\vec{Z}_X + \vec{H}_Y.\vec{Z}_Y \\ \hline (\vec{H}_X.\vec{Z}_X + \vec{H}_Y.\vec{Z}_Y)^\dagger & \vec{H}_W \cdot \frac{\vec{\tau}}{2} \end{array}\right) \qquad (8.7)$$

In this equation, all H fields have vanishing vacuum expectation values, the constant $\sqrt{15}\,\sigma_o\,T_B$ being separated out. The Higgs bosons have the same charge as the corresponding gauge bosons, which also belong to the adjoint representation :

(i) the 8 H_G^a's (coloured and thus presumably confined), H_W^3 and H_B have zero charge.

(ii) $$H_W^{(+)} = \frac{1}{\sqrt{2}}(H_W^1 - iH_W^2) = \left(H_W^{(-)}\right)^\dagger$$

has charge +1.

(iii) the 3 complex H_X and the 3 complex H_Y (colour triplets) have charge -4/3 and -1/3 respectively.

The gauge fields which are coupled to those Higgs fields which have non vanishing expectation values acquire masses. The Higgs-gauge couplings are obtained from the term

$$\frac{1}{2}\,\mathrm{Tr}\,[(D_\mu H)(D^\mu H)]$$

of the Higgs Lagrangian, where D_μ is the covariant derivative

$$D^\mu H = \partial^\mu H + ig\, A^\mu_a\, [T^a, H] \quad .$$

We have used hermitian fields so that D is hermitian. The gauge boson mass term is identified as

$$\frac{1}{2} A^\mu_a\, M^2_{ab}\, A_{\mu,b} = -\frac{g^2}{2} A^\mu_a A_{\mu b}\, \mathrm{Tr}\Big([T^a, H_o]\,[T^b, H_o]\Big)$$

with H_o given by Eq.(8.6).

The mass matrix thus reads

$$M^2_{a,b} = g^2\, \mathrm{Tr}\,\Big([T^a, H_o]\,[T^b, H_o]^\dagger\Big) \quad . \tag{8.8}$$

The 12 generators associated with the gluons, the W's and the B obviously commute with H_o . These gauge bosons thus remain massless, as they must do. The masses of the other 12 X and Y bosons are easily computed from Eq.(8.8) and the explicit representations of the corresponding generators (Eqs.(7.4) and (7.5)). One finds

$$m^2_X = m^2_Y = \frac{25}{8} g^2 \sigma^2_o \quad . \tag{8.9}$$

The reason why all these masses are equal is that, at this stage, SU(3) ⊗ SU(2) is exact.

The 3 H_X and the 3 H_Y remain massless in the Landau gauge. They can be gauged away (unitary gauge - see the beginning of Section 4), and their 12 degrees of freedom are transferred to the 12 gauge bosons which become massive. The 12 physical Higgs bosons H_G , H_W , H_B acquire masses, which we compute by taking the part of the Higgs potential (8.4) quadratic in these fields, using the representation (8.7) of H . One finds :

$$\begin{aligned} m^2_{H_G} &= \frac{5}{4} \lambda_2\, \sigma^2_o \\ m^2_{H_W} &= 5\, \lambda_2\, \sigma^2_o \\ m^2_{H_B} &= (15\lambda_1 + \frac{7}{2} \lambda_2)\, \sigma^2_o \end{aligned} \tag{8.10}$$

The 12 physical Higgs bosons have extremely high masses of the same order as the X and Y masses also proportional to σ_o (see Section 9 for orders of magnitude). The choice λ_2 very small seems unnatural, and anyway such an artificial adjustment would be destroyed by higher loop corrections to the Higgs potential.

b) Second stage of breaking $SU(3) \otimes SU(2) \otimes U(1) \rightarrow SU(3) \otimes U_{e.m.}(1)$

The second stage of SSB breaks $SU(2) \otimes U(1)$ down to $U_{e.m.}(1)$, and also gives masses to the fermions, which are still massless after the first stage exactly as in the unbroken Salam-Weinberg model. By the way, it happens that there is no Yukawa coupling at all between the fermions of the $\bar{5}$ and 10 and the Higgs fields in the adjoint representation (see below), so that not only H_o , but all Higgs bosons of the adjoint decouple from fermions.

We recall that the form of a Yukawa coupling of fermions to a generic Higgs field Φ is

$$\bar{f}_R \, f_L \, \Phi + h.c \qquad .$$

The problem is to determine which Higgs representation can be involved. The 5 multiplet ψ is

$$\psi = \{d_i , e^+ , -\nu^c\}_R$$

and the conjugate

$$\psi^c = \{d^i , e^- , -\nu\}_L \qquad .$$

The 10 multiplet is

$$M = \{u^i , u_i , d_i , e^+\}_L$$

and its conjugate is

$$M^c = \{u_i , u^i , d^i , e^-\}_R \qquad .$$

Terms like $\bar{f}_R f_L$ are thus to be found in(*)

$$\bar{\psi} \otimes M \qquad \text{or} \qquad \overline{M^c} \otimes M$$

We recall that the charge conjugate of a spinor f is defined as

$$f^c = C(\bar{f})^T$$

so that

$$\overline{M^c} = \overline{(C(\bar{M})^T)} = M^T \gamma_o C^\dagger \gamma_o = -M^T \gamma_o^* \gamma_o^T C^\dagger$$

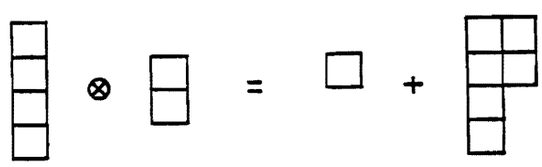
$$= M^T C^* \quad .$$

We have used the fact that in any gamma matrix representation $\gamma_o = \gamma_o^\dagger$ and $C = -C^T$.

The only Higgs representations giving rise to fermion mass terms are then to be found in

(i) $\bar{5} \otimes 10 = 5 + \overline{45}$

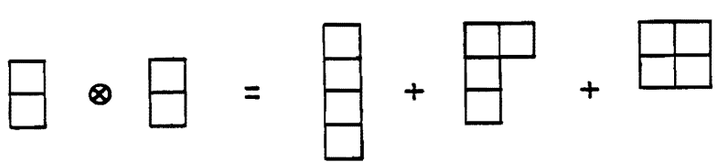
whose Young tableaux (see Appendix) are



(8.11)

(ii) or $10 \otimes 10 = \bar{5} + 45 + 50$

with Young tableaux



(8.12)

We first see that the adjoint is absent, the candidate Higgs representations being 5, 45 and 50. Moreover the representation must contain at least the

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(*) $\bar{\psi} \otimes \psi$, $\bar{M} \otimes M$ and $\overline{M^c} \otimes \psi$ Yukawa couplings are forbidden by chirality. $\overline{\psi^c} \otimes \psi$ can couple to reps 10 and 15. In the latter case the neutrino could acquire a Majorana mass.

(1,2) representation of $SU(3) \otimes SU(2)$ in order to give SU(3) invariant masses to the fermions. This leads to reject the 50 representation which decomposes under $SU(3) \otimes SU(2)$ according to

$$50 = (8,2) + (6,3) + (\overline{6},1) + (3,2) + (\overline{3},1) + (1,1) \quad .$$

The 5 and 45 representations have the following $SU(3) \otimes SU(2)$ decompositions

$$5 = (3,1) + (1,2)$$

$$45 = (8,2) + (6,1) + (\overline{3},3) + (3,2) + (\overline{3},1) + (3,1) + (1,2) \quad .$$

The simplest choice is to assume the existence of one multiplet {h} of Higgs bosons in the fundamental representation.

The Higgs potential is minimum for a non vanishing value ω of the Higgs fields {h}, that is (see § 8.a)i)) $<\sum_{\alpha=1}^{5} h^{\alpha} h_{\alpha}> = \omega^2$. The value of ω will be far smaller than that of σ_o characterizing the $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)$ breaking, in order for the first stage breaking to be far stronger than the second one. As we have seen in Section 3, the order of magnitude of ω is 200 GeV, whereas, as will be shown in Section 9, σ_o is of order 10^{14} GeV. The complete Higgs potential with both the adjoint H and the fundamental {h} representations present, and coupled to each other, is (assuming again $h \to -h$ and $H \to -H$ symmetries) :

$$V = -\frac{\mu^2}{2} \operatorname{Tr} H^2 + \frac{\lambda_1}{4} \left(\operatorname{Tr} H^2\right)^2 + \frac{\lambda_2}{4} \operatorname{Tr} H^4 - m^2 h^{\dagger} h + \frac{\lambda}{2} (h^{\dagger} h)^2$$
$$+ \frac{\alpha}{2} (h^{\dagger} h) \operatorname{Tr} H^2 + \frac{\beta}{2} h^{\dagger} H^2 h \quad . \tag{8.13}$$

Suppose first $\alpha = \beta = 0$. The situation has been studied in part a) of this section. For positive m^2 , the minimum is obtained for

$$<h^{\dagger} h> = \omega^2 = \frac{m^2}{\lambda}$$

the direction of the vector ground state h_o in the 5 dimensional space being arbitrary. As a function of H , the potential is minimum for a field configuration H_o , which, after a suitable SU(5) transformation reads

$$H_o = \frac{\mu}{\sqrt{\frac{15}{2}\lambda_1 + \frac{7}{4}\lambda_2}} \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & 3/2 & \\ & & & & 3/2 \end{pmatrix} . \qquad (8.14)$$

It is clear that in order for SU(3) symmetry to be preserved by the 2nd stage breaking, the vector h_o has to point in a direction contained in the plane of the two last coordinates. Hence h_o has to be

$$h_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ h_4^o \\ h_5^o \end{pmatrix} . \qquad (8.15)$$

Once it is of this form, an SU(2) transformation which leaves H_o unchanged, allows to write it

$$h_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega = \sqrt{\frac{m^2}{\lambda}} \end{pmatrix} , \qquad (8.16)$$

so that the $SU(2) \otimes U(1)$ SSB generated by $\{h\}$ is identical to that of the Salam-Weinberg model, as described in Section 3. Therefore, the crucial point is to insure that the minimum of the complete potential V of Eq.(8.13) is actually obtained for h_o in the form (8.15), when the couplings α and β of the h and H fields are non zero.

Remark

One should notice that even in the case $\alpha = \beta = 0$, h and H get coupled by radiative corrections (e.g. gauge boson exchanges). In the absence of such couplings, the part of V (Eq.(8.13)) which contains only $\{h\}$ has SU(5) symmetry, a larger symmetry than that of the rest of the Lagrangian after the first stage breaking. As stated at the beginning of the present section, we might have to face a situation where pseudo-Goldstone bosons (massless physical particles not associated with a broken symmetry) remain in the spectrum[22]. This is easy to understand in our particular case. Suppose h_o has the form (8.15) in the absence of h-H couplings. Among the 4 real fields contained in h_4 and h_5 , 3 fields are "eaten" by the 3 gauge bosons $\vec{W}$,

and one field remains as a massive Higgs boson, as in the Salam-Weinberg model. But the 6 real fields of h_1, h_2, h_3 remain massless. There is no mystery about what happens. In fact, in the absence of coupling, the generic situation for h_o after the first stage breaking is *not* (8.15), but rather, say $\begin{pmatrix} 0 \\ 0 \\ h_3^o \\ 0 \\ h_5^o \end{pmatrix}$, since the h fields know nothing about which axis among the 5 SU(5) axis are the 3 SU(3) axis and the 2 SU(2) axis. In this case, both SU(3) and SU(2) are separately broken. 5 real Higgs fields among the 6 of h_1, h_2, h_3 disappear, and 1 remains as a massive particle, while as usual 3 disappear among the 4 of h_4, h_5, the last one being massive. So, in a generic, uncoupled situation, one in fact does *not* expect pseudo-Goldstone bosons to be present. If the couplings are turned on (by radiative corrections or explicitly by non zero values of α and β), the h fields which do not disappear anyway get large masses ($\sim\sigma_o$) due to the terms $h^\dagger h\, H_o^2$ and $h^\dagger H_o^2\, h$.

A complete treatment of the coupled potential (8.13) can be found in Ref.[25]. There it is shown that for β positive, the couplings of $\{h\}$ to H actually provide ones with the orientation required for $\{h_o\}$ (Eq.(8.15)). The desired breaking $SU(5) \xrightarrow{H} SU(3) \otimes SU(2) \otimes U(1) \xrightarrow{h} SU(3) \otimes U_{e.m.}(1)$ is thus obtained in a natural way(*). The minimum of the potential is finally realized for the following Higgs configuration

$$h_o = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega \end{pmatrix}$$

$$H_o = \sigma_o \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \frac{3}{2}(1+\varepsilon) & \\ & & & & \frac{3}{2}(1-\varepsilon) \end{pmatrix} \tag{8.17}$$

(*) Here "natural" means that the desired breaking does not require any fine tuning of the parameters, so that it survives small variations and presumably to higher order effects. In our case, only inequalities on the Higgs potential couplings have to be fulfilled. What will be seen to be unnatural is the value of ω/σ_o (hierarchy problem, see Section 9).

The terms proportional to ε in H_o transform like an adjoint representation of SU(2). They reflect the SU(2) breaking induced by h_o . As shown in Section 3 (value of $\rho = G_N/G_F$ in excellent agreement with a breaking by an SU(2) Higgs doublet), the weak interaction phenomenology requires $\sigma_o\varepsilon$ to be much smaller than ω . Computation shows that the condition $\varepsilon << 1$ is in fact insured provided ω/σ_o is very small. The problem of how obtaining this "hierarchy" of vacuum expectation values in a "natural" way is a completely opened question[26]. One finds that it requires very strong cancellations in the expression for M_W, between terms which are a priori of order $M_{X,Y}$; moreover these cancellations, when imposed at zeroth order in perturbation, are not preserved by higher order loop contributions (the coupling constants are very large as compared to 10^{-13}). The so-called hierarchy problem is precisely that the potential parameters have to be adjusted up to many decimals at each order in perturbation. As a conclusion about this breaking scheme of SU(5), we may say that, while the two required steps may be achieved in a very simple and elegant way, the need for a fine tuning of the parameters is an up to now unsolved problem of the model (including other groups than SU(5), like SO(10) for example).

Fermion masses

We come to the predictions of SU(5) for the fermion masses, generated by the second stage SSB. The most general Yukawa couplings of the Higgs bosons in the 5 rep. to the fermions of the $(\bar{5} + 10)$ rep. take the form (Eqs.(8.11,12)) :

$$L^{\text{Higgs}(5)} = G\, h^{\alpha}\, \bar{\psi}^{\beta}\, M_{\alpha\beta} - \frac{1}{4}\, \tilde{G}\, \varepsilon^{\alpha\beta\gamma\delta\eta}\, h_{\alpha} (M_{\beta\gamma})^{T}\, C^{*} M_{\delta\eta} + \text{h.c} \quad . \qquad (8.18)$$

In order to understand the structure of the second term (the normalization is a matter of convenience), one remarks that $\bar{5}$ can be obtained as the totally antisymmetric product of four reps.5 (i.e. of two reps.10). So, ε being the fully antisymmetric tensor with 5 indices, the second term is of the form $\sum_{\alpha} h_{\alpha}\phi^{\alpha}$, the singlet piece of $5 \otimes \bar{5}$.

Since only h_5 has a non vanishing vacuum expectation value ω , the first term is readily seen to yield the following mass term :

$$\begin{aligned} M_1 &= G\omega\, \bar{\psi}^{\beta}\, M_{5\beta} + \text{h.c} \\ &= -\frac{G\omega}{\sqrt{2}} \left(\sum_i \bar{d}^{i}_{R}\, d_{iL} + \overline{e^{+}_{R}}\, e^{+}_{L} \right) + \text{h.c} \end{aligned} \qquad (8.19)$$

The coupling constant G may be complex, $G = |G| e^{i\varphi}$. Any fermion f appears in M_1 under the form

$$G\, \bar{f}_R f_L + G^* \bar{f}_L f_R = |G| \left(\bar{f}_R e^{i\varphi} f_L + \bar{f}_L e^{-i\varphi} f_R \right)$$

so that each chiral fermion field f_L , f_R may be redefined, $f_L \to e^{-i\varphi} f_L$ or $f_R \to e^{i\varphi} f_R$ in such a way that M_1 reads

$$M_1 = -\frac{|G|\omega}{\sqrt{2}} \left(\sum_i \bar{d}^i d_i + \overline{e^+}\, e^+ \right) \quad . \tag{8.20}$$

The second mass term is

$$M_2 = -\frac{1}{4} \tilde{G} \omega \varepsilon^{5\beta\gamma\delta\eta} \left[M_{\beta\gamma} \right]^T C^* M_{\delta\eta} + h.c \quad . \tag{8.21}$$

The right hand side of this equation is obviously symmetric under $\beta \leftrightarrow \gamma$ and $\delta \leftrightarrow \eta$. We remark that it is also symmetric under the exchange of the two pairs $(\beta\gamma)$ and $(\delta\eta)$. So is the ε tensor, and moreover we have :

$$(M_{\beta\gamma})^T C^* M_{\delta\eta} = \left[(M_{\beta\gamma})^T C^* M_{\delta\eta} \right]^T = -(M_{\delta\eta})^T C^\dagger M_{\beta\gamma}$$

$$= (M_{\delta\eta})^T C^* M_{\beta\gamma} \quad .$$

We have made use successively of the facts that the l.h.s., a scalar in Dirac space, is invariant under transposition, that two fermion fields anticommute and that in any gamma matrix representation $C^T = -C$. Hence the four terms in (8.21) where one index among $\beta, \gamma, \delta, \eta$ is equal to 4 are equal. We may thus write

$$M_2 = -\tilde{G} \omega \varepsilon^{ijk} (M_{ij})^T C^* M_{k4} + h.c \quad ,$$

where $i,j,k = 1,2,3$. Using the explicit particle content of the 10, namely $M_{k4} = u_{kL}/\sqrt{2}$ and

$$M_{ij} = \varepsilon_{ijm} \frac{u_L^m}{\sqrt{2}} = \varepsilon_{ijm}\, C\, \frac{(\overline{u_R^m})^T}{\sqrt{2}} \quad ,$$

$C^\dagger C = 1$, we obtain

$$M_2 = -\frac{\tilde{G}\omega}{2} \varepsilon^{ijk} \varepsilon_{ijm} \overline{u_R^m} u_{kL} + h.c$$

$$= -\tilde{G}\omega \overline{u_R^m} u_{mL} + h.c \quad .$$

By phase transformations of the fields u_L and u_R , made in order to absorb the phase of $\tilde{G}$, as above for M_1 , we finally get

$$M_2 = -|\tilde{G}|\,\omega \sum_m \bar{u}^m u_m \quad .$$

This means term M_2 concerns only u-quarks and thus leads to no prediction at all since $\tilde{G}$ is arbitrary. On the contrary, the equation (8.20) predicts that the down quark and the electron have the same mass. This degeneracy is a consequence of the SU(4) invariance of $\{h_o\}$. It is however broken by radiative corrections (see Section 9), the quarks being subject to stronger interactions than the electrons, below the scale σ_o where SU(5) is broken.

In what follows, we outline what happens if one multiplet of Higgs in the 45 (instead of the 5) is used. This 45 can be constructed from the product $5 \otimes \overline{10} = 45 + \bar{5}$. Since $\overline{10} = [\bar{5} \otimes \bar{5}]_{\text{antisym.}}$, the 45 is represented by a third rank tensor $T_\alpha^{\beta\gamma}$, antisymmetric with respect to the upper indices. Furthermore $\sum_\alpha T_\alpha^{\alpha\gamma}$, which would clearly represent the $\bar{5}$, has to vanish. Let us now assume that SSB occurs (with the 45) in such a way that $SU(3) \otimes SU(2) \otimes U(1)$ is broken down to $SU(3) \otimes U_{e.m.}(1)$. The Higgs fields which get non vanishing vacuum expectation values must have zero charge. The only zero charge components of T are :

$$T_i^{k5} = -T_i^{5k} \qquad (i,k = 1,2,3) \text{ and } \quad T_4^{45} = -T_4^{54}$$

as one verifies from the general relation

$$Q(T_\alpha^{\beta\gamma}) = (Q_5)_{\alpha\alpha} - (Q_5)_{\beta\beta} - (Q_5)_{\gamma\gamma} \quad ,$$

with Q_5 being the charge matrix in the representation 5 (Eq.§6.10)). The Higgs expectation values must be SU(3) symmetric, so that we set :

$$\langle T_i^{k5} \rangle = \omega\,\delta_i^k = -\langle T_i^{5k} \rangle$$

and the trace condition on T finally imposes

$$\langle T_4^{45} \rangle = -3\omega = -\langle T_4^{54} \rangle \quad .$$

Since the 45 representation appears both in $5 \otimes \overline{10}$ and $10 \otimes 10$, there are

again two possible Yukawa couplings, namely

$$G\,\bar{\psi}^{\alpha}\,M_{\beta\gamma}\,T^{\beta\gamma}_{\alpha} + h.c$$

and

$$\tilde{G}\,\varepsilon^{\alpha\beta\gamma\delta\eta}\,(M_{\alpha\beta})^{T}\,C^{*}M_{\rho\gamma}\,T^{\rho}_{\delta\eta} + h.c$$

Substituting the expectation values for T, one obtains from the first term

$$2G\,\omega\left(-3\,\bar{\psi}^{4}\,M_{45} + \sum_{i}\bar{\psi}^{i}\,M_{i5}\right) + h.c \;=\; \sqrt{2}\,G\,\omega\left(-3\,\overline{e^{+}_{R}}\,e^{+}_{L} + \sum_{i}\bar{d}^{i}_{R}\,d_{iL}\right) + h.c$$

The second term yields

$$-6\,\tilde{G}\omega\,\varepsilon^{ijk}\,(M_{ij})^{T}\,C\,M_{4k} + 2\,\tilde{G}\omega\,\varepsilon^{\alpha\beta\gamma k5}\,(M_{\alpha\beta})^{T}\,C\,M_{k\gamma} + h.c \;=\; 0 \quad ,$$

as one verifies by manipulations similar to those made for the breaking by rep.5 . After suitable phase transformations on the fields, the mass term becomes

$$M \;=\; -\sqrt{2}\,|G|\,\omega\left(3\,\overline{e^{+}}\,e^{+} + \sum\bar{d}^{i}\,d_{i}\right) \quad . \tag{8.23}$$

The prediction of SU(5) for a rep.45 (alone) of Higgs bosons is thus that the electron should have a mass three times larger than the down quark mass, while the up quark should be massless. Such a situation before mass renormalization does not appear as very appealing. Of course one may mix both reps.5 and 45 of Higgs bosons, and play with the parameters, but very little of the predictions is left.

c) Generation mixing

So far we have considered the case of only one family. In the SU(5) model, nature is supposed to contain at least 3 such families. Let $\{\psi'_a\}$ and M'_a be the corresponding representations 5 and 10 where the subscript a distinguishes between various generations. The prime on ψ and M means that, in general, ψ'_a and M'_a are linear combinations of the ψ and M containing the mass eigenstates, as we shall see now. In the presence of several families, the most general fermion Lagrangian is

$$L = i\bar{\psi}'_a \not{D} \psi'_a + i\,\mathrm{Tr}(\bar{M}'_a \not{D} M'_a) + h\,\bar{\psi}'_a G_{ab} M'_b + h\, M'^{\,T}_a C\, \tilde{G}_{ab} M'_b + \mathrm{h.c.}, \quad (8.24)$$

where summation over the generation index is understood. The term involving the covariant derivative is by definition proportional to unity in generation space. The Higgs part, where all SU(5) indices have been omitted in the couplings to the Higgs bosons {h}, contains matrix couplings (in generation space) G_{ab}, $\tilde{G}_{ab}$, which generalize the couplings G and $\tilde{G}$ of the previous section. They have no reason to be diagonal, and are in general arbitrary complex matrices. When h is replaced by its vacuum expectation value, one obtains two fermion mass matrices, and as announced, the fermions of ψ' and M' are not mass eigenstates. Explicitly, the mass matrices are

$$L_{mass} = -\frac{G_{ab}\omega}{\sqrt{2}}\left(\overline{d'^{i}_a}\, d'_{ib_L} + \overline{e^{+'}_a}\, e^{+'}_{b_L}\right) - \overline{u'^{k}_a}\, u'_{kb_L}\, \frac{\tilde{G}_{ab} + \tilde{G}_{ba}}{2}\,\omega + \mathrm{h.c} \quad (8.25)$$

u'_a, d'_a, e'_a here stand for charge 2/3, -1/3 quarks and charged lepton respectively. They are linear combinations (to be determined from G and $\tilde{G}$) of, resp., the (u,c,t,...), (d,s,b,...) quarks and (e,μ,τ,...) leptons, which are the physical mass eigenstates fermions. The two non diagonal mass matrices in (8.25) are :

$$M_{ab} = G_{ab}\,\frac{\omega}{\sqrt{2}} \quad (8.26)$$

and

$$\tilde{M}_{ab} = (\tilde{G}_{ab} + \tilde{G}_{ba})\,\frac{\omega}{2} \quad (8.27)$$

We have to show that they can be diagonalized through global unitary transformations acting only on the generation indices. Being global, they do not affect the "kinetic" terms $\bar{f}\not{D} f$. Since the two matrices in (8.25) concern d's and e's on one hand, u's on the other, we first diagonalize M, and then diagonalize $\tilde{M}$.

We first recall that any regular matrix A can be put into diagonal form a by a bi-unitary transformation (V,U)

$$V\,A\,U^{\dagger} = a \quad (8.28)$$

a is furthermore real and positive definite. The proof is as follows. The matrix $A^{\dagger}A$, being hermitian, can be diagonalized by a unitary matrix U :

$$U A^{\dagger} A U^{\dagger} = \mathcal{A}^2 \quad ,$$

where $\mathcal{A}$ is defined as the square root of $\mathcal{A}^2$ whose all elements are positive. A being regular, $\det(\mathcal{A}) = |\det A| \neq 0$, and $\mathcal{A}^{-1}$ exists. We define a second unitary matrix $V = \mathcal{A}^{-1} U A^{\dagger}$ and Eq.(8.28) follows.

Let now (V,U) be the pair of matrices in the generation space which diagonalizes M of Eq.(8.26). The new fields

$$\begin{aligned} \psi'' &= V \psi' \\ M'' &= U M' \end{aligned} \tag{8.29}$$

are such that d''_a and $e^{+''}_a$ are mass eigenstates. We drop the double prime for them and write

$$m_{d_a} = m_{e_a} = m_a \quad .$$

The new u-quarks, u''_a are still unphysical since (V,U) in general does not diagonalize $\tilde{M}$ of Eq.(8.27). From (8.29) and (8.24), the mass matrix between u'' states is

$$N = U^* \tilde{M} U^{\dagger} \quad ,$$

and we diagonalize it by a new bi-unitary transformation $(\tilde{V},\tilde{U})$ which acts on charge 2/3 quarks only, not to destroy what is already achieved. The new u-type quarks are mass eigenstates, with masses $\tilde{m}_a$.

In order to construct explicitly the physical u-quarks, we remark that the symmetry of $\tilde{M}_{ab}$ (Eq.(8.27)) allows a simplification. It implies that N also is symmetric, so that

$$N N^{\dagger} = N^T N^* \quad .$$

In terms of the diagonal form $\tilde{M}$ of N , which by definition verifies $N = \tilde{V}^{\dagger} \tilde{M} \tilde{U}$, this equality means

$$\tilde{V} \tilde{U}^T \tilde{M}^2 = \tilde{M}^2 \tilde{V} U^T \quad .$$

It shows that the matrix $S = \tilde{V}\,\tilde{U}^T$, which commutes with the diagonal M^2 , the elements of which are by assumption all different (no degeneracy in flavour space), is also diagonal. Since it is unitary,

$$S_{ab} = e^{i\varphi_a}\,\delta_{ab} \quad .$$

We then replace $\tilde{V}$ by $S\tilde{U}^*$, and get for the mass matrix between u-type quarks

$$\bar{u}'_R\,\tilde{M}\,u'_L = \bar{u}''_R\,N\,u''_L = \bar{u}''_R\,(\tilde{U}^T S^\dagger)\,\tilde{M}\,\tilde{U}\,u''_L \quad ,$$

which shows that the left handed and right handed mass eigenstates are

$$\begin{aligned} u_{kL} &= \tilde{U}\,u''_{kL} \\ u_{kR} &= S\,\tilde{U}^*\,u''_{kR} \quad . \end{aligned}$$

Here we have restored the colour index k.

Finally, since all neutrinos are massless (in the model), any linear combination of neutrino fields is a mass eigenstate. It is a matter of convention to set

$$\nu_a \equiv \nu''_a$$

According to their definition in terms of the original fields, the physical fermions fall in either of the three classes (d,e,ν) or u_{kR} or u_{kL} . Since the interactions with the gauge bosons were written in terms of the original fields (with primes), these interactions which involve different class physical fermions contain Cabbibo type mixing angles and (or) phases. Let us make this point more explicit. The fermion-gauge boson interactions, in terms of the original fields, are (Eqs.(7.13,18) :

$$-g\Bigg\{G^\mu_a \sum_{\text{quarks}} \bar{q}'\,\gamma_\mu \frac{\lambda^a}{2}\,q' + \vec{W}^\mu \sum_{\text{fermions}} \bar{f}'_L\,\gamma_\mu \frac{\vec{\tau}}{2}\,f'_L + \sqrt{\frac{3}{5}}\,B^\mu \sum_{\text{fermions}} \bar{f}'\,\gamma_\mu \frac{Y}{2}\,f'$$

$$+\Bigg[\frac{X^\mu_i}{\sqrt{2}}\left(\bar{d}'^{\,i}\,\gamma_\mu\,e^{+\prime} + \varepsilon^{ijk}\,(\bar{u}')_j\,\gamma_\mu\,u'_{kL}\right) + \frac{Y^\mu_i}{\sqrt{2}}\left(\bar{\nu}\,\gamma_\mu\,d'^{\,i}_L - \bar{u}'^{\,i}\,\gamma_\mu\,e^{+\prime}_L + \varepsilon^{ijk}\,(\bar{u}')_j\,\gamma_\mu\,d'_{kL}\right) + \text{h.c}\Bigg]\Bigg\} \quad .$$

A sum over all generations is understood everywhere. The first transformation (V,U) which acts over all types of fermions leads to the same interactions in terms of the d, e and u" fields. But the second one, defined by $(\tilde{V},\tilde{U})$, or equivalently $(S\tilde{U}^*, \tilde{U})$, distinguishes the u" from the d and e fields, and the u''_L from the u''_R. As a consequence the interactions with the $W^{\pm}$ and X,Y gauge bosons, which involve u''_L and u''_R linearly, are modified.

Recalling that $d = d''$ and $u_L = \tilde{U} u''_L$ are physical particles, we obtain for the charged weak current :

$$\frac{1}{\sqrt{2}} W^+_\mu \bar{u}'_L \gamma^\mu d'_L + h.c = \frac{1}{\sqrt{2}} W^+_\mu \bar{u}''_L \gamma^\mu d''_L + h.c$$

$$= \frac{1}{\sqrt{2}} W^+_\mu \bar{u}_L \gamma^\mu \tilde{U} d_L + h.c \quad . \qquad (8.30)$$

The unitary matrix $\tilde{U}$ is the usual generalized Cabibbo mixing matrix. The charged weak current couples the charge 2/3 physical quarks $(u,c,t,...)_L$ to the linear (charge -1/3) quark combinations $\tilde{U}(d,s,b,...)_L$. $\tilde{U}$ depends on F^2 real parameters if there are F families. However, the 2F quark fields have 2F-1 arbitrary relative phases which can be used to absorb 2F-1 phases of the U matrix elements in the quark field definitions. Hence the number of physically relevant parameters in $\tilde{U}$ is $F^2 - (2F-1) = (F-1)^2$. It is 1 for F = 2 (the Cabibbo angle), 4 for F = 3 (3 Cabibbo-like angles and 1 phase, responsible for CP violations in weak decays[27]).

In performing the relevant unitary transformations in the piece of the interaction Lagrangian which violates baryon number conservation, one has to be careful about the notations. Consider for example the quantity $(\bar{u}')_j \gamma_\mu u'_{kL}$ which couples to X^μ. From our conventions (see the beginning of Section 5), $(\bar{u}')_j = \overline{u'^j}$ is the Dirac conjugate of the spinor $u'^j = (u'_j)^c$. Since for any fermion $(f_R)^c = (f^c)_L$, the current $(\bar{u}')_j \gamma_\mu u'_{kL}$ can be written

$$\overline{u'^j} \gamma_\mu u'_{kL} = \overline{(u'^j)_L} \gamma_\mu u'_{kL}$$

$$= \overline{(u'_{jR})^c} \gamma_\mu u'_{kL} \quad .$$

As a consequence, when we transform the u'-fields into the physical u-fields, we have to take care of the different transformations occurring for the left

and right u" :

$$u'_L = U^\dagger u''_L = U^\dagger \tilde{U}^\dagger u_L$$

but

$$u'_R = U^T u''_R = U^T \tilde{U}^T S^\dagger u_R \quad ,$$

so that

$$(u'_R)^c = U^\dagger \tilde{U}^\dagger S^T (u_R)^c$$

and

$$\overline{(u'_R)^c} = \overline{u^c_R}\, S^* \tilde{U}\, U \quad .$$

From these identities, we find :

$$(\bar{u}')_j \gamma_\mu u'_{kL} = (\bar{u})_j \gamma_\mu S^* u_{kL} \quad .$$

The complete interaction with the X,Y bosons finally reads in terms of the physical fermion fields :

$$- g \frac{X^\mu_i}{\sqrt{2}} \left(\bar{d}^i \gamma_\mu e^+ + \varepsilon^{ijk} (\bar{u})_j \gamma_\mu S^* u_{kL} \right)$$
$$- g \frac{Y^\mu_i}{\sqrt{2}} \left(\bar{\nu} \gamma_\mu d^i_L - \bar{u}^i \gamma_\mu \tilde{U} e^+_L + \varepsilon^{ijk} (\bar{u})_j \gamma_\mu S^* \tilde{U} d_{kL} \right) + h.c \qquad (8.31)$$

Recall that in this formula, the symbols u,d,e,ν are generic for respectively charge 2/3 , -1/3 quarks, charged leptons and neutrinos. We see that in addition to the $\tilde{U}$ matrix which already appeared in the weak charged currents, we have a new matrix S in the currents. In the present case, as we have shown, S is diagonal and its phases are not observable at the tree diagram level. Since furthermore, we know from experiment that $\tilde{U}$ is closed to unity (small Cabibbo angles and CP violating phase), the interaction described by (8.31) is nearly identical to the original one (7.18). This simplicity disappears for more complicated Higgs systems : S is no more diagonal because the original mass matrix $\tilde{M}_{ab}$ is no more symmetric. Even with $\tilde{U} \simeq \mathbb{1}$, (8.31) is no more diagonal in generation space.

Let us make a final remark : we have stated that the first family was composed of

- one $\bar{5}$ containing the physical d^c_L , e^-_L and ν_e ;

- one 10 containing the physical d_L , e_L^+ , a combination of the various left handed charge 2/3 quarks $(u,c,t,...)_L$ and another combination of the corresponding left handed antiquarks.

In the same way, the second family is composed of a $\bar{5}$ made of s_L^c , μ_L^- and ν_μ , and of a 10 with s_L , μ_L^+ and 2 combinations of respectively quarks and antiquarks, and so on. Concerning these statements, the following comments are in order.

(i) As already said above, any linear combination of massless ν's is a mass eigenstate, which allows ones to define the neutrino states in such a way that weak interactions induce only $\nu_e \to e$, $\nu_\mu \to \mu$, transitions. The pairing (e,ν_e) , (μ,ν_μ) , ... is thus only a labelling convention for the ν's . The pairing $\left(e , \frac{\nu_e+\nu_\mu}{\sqrt{2}}\right)$ and $\left(\mu , \frac{\nu_e-\nu_\mu}{\sqrt{2}}\right)$, say, involves exactly the same physics although in a rather awkward way.

(ii) The mixing between the u's is severely constrained by existing data on weak interactions. For example the matrix $\tilde{U}$ restricted to the two first families is nearly unitary (success of the original Cabibbo model with only one mixing angle θ_c). So, the other mixing angles contained in $\tilde{U}$ must be small.

(iii) In principle, one may associate the pair (e,ν_e) with any quark family. Such a choice would lead to the same present energy phenomenology except for the fermion masses(*). But this unusual association would however clearly show up in X and Y interactions. With the choice (e,s) , (μ,d) for example, the proton decays $p \to \mu^+\pi^o$ and $p \to e^+K_o$ rather than $p \to e^+\pi^o$ and $p \to \mu^+K_o$ become Cabibbo allowed.

(iv) There is however no possibility, in the simplest SU(5) model for, say, the first multiplet to be made of the down quarks and of some linear combination of the physical leptons.

(*) In Section 9 below, it is shown that after continuation down to present energies, the (asymptotic) prediction $m_\tau = m_b$ compares well with experiment (under the assumption of 3 families only). But the other mass predictions, in particular $m_d = m_e$, have problems. It is thus not impossible, though a little bit bizarre, to make the associations (e,s) and (μ,d).

d) B-L invariance

As we have seen (Eq.(8.31)), the X and Y gauge bosons connect quarks and leptons and thus the baryon and lepton numbers B and L are not conserved. However, B-L is conserved in the simplest SU(5) model as we shall see right now. The unbroken theory exhibits an additional global abelian invariance which we call $U_x(1) = e^{-i\beta x}$. U_x commutes with the SU(5) transformations. The gauge bosons and the Higgs bosons in the 24 are real (the 24 is a real representation) so they must be U_x singlets.

Let $x_{5,\bar{5}}$ and $x_{10,\overline{10}}$ be the eigenvalues of X for the fermions of the $5,\bar{5}$ and $10,\overline{10}$ respectively, x_5^h that of the Higgs bosons in the 5. The invariance of $\bar{\psi}_5 D_\mu \psi_5$ and $\bar{\psi}_{10} D_\mu \psi_{10}$ requires $x_5 = -x_{\bar{5}}$ and $x_{10} = -x_{\overline{10}}$. In order for the couplings of the fermions to the Higgs bosons of the 5, namely for terms of the form (Eq.(8.18)) :

$$h_{\bar{5}}\, \psi_{\bar{5}}\, M_{10} \qquad \text{and} \qquad h_5\, M_{10}^T\, C M_{10}$$

to be invariant, one has

$$-x_5^h - x_5 + x_{10} = x_5^h + 2x_{10} = 0 \quad .$$

Hence setting arbitrarily

$$x_{10} = 1 \quad ,$$

one obtains

$$x_5 = 3 \tag{8.32}$$

$$x_5^h = -2$$

The breaking of SU(5) by the adjoint Higgs representation preserves the $U_x(1)$ symmetry because these Higgs are U_x invariant. This is not so for the second stage breaking since the non vanishing value ω of h_α is shifted by $e^{-i\beta x_5}$ under a U_x transformation. Now the hypercharge Y of SU(5) also gets broken at the second stage. Using the explicit form of Y in a rep.5,

$$Y = \begin{pmatrix} -2/3 & & & & \\ & -2/3 & & & \\ & & -2/3 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{8.33}$$

and $x_5^h = -2$ (Eq.(8.32)), we see that, for the Higgs bosons in the rep.5, (X+2Y) has the eigenvalues

$$(X_5^h + 2Y) = \begin{pmatrix} -10/3 & & & & \\ & -10/3 & & & \\ & & -10/3 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} .$$

Thus X+2Y remains unbroken when the h fields acquire a non vanishing expectation value in the last two directions(*). Let us now compute X+2Y for the fermions. Calling B and L the usual baryon and lepton numbers, we obtain

$$\left(\frac{X+2Y}{5}\right)_{d_L^c} = -\frac{1}{3} = B_{d^c}$$

and (8.34)

$$\left(\frac{X+2Y}{5}\right)_{e_L^-,\nu} = -1 = -L_{e^-,\nu} .$$

For the fermions in the representation 10, we get

$$\left(\frac{X+2Y}{5}\right)_{u_L^c} = -\frac{1}{3} = B_{u^c}$$

$$\left(\frac{X+2Y}{5}\right)_{u_L,d_L} = \frac{1}{3} = B_{u,d} \tag{8.35}$$

$$\left(\frac{X+2Y}{5}\right)_{e^+} = 1 = -L_{e^+} .$$

(*) One may wonder where is the Goldstone boson associated with the spontaneous breaking of the global U_x symmetry. In fact, as X+2Y is unbroken, there is only *one* Goldstone boson, associated with X-2Y, say (and not two associated with X and Y), and this boson is "eaten" by the B boson, which is massive. This miraculous escape from the Goldstone theorem is known as the 't Hooft mechanism.

Hence $B-L = \frac{X+2Y}{5}$ is conserved. Moreover, since $x_{adj.} = 0$, $\left(\frac{X+2Y}{5}\right)_{adj.} = \frac{2}{5} Y_{adj.}$ from which we learn that

$$(B-L)_{gluons} = (B-L)_{\vec{W}} = (B-L)_B = 0 \quad ,$$

whereas

$$(B-L)_{X,Y} = \frac{2}{3} = -(B-L)_{X^*,Y^*} \quad . \tag{8.36}$$

The same is true for the corresponding $H_{X,Y}$ Higgs bosons. Of course the non vanishing of $(B-L)_{X,Y}$ is connnected to the fact that the X,Y gauge bosons (as well as the Higgs $H_{X,Y}$) couple on one hand to 2 quarks ($B = 2/3$, $L = 0$) and on the other to one antilepton ($B = 0$, $L = -1$) and one antiquark ($B = -1/3$, $L = 0$) . For example the transition

$$u_L + u_R \rightarrow Y \rightarrow e_L^+ + d_R^c$$

is allowed by the Lagrangian of Eq.(8.31) and indeed leads to the possibility of proton decay.

Remark

B-L is not conserved for any Higgs system. The addition of a rep.45 or 50 is innocent in this respect, but suppose instead that we have an additional rep.15 of Higgs bosons. The U_x invariance of the Yukawa coupling $\bar{\psi}_5^c \psi_5 H_{15}^\dagger$ requires $X_{15}^h = 6$. Assume now that the zero charge, colour singlet, SU(2) triplet component of H_{15} , $(H_{15})_{55}$, acquires a non vanishing expectation value. This has several consequences :

i) the relation $\frac{m_W^2}{\cos^2\theta \; m_Z^2} = 1$ is violated ;

ii) the neutrino acquires a Majorana mass from

$$<(\bar{\psi}^c)_\alpha \psi_\beta (H_{15})^{\alpha\beta}> = \bar{\nu}_L (\nu_L)^c <(H_{15})^{55}> \quad ;$$

iii) B-L is violated as X+2Y is equal to 6+4 for $(H_{15})_{55}$. This point is related to point ii): B-L invariance forbids a Majorana mass at all orders.

In such a case there is a physical Goldstone boson associated with (B-L) SSB. The only way to get rid of it is to make the original Lagrangian not U_x invariant (e.g. by adding the trilinear coupling $h_5 h_5 H_{15}^\dagger$).

9. RENORMALIZATION OF THE SU(5) PREDICTIONS

This chapter intends to show how much the predictions of exact SU(5) are altered at present energies by SU(5) breaking. The computational tool is the renormalization group[28].

a) Coupling constant renormalization

Let g_1, g_2 and g_3 be defined from the part of the Lagrangian containing the B, the W's and the gluons :

$$L = -g_3\, G_a^\mu \sum_{\text{quarks}} \bar{q}\, \gamma_\mu \frac{\lambda^a}{2}\, q$$
$$- g_2\, \vec{W}^\mu \sum_{\text{fermions}} \bar{f}_L\, \gamma_\mu \frac{\vec{\tau}}{2}\, f_L$$
$$- g_1 \sqrt{\frac{3}{5}}\, B^\mu \sum_{\text{fermions}} \bar{f}\, \gamma_\mu \frac{Y}{2}\, f \quad . \qquad (9.1)$$

Similarly g_x is the X or Y coupling to the fermions. In absence of SSB, SU(5) symmetry means $g_1 = g_2 = g_3 = g_x$. In the case of SSB, these equalities can be true only at very high energies, above the mass scale σ_o at which breaking occurs. To be more precise, let us recall that coupling constants are defined for a given scale μ (the value of the external momenta at which the renormalization conditions are imposed). If μ, which is arbitrary, is varied, the couplings and masses have to be varied in order for the physics to remain unchanged. This invariance of physics with respect to μ leads to the so-called renormalization group equations (R.G.E). g_1, g_2, g_3 and g_x are thus functions of μ ("running coupling constants"), and they all become equal at very large μ's.

The RGE for the couplings have the form :

$$\mu \frac{\partial}{\partial \mu} g_a(\mu) = \beta_a\, ([g_i(\mu)], \text{ masses, Higgs self couplings and couplings to fermions}).$$

The β functions are computable in perturbation. For μ much larger than all particle masses ($\mu \gg \sigma_o$), the situation is identical to the symmetric one (where all particles are massless) : the four β_a's are thus equal, and accordingly the g's remain consistently equal in this region. As μ decreases, the masses of the X and Y become non negligible, so that the values of the various β's become different. As a consequence, the corresponding g's have different

evolutions between the region $\mu \approx \sigma_o$ and present energies. Before discussing these evolutions, we recall how the renormalization program is achieved. The Green's functions $\Gamma_o^{(*)}$ computed with Feynman rules for given bare mass m_o and coupling g_o are in general represented by divergent integrals. These integrals are made convergent by some regularization procedure, such as for example the introduction of a cutt-off parameter Λ in the loop integrals over internal momenta. A renormalized Green's function Γ, depending on renormalized mass m and coupling g, can be defined as

$$\Gamma(g,m) = \lim_{\Lambda\to\infty} Z\ \Gamma_o(g_o,m_o,\Lambda) \qquad . \tag{9.2}$$

Z is a product of renormalization constants$^{(**)}$. It contains

- one factor $Z_A^{1/2}$ for each external gauge boson leg,
- one factor $Z_F^{1/2}$ for each external fermion leg,
- other factors associated with other external legs we have not to consider here.

The limit $\Lambda \to +\infty$ is taken at fixed g and m, g_o and m_o being redefined as functions of g, m and Λ according to

$$g = \lim_{\Lambda\to\infty} Z_g^{-1}\ g_o$$

$$m = \lim_{\Lambda\to\infty} Z_m^{-1}\ m_o \qquad .$$

The renormalization constants (Z_g, Z_m, Z_A, Z_F,...) are functions of g, m and Λ (or g_o, m_o and Λ). They diverge as Λ goes to infinity and are determined by the requirement that the limit $\Lambda\to\infty$ of Eq. (9.2) exists for all 1.P.I. Green's functions. Proving renormalizability consists in showing that it is actually possible to compute these renormalization constants order by order in g from the requirement that all 1.P.I. Green's functions are finite. The renormalization constants are determined up to parts which remain finite as $\Lambda\to\infty$. Accordingly, the Green's functions are obtained up to some constants (with respect to external

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(*) It is sufficient to consider the so-called "1-particle irreducible" (1.P.I.) Green's functions (no way to decompose the diagrams into two disconnected pieces on cutting 1 internal line only). All Green's functions can then be built without introducing any new problem of convergence.

(**) They are constants with respect to external momenta, but functions of the parameters (see below).

momenta). These constants are fixed by the "renormalization conditions", i.e. by specifying the value of a finite number of 1.P.I. Green's functions at given values of the external momenta, usually defined as euclidian values such that $p_i^2 = -\mu^2$. By the renormalization conditions, g and m become functions of the particular mass scale μ chosen. Hence, they are "running" coupling and mass $g(\mu)$ and $m(\mu)$.

One has some freedom in choosing which set of 1.P.I. Green's functions is used to impose the renormalization conditions. (It is sufficient to fix the values of all the propagators and of one vertex). Since the coupling constants (except g_1) appear as the coefficient of the term of the Lagrangian which is cubic in the gauge fields A_μ, the 3 A vertex may be conveniently used in the following way. In the Lagrangian, the cubic term has the form

$$g_a \; f_{\alpha\beta\gamma} \; P_{\mu\nu\rho} \; A^\mu_\alpha A^\nu_\beta A^\rho_\gamma \tag{9.3}$$

where f is the group structure constant and, in momentum space, $P_{\mu\nu\rho}$ is a Lorentz tensor linear in the boson momenta. The 3 A vertex (the 3 A's are all either gluons or W's or X,Y) contains a term of this type, which at the renormalization point may be written

$$F_a(p_i^2 = -\mu^2) \; f_{\alpha\beta\gamma} \; P_{\mu\nu\rho} \quad , \tag{9.4}$$

plus other terms with different Lorentz tensors in the p_i's, which are finite and do not require renormalization. We then set the renormalization condition

$$g_a(\mu) = F_a(\mu^2) \quad , \tag{9.5}$$

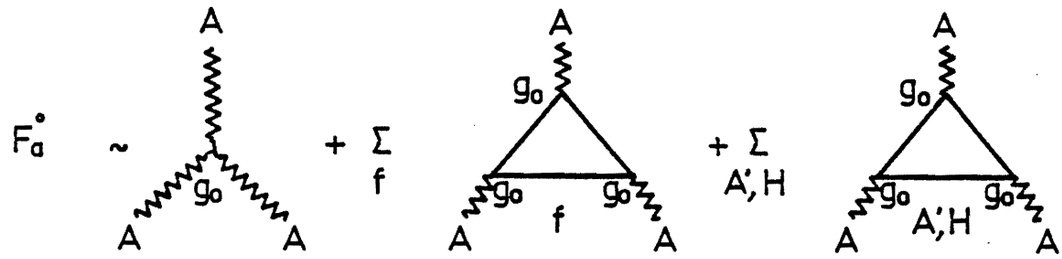
and from Eq. (9.2) for the 3 A vertex (where group and Lorentz indices where omitted), we get

$$g_a(\mu) = \lim_{\Lambda\to\infty} Z_A^{3/2} \; F_a^o \quad , \tag{9.6}$$

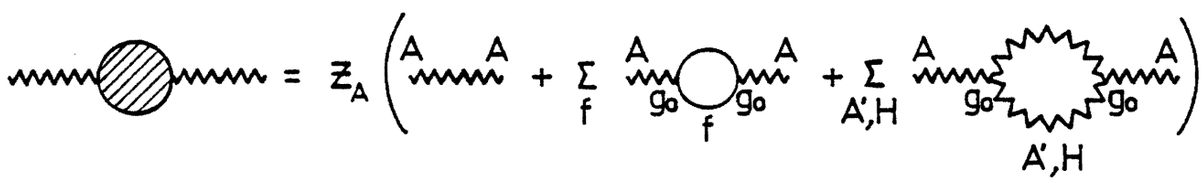
where Z_A and F_a^o have to be calculated in perturbation as functions of g_o, m_o and Λ. At lowest order, $g_a(\mu) = g_o$ and the coupling constants are SU(5) symmetric. The difference, by one order of magnitude, between the strong and electromagnetic coupling constants observed at present energies $\mu \simeq \mu_o$ is actually due to radiative corrections. The point is that the usual perturbative

expansion is not valid due to the presence of large logarithms $\ln \frac{m_{X,Y}}{\mu_o}$. The role of the R.G. is precisely to sum up all these logs, leading to a final result quite different from the lowest order.

At next order (one loop), F_a^o is given by the graphs (computed at $p_i^2 = -\mu^2$)



and Z_A is computed from the A boson inverse propagator



and a renormalization condition imposed to it at $p^2 = -\mu^2$. In both cases, the third graph involves all the (Higgs and gauge) bosons which couple to the particular A considered. For example, if A is a gluon, the loop may involve gluons, X or Y bosons, or Higgs bosons. But it never involves W's or B's. Since the fermion and boson propagators occur only in graphs of order g_o^3 , at this order we may use the renormalized masses as well as the bare ones in the graph computation. Finally, as far as the dependence on Λ is concerned, all loop integrals happen to be logarithmically divergent, leading to terms $\mathrm{Log}\,\frac{\Lambda^2}{M^2}$, where M^2 is some typical scale. For each graph, the only scales are the masses m_L appearing in the loop, and μ, so that all the results obtained through equations (9.6) to (9.8) can be gathered under the form :

$$g_a(\mu) = g_o + \frac{g_o^3}{8\pi}\left[\sum_L b_L \ln \frac{\Lambda^2}{\mu^2 + x m_L^2} + f_L\left(\frac{\mu}{\Lambda}, \frac{m_L}{\mu}\right)\right] + O(g_o^5) \quad . \tag{9.9}$$

Note that, as there is only one bare coupling g_o, the four $g_a(\mu)$ are not independent. Once the value of one of them has been fixed, the values of the others are determined. f_L has a limit $\tilde{f}_L\left(\frac{m_L}{\mu}\right)$ as $\Lambda \to \infty$. It can be shown that $\tilde{f}_L$ is moreover regular in both limits $\frac{m_L}{\mu} \to 0$ and $\frac{m_L}{\mu} \to \infty$. x is a constant of order 1 coming from the loop integrations. The sum runs over all possible

loops, m_L being the corresponding (bare or renormalized) mass(*). One sees from Eq. (9.9) that for μ much larger than all the masses of the problem, the relation between $g_a(\mu)$ and g_o is that of the massless (unbroken) theory, the same for all a's :

$$g_a(\mu) = g_o + \frac{(g_o)^3}{8\pi}\left[\sum_L b_L \ln\frac{\Lambda^2}{\mu^2} + f_L\left(\frac{\mu}{\Lambda}, 0\right)\right] + O(g_o^5) \quad .$$

This is the result announced : $g_1(\mu)$, $g_2(\mu)$, $g_3(\mu)$ and $g_x(\mu)$ are equal for very high μ's.

Suppose now that we change the renormalization scale from μ to μ'. From equation (9.9) the relation between $g_a(\mu)$ and $g_a(\mu')$ reads for μ close to μ', after taking the limit $\Lambda \to \infty$:

$$g_a(\mu) = g_a(\mu') + \frac{1}{8\pi}\sum_L g^3(L)\left[b_L \ln\frac{\mu'^2+xm_L^2}{\mu^2+xm_L^2} + \tilde{f}_L\left(\frac{m_L}{\mu}\right) - \tilde{f}_L\left(\frac{m_L}{\mu'}\right)\right] + O(g^5) \quad . \qquad (9.10)$$

In this equation, for each loop L $g^3(L)$ has to be interpreted as a cubic monomial of the renormalized coupling constants, that is $g^3(L) = g_a^3(\mu')$ or $g_a^2(\mu')g_b(\mu')$ or $g_a(\mu')g_b^2(\mu')$ or $g_b^3(\mu')$, according to which vertices appear in L. For example, if the graph involves three-gluon vertices, $g^3(L)$ is to be reinterpreted as $g_3^3(\mu')$. The evolutions of the four coupling constants are thus coupled to each other. However, as stated above, the 1 loop graphs for F^o_{gluon} (F^o_W) never involve W bosons (gluons) or B bosons as internal lines. Hence, at this order, the evolutions of $g_3(\mu)$ and $g_2(\mu)$ decouple from each other and only depend on themselves and on $g_x(\mu)$.

For μ' and μ very different, Eq. (9.10) becomes useless (large $\ln(\mu'/\mu)$), and we must use the R.G.E. Recalling the definition of the β function

$$\beta(g(\mu)) = \mu\frac{\partial}{\partial\mu}g(\mu) \quad , \qquad (9.11)$$

we are going to use (9.10) in order to differentiate $g(\mu)$ with respect to μ, which gives the 1 loop contribution to β, and then to integrate the differential equation to get $g(\mu)$. We remark that since $\tilde{f}_L$ is regular in both limits $m_L/\mu \to 0$ and $m_L/\mu \to \infty$, $\mu\frac{\partial}{\partial\mu}\tilde{f}_L\left(\frac{m_L}{\mu}\right)$ vanishes in both cases and gives no contribution to β provided μ is chosen far away from all masses. So, the contribution to β from a given loop is :

(*) Strictly speaking, some loops involve particles of different masses. Eq. (9.9) is thus somewhat symbolic. This complication however does not modify the conclusions.

i) $-\frac{g(\mu)^3}{4\pi} b_L$ for $\mu^2 >> m_L^2$, as in a massless theory

ii) $O\left(\frac{\mu^2}{m_L^2}\right)$ for $\mu^2 << m_L^2$.

Hence extremely heavy particles decouple, which is a special case of the Appelquist and Carrazzone theorem[29]. That means that for μ far below M_X, M_Y the X and Y contributions to the evolutions of g_2 and g_3 are negligible. If at the same time μ is well above M_W and M_Z, these masses can be neglected, and g_2 evoluates as an unbroken SU(2) coupling constant.

In order to compute the evolution of $g_1(\mu)$, we have to use another Green's function. We choose the gauge boson-fermion vertex (more precisely that part of it which couples to the fermion via a $\gamma_\mu B^\mu$ coupling). Here, in addition to Z_B, we need the fermion renormalization constant Z_F, which can be computed from the renormalized fermion inverse propagator. That the evolution of g_1 decouples from the others (namely g_2 and g_3) is not as simple as it was for g_2 and g_3. Nevertheless it holds (in the one loop approximation) due to the Ward identities which relate the fermion propagator to the fermion-B field vertex. They have the consequence that the renormalized gauge coupling g_1 is related to the bare one simply through

$$g_1 = \sqrt{Z_B}\, g_o \quad .$$

(Note that any reference to the external fermion has disappeared). So we have to consider only the B propagator (to compute Z_B) and the decoupling holds as for g_2 and g_3.

Summarizing : for $M_{W,Z} << \mu << M_{X,Y}$, the X and Y contributions to the β's vanish and the W,Z masses can be neglected. The three coupling constants $g_3(\mu)$, $g_2(\mu)$ and $g_1(\mu)$ "run" as independent SU(3), SU(2) and U(1) coupling constants (SU(2) and U(1) are still unbroken).

As a side remark, let us point out that a great variety of renormalization conditions can be used. They all lead to equivalent results (in perturbation), the results given by one renormalization scheme being obtained from those given by another one through a change of the coupling constants and masses. This is one of the aspects of the R.G. invariance. However this does not mean that the β functions obtained in different schemes are equal (β is even gauge dependent).

Only when all particles have zero mass, the first two coefficients of the expansion of $\beta(g)$ are universal[30]. The same "universality" holds in the so-called minimal schemes (MS) where the renormalization is done in a way completely independent of the masses. Note that this makes the MS not directly suitable for our purpose, as it leads to mass independent β-functions, which apparently violates the decoupling theorem. In fact, in such schemes, the perturbative expansion is not valid at low energies, even after the R.G. "improvement", due to the presence of large logarithms, $\ln \frac{M_{X,Y}}{\mu}$. The problem of the large logs and that of the lack of decoupling are related to each other : if all the logs are summed up, the decoupling is restored[31].

For any pure SU(N) gauge group (and for U(1) setting N=0 in the formula below), b_N is found to be[32].

$$b_N = \frac{1}{12\pi} (11N-2\ \mathrm{Tr}(T_f^a)^2 - \mathrm{Tr}(T_H^a)^2) \quad . \tag{9.12}$$

The first term in the bracket is the gauge boson contribution in the loops. a is not summed over. The T_f^a's (T_H^a's) are the gauge group generators in the representation of the fixed chirality massless fermions (complex massless Higgs). The trace of $(T_f^a)^2$ has the same value for SU(3), SU(2) and U(1), since the T_f^a's are generators of SU(5), normalized as such. Taking SU(3) for definiteness one has

$$T^a = \frac{\lambda^a}{2} \quad , \qquad \mathrm{Tr}\,(T^a)^2 = \frac{1}{2} \quad ,$$

for each fixed chirality SU(3) triplet. There are 4 such triplets per family : $\{u_L^i\}$, $\{d_L^i\}$, $\{u_{i,L}\}$, $\{d_{i,L}\}$. So, for F SU(5) families of negligible mass fermions, the fermion contribution to b_N is

$$b_N^f = \frac{-4F}{12\pi} \quad .$$

The Higgs boson contribution is intrinsically small. We have to work in a manifestly renormalizable gauge, such as the 't Hooft, or R_ξ gauge (see section 4). In this gauge, there are two classes of Higgs bosons. A first class has very large masses, of the order of $M_{X,Y}$. This is the case first for the 12 physical Higgs bosons of the adjoint representation. It is also the case for the 12 unphysical ones because in this gauge (section 4), the unphysical bosons have masses proportional to $m_{X,Y}/\xi^{1/2}$ (we keep ξ finite,

for example the Feynman-'t Hooft gauge $\xi = 1$). Finally, the 6 coloured fields of the fundamental representation {h} have masses of order $m_{X,Y}$ because, as explained in section 8, they couple to the adjoint H. All these very massive Higgs bosons decouple and give no sizable contribution to β for $\mu \ll m_{X,Y}$. We are left with 4 real fields in the fundamental Higgs representation. One is the physical Higgs, $m \sim m_{W,Z}$, the 3 others are unphysical and have masses of order $m_{W,Z}/\xi^{1/2}$. So, only 4 out of the (24 + 10) real Higgs bosons may give contributions to the β functions(*). For $\mu \gg m_{W,Z}$, their masses can be neglected. In this limit, their contributions are given by

$$\text{(i)}\quad b = -\frac{1}{12\pi}\,\frac{3}{5}\,\left[\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right] = -\frac{1}{12\pi}\,\frac{3}{10} \qquad \text{for } U(1)$$

$$\text{(ii)}\quad b = -\frac{1}{12\pi}\,\mathrm{Tr}\left(\frac{\tau_a}{2}\right)^2 = -\frac{1}{12\pi}\,\frac{1}{2} \qquad \text{for } SU(2)$$

$$\text{(iii)}\quad b = 0 \qquad \text{for } SU(3)$$

They are indeed small as compared to those of the fermions and gauge bosons. Collecting all the contributions to the coefficients b_N, we finally obtain

$$b_N = \frac{1}{12\pi}\left[11N - 4F - \begin{Bmatrix} \frac{3}{10} \\ \frac{1}{2} \\ 0 \end{Bmatrix}\right]\begin{Bmatrix} U(1) \\ SU(2) \\ SU(3) \end{Bmatrix} \quad . \tag{9.13}$$

We note that F cancels in differences $b_N - b_{N'}$, and, neglecting the Higgs boson contributions, that the b's increase linearly with N.

We are now ready to study the evolutions with μ of the couplings. Setting $\alpha_i(\mu) = \frac{g_i^2(\mu)}{4\pi}$, the solution of the R.G. equation (9.11) for α_i is at the 1 loop order the celebrated relation

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(M)} + b_i \ln\frac{\mu^2}{M^2} \quad . \tag{9.14}$$

(*) We recall that any way the so-called unphysical Higgs bosons (together with the Fadeev-Popov ghosts), which have ξ-dependent masses, have only the effect of compensating unwanted poles at $k^2 = m^2/\xi$ in the transverse part of the gauge boson propagators. For ξ finite, the contributions of this transverse part and of the unphysical Higgs separately vanish. For ξ infinite (Landau-'t Hooft), the poles are $k^2 = 0$, the decoupling does not hold, but the two contributions cancel each other.

This result is valid in any region for μ and M , $M_W << \frac{\mu}{M} << M_X$, provided $\alpha_i(\mu)$ and $\alpha_i(M)$ are both small as compare to 1. α_3 is the usual strong interaction running coupling constant, $\alpha_2 = \frac{\alpha}{\sin^2\theta_W}$ and $\alpha_1 = \frac{5}{3}\frac{\alpha}{\cos^2\theta_W}$. Note that α , the fine structure constant, and the Weinberg angle θ_W are also "running". We have :

$$\alpha^{-1}(\mu) = \alpha_2^{-1}(\mu) + \frac{5}{3}\alpha_1^{-1}(\mu)$$

$$tg^2\theta(\mu) = \frac{3\alpha_1(\mu)}{5\alpha_2(\mu)} \quad . \tag{9.15}$$

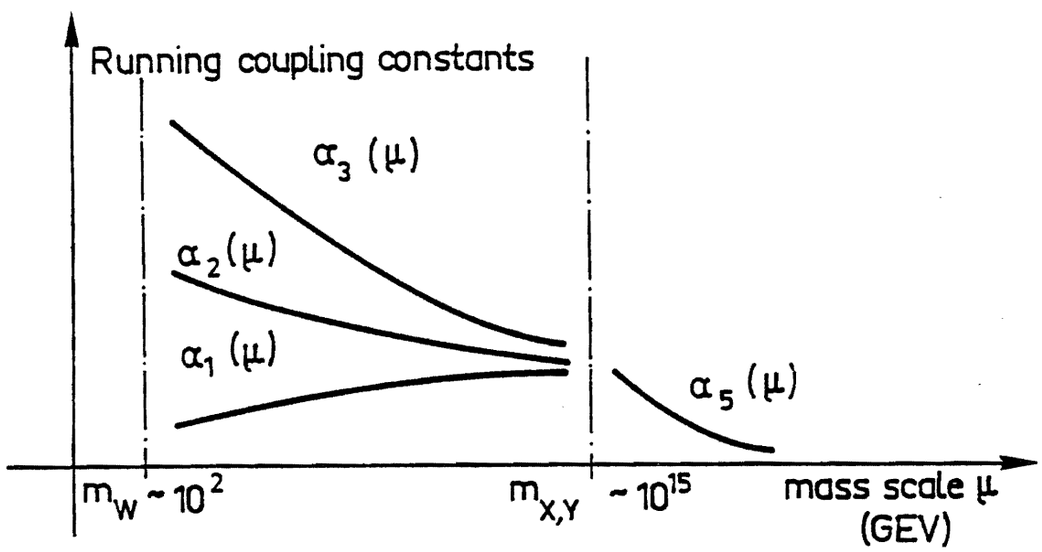


Fig. 9.1 - Sketch of the evolutions of the running coupling constants $\alpha_i(\mu)$. In between $m_{W,Z}$ and $m_{X,Y}$, the α's of SU(3), SU(2), U(1) evoluate independently. As $\mu \sim m_{X,Y}$ is approached, the X,Y boson contributions become important and tend to restore the complete symmetry. Above $m_{X,Y}$, all couplings are close to $\alpha_5(\mu)$.

The evolutions of the three couplings are sketched on Fig. 9.1. At a mass scale a little above the W and Z masses, one has $\alpha_3 > \alpha_2 > \alpha_1$. (The latter inequality is due to $tg^2\theta < \frac{3}{5}$). From $b_3 > b_2 > 0$ and $b_1 < 0$, we get that α_3 decreases faster than α_2, and that α_1 increases. These 3 coupling constants are thus in a good position to meet(*). As one deduces from the decoupling properties, the

(*) Conversely, grand unification (like SU(5)) "explains" why the strong interactions associated with a gauge group SU(3) are stronger than the e.m. interactions associated with a gauge group $U_{e.m.}(1) \subset SU(2) \otimes U(1)$.

three curves come close to each other in the region $\mu \sim m_{X,Y}$. A precise description of this region requires taking the effect of the X and Y mass carefully into account. For still larger μ's the three curves have essentially merged and there is only one rapidly decreasing ($b_5 \simeq \frac{55}{12\pi}$) SU(5) coupling constant left.

From the experimental data on two of the couplings, one can now compute the X and Y mass, and the value of the SU(5) coupling constant in the asymptotic region. This allows one to then predict the value of the third coupling constant. In other words, one can compute $\sin\theta$, say, in terms of α and the presently known (?) α_s.

As a first rough estimate,(*) one does as if the three approximate expressions (9.14) for α_1, α_2 and α_3 met at some unification point $M_{GU} \simeq M_{X,Y}$. Let α_{GU} be this common value. One has

$$\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_{GU}} + b_3 \ln \frac{\mu^2}{M_{GU}^2}$$

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha_2(\mu)} + \frac{5}{3\alpha_1(\mu)} = \frac{8}{3\alpha_{GU}} + \left(\frac{5}{3} b_1 + b_2\right) \ln \frac{\mu^2}{M_{GU}^2} \quad .$$

Thus

$$\frac{1}{\alpha(\mu)} - \frac{8}{3\alpha_s(\mu)} = \frac{1}{12\pi} \left(\frac{5}{3} b_1 + b_2 - \frac{8}{3} b_3\right) \ln \frac{\mu^2}{M_{GU}^2} \quad . \tag{9.16}$$

The (negligible mass) fermion contribution cancels out in this combination (as it has to) and we obtain the simple formula

$$\frac{1}{\alpha(\mu)} - \frac{8}{3\alpha_s(\mu)} = \frac{11}{2\pi} \ln \frac{M_{GU}^2}{\mu^2} \tag{9.17}$$

($\frac{219}{40\pi} \ln \frac{M_{GU}^2}{\mu^2}$ had we included the Higgs contribution). We will apply this formula for μ a little above the W and Z masses. The value for α_s is roughly known thanks to deep inelastic scattering experiments, whose results are compatible with

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(*) In this approximation we do not commit ourselves with any assumption about the nature of the (simple) unifying group. The only assumption made is that the breaking of the unifying group down to SU(3) $\otimes$ SU(2) $\otimes$ U(1) occurs in one step only.

$$\alpha_s(\mu) = \frac{b_3}{\ln \mu^2/\Lambda^2} = \frac{12\pi}{(33-4F)\ln \mu^2/\Lambda^2} \tag{9.18}$$

and $\Lambda \sim 500$ MeV. In this formula F is the number of families which can be actually excited at the energy scale μ. In order to compute α_s at $\mu \simeq M_W$, we thus have to take into account all quark thresholds opened below M_W. As a first approximation, one considers a new quark as infinitely heavy below the corresponding threshold and as massless above it. This means that F in formula (9.18) changes by half a unit at each threshold. If α_s is still represented by Eq. (9.18), Λ also has to change at each threshold, faking the associated non perturbative phenomena, in order to keep $\alpha_s(\mu)$ continuous.

$\alpha(\mu)$ is known on shell with an extreme precision

$$\alpha(\mu^2 = -m_e^2) = \frac{1}{137} \quad .$$

We have to extrapolate this value in the euclidean region, up to μ of the order of m_W, m_Z where we start our analysis. One finds :

$$\alpha(M_W) = \frac{1}{128} \quad .$$

This value, together with that of $\alpha_s(M_W)$ obtained from Eq. (9.18) and the relation (9.17) leads to :

$$M_{X,Y} \simeq M_{GU} \simeq 4.10^{15} \text{ GeV} \quad ,$$

and one deduces from any of the evolution equations

$$\alpha_{GU} \simeq \frac{1}{42}$$

More refined computations have been performed which treat correctly the mass effects[33]. A detailed discussion is given in Ref.[34] . The result for $M_{X,Y}$ is

$$M_{X,Y} = (6 \pm 3)\ 10^{14} \text{ GeV}$$
$$(\text{for } \Lambda = 0.5 \text{ GeV}) \ ,$$

i.e. one order of magnitude below the previous result. The error quoted is the one estimated in Ref.[34]. With the latter value of $M_{X,Y}$ and Eq. (9.15), one predicts

$$\sin^2\theta_W(M_W) = \frac{3}{8}\left(1 - \frac{5(\alpha_2(M_W)-\alpha_1(M_W))}{5\alpha_2(M_W)+3\alpha_1(M_W)}\right) = \frac{3\alpha_1(M_W)}{5\alpha_2(M_W)+3\alpha_1(M_W)}$$

$$\simeq 0.206 \pm 0.006 \quad .$$

This theoretical estimation is to be compared with the present world average experimental value[4]

$$\sin^2\theta_W = 0.229 \pm 0.014 \quad . \tag{9.19}$$

It is not clear whether or not there is a discrepancy between these two values, which by the way differ only by about 1.5 standard deviation. In particular the present experimental determinations neglect radiative corrections.

b) Fermion mass renormalization[35]

The predictions for fermion mass ratios, like $m_e = m_d$ for SSB by a 5 of Higgs bosons, are also strongly altered by radiative corrections. As a matter of fact, only fermion masses relative to some superhigh energy scale may satisfy the lowest order predictions. This is certainly not the case for the mass defined as the position (computed perturbatively) of the fermion propagator pole (whatever it means for a confined particle, especially if it is light). The scale then involved is the physical mass of the particle, which is small. On the contrary, fermion masses defined from the value of some Green function evaluated at a scale μ much larger than $m_{X,Y}$ verify the naive SU(5) predictions properties (at least for the simplest Higgs systems). Such definitions are the followings :

(i) the fermion mass is defined from the Yukawa Higgs-fermion-fermion vertex $G(p_i)$ at the point $p_i^2 = -\mu^2 >> m_{X,Y}^2$. As shown in section 8,b the fermion masses appear, at the tree level, as products of a Yukawa coupling constant G by the vacuum expectation value ω of some Higgs field ϕ, up to a Clebsch-Gordan coefficient A (or a sum of such products)

$$m = A\,\omega\,G \quad . \tag{9.20}$$

In the renormalized theory, this relation becomes

$$m(\mu^2) = A\,\omega\,(\mu^2)\,G\,(p_i^2 = -\mu^2) \quad . \tag{9.21}$$

The vacuum expectation value is evaluated from a potential renormalized at the point $\phi = \mu$, and the Yukawa vertex is evaluated at the point $p_i^2 = -\mu^2 >> m_{X,Y}^2$. The ratio of, say, the electron mass to the down quark mass is then

$$\frac{m_e(\mu^2)}{m_d(\mu^2)} = \frac{A_e}{A_d} \times \frac{G_e(\mu^2)}{G_d(\mu^2)} \tag{9.22}$$

$A_e = A_d$ in the case of SSB by a 5 of Higgs bosons. For $\mu^2 \to \infty$ the Yukawa couplings are SU(5) symmetric (exactly as the gauge couplings) so we recover the lowest order SU(5) result for the mass ratio. This would not be true however in a more general case where several Higgs multiplets give masses to the fermions[35].

(ii) the fermion mass is defined from the fermion propagator

$$S_F(p) = \frac{A(p^2)}{\not{p} - m(-p^2)} \tag{9.23}$$

by the value of the function $m(-p^2)$ at $p^2 = -\mu^2$, $\mu^2 >> m_{X,Y}^2$. This definition is useless in the involved cases above mentioned where the fermion masses do not obey the naive broken SU(5) prediction as $\mu^2 \to \infty$.

In subsequent uses of masses in perturbative computations, one sets $m = m(\mu_o)$ if the typical scale involved is of order μ_o. If we were to use $m(\mu)$, $\mu >> \mu_o$, the perturbative expansion would not be valid due to large $\log\left(\frac{\mu}{\mu_o}\right)$. In order to compare SU(5) predictions with "experimental" determination of fermion masses we have to scale them from their region of validity $\mu >> m_{X,Y}$ down to the low energy region where they are measured. Here again, the renormalization group makes the job of resumming the large logs.

The renormalization group equation for the masses is

$$\frac{\partial\, m(\mu)}{\partial \ln \mu} = m(\mu)\, \gamma_m(\alpha(\mu)) \quad , \tag{9.24}$$

where γ_m, called the mass anomalous dimension, is computed in perturbation. The solution of the RGE is

$$m(\mu) = m(\mu_o)\, e^{\int_{\ln\mu_o}^{\ln\mu} \gamma_m(\alpha(\mu))\, d(\ln\mu)} \quad ,$$

or
$$m(\mu) = m(\mu_o) \quad e^{\int_{g(\mu_o)}^{g(\mu)} \frac{\gamma_m(g')}{\beta(g')} dg'} \tag{9.25}$$

where use has been made of the RGE for g,

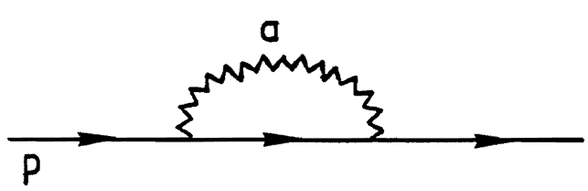
$$d(\ln(\mu)) = \frac{dg(\mu)}{\beta(g)} \quad .$$

At the 1-loop level, $\beta = -b \frac{g^3}{4\pi}$ and $\gamma_m = \frac{g^2}{4\pi} \gamma_m^o$. b and γ_m^o are constants if all the masses are neglected in the computation. The result of (9.25) is

$$m(\mu) = m(\mu_o) \; e^{-\frac{\gamma^o}{b} \int_{g(\mu_o)}^{g(\mu)} \frac{dg}{g}} = m(\mu_o) \left(\frac{\alpha(\mu)}{\alpha(\mu_o)}\right)^{-\frac{\gamma^o}{2b}} \quad . \tag{9.26}$$

If we choose to define the fermion masses from the fermion propagators, we have to compute all fermion self energies in order to obtain the γ_m's. For μ^2 larger than the X and Y masses all (fermion) γ_m's are equal due to the SU(5) symmetry being exact. The masses of, say, the electron and the down quarks evoluate with μ,but in the standard Higgs scheme their ratio remains fixed at 1. As the value of μ is decreased well below $M_{X,Y}$, the X and Y contributions go to zero and the various mass parameters follow different evolutions.

As for the coupling constant renormalization, we make an approximate treatment, neglecting all masses exept the X,Y mass, taken as infinite for μ's below $m_{X,Y}$ (in fact up to $\mu = m_{X,Y}$), and assuming that the SU(5) prediction is verified at $\mu = m_{X,Y}$. At first order we have to consider only one loop diagrams like



where a can be a gauge or a Higgs boson (Higgs bosons will be however subsequently neglected). Renormalization is achieved by imposing the renormalization condition

$$S_F(p)\Big|_{p^2=-\mu^2} = \frac{1}{\not p - m(\mu^2)} \quad .$$

At a different point, the propagator has become

$$S_F(p)\Big|_{p^2=-\mu'^2} = \frac{A(\mu'^2)}{\not p - m(\mu'^2)} \quad ,$$

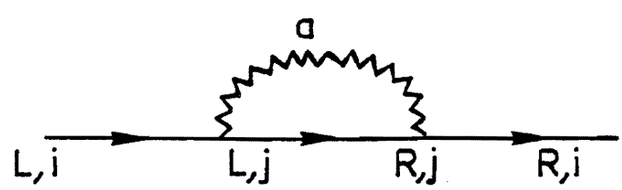
with, from the definition of γ_m , $m(\mu'^2) \approx m(\mu^2)(1-\gamma_m(\alpha(\mu^2))\ln\frac{\mu}{\mu'})$ at $\mu' \sim \mu$. The computation is simpler in the Landau gauge, where it can be shown that the fermion wave function renormalization Z_F is finite. If furthermore all masses are neglected, Z_F is a constant, fixed to one by the renormalization condition. That means $A(p^2) \equiv 1$. The computation of the diagram then gives :

$$\gamma_m^o = -\frac{3}{2\pi} \sum_{a,j} (T_R^a)_{i,j} \, (T_L^a)_{j,i} \tag{9.27}$$

The T_R^a's (T_L^a's) are the SU(3), SU(2), U(1) generators in the representation of the right (left) fermions. One sums over all possible gauge bosons. The reason for the occurrence of T_L, T_R is the following. In order to get $m(\mu')$, one computes $S_F^{-1}(p)$ from the self-energy diagram, and, due to $A(p^2) = 1$

$$m(\mu') = \frac{1}{4} \, \mathrm{Tr} \, S_F^{-1}(p)\Big|_{p^2=-\mu'^2} \quad .$$

The trace is taken on Dirac indices, and the only contribution to $m(\mu')$ comes from the case where the incoming and outgoing fermions have opposite chiralities. Since chirality is conserved at the gauge boson-fermion vertex, the transition from L to R in the fermion propagator goes as depicted on the next figure



The internal fermion propagator changes the chirality (by its mass term) and conserves the gauge index. The contrary occurs at each of the two vertices. The initial gauge index i is finally conserved since

$$\sum_{a,j} T_{ij}^a \, T_{jk}^a = \delta_{ik} \times \mathrm{cst} \quad .$$

Neglecting the large mass X,Y gauge boson contributions, we only have to

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(*) The γ_m's are gauge dependent. However this dependence finally cancels in mass ratios (e.g. $\frac{m_e(\mu)}{m_d(\mu)}$).

consider those of the SU(3), SU(2) and U(1) gauge bosons. The results are the followings.

i) SU(3) contributes to the anomalous dimension only for quarks because the leptons are colourless. One finds :

$$\gamma_o^{(3)} = -\frac{3}{2\pi}\frac{1}{N}\sum_a \mathrm{Tr}\,(T^aT^a) = -\frac{3}{2\pi}\frac{N^2-1}{2N}$$

$$= -\frac{2}{\pi} \qquad (N=3) \tag{9.28}$$

ii) SU(2) gives no contribution as $(T_R)^{SU(2)} = 0$.

iii) U(1) contributes by

$$\gamma_o^{(1)} = -\frac{3}{2\pi}\times\left(\frac{3}{5}\right)\times\frac{Y_R}{2}\times\frac{Y_L}{2}$$

$$\left.\begin{aligned} &= -\frac{1}{10\pi} && \text{for charge } \tfrac{2}{3} \text{ quarks} \\ &= +\frac{1}{20\pi} && \text{for charge } -\tfrac{1}{3} \text{ quarks} \\ &= -\frac{9}{20\pi} && \text{for charged leptons} \end{aligned}\right\} \tag{9.29}$$

Using the above results for γ_m^o , the values of b_N (Eq. (9.13)) and the solution (Eq. (9.26)) of the RGE for m the SU(5) predictions for fermion mass ratios thus get modified into (neglecting the Higgs contributions to the γ functions)

$$\frac{m_d(\mu)}{m_e(\mu)} = \frac{m_s(\mu)}{m_\mu(\mu)} = \frac{m_b(\mu)}{m_\tau(\mu)} = \left(\frac{\alpha_s(\mu)}{\alpha_{GU}}\right)^{\frac{12}{33-4F}} \left(\frac{\alpha_1(\mu)}{\alpha_{GU}}\right)^{\frac{3}{4F}} . \tag{9.30}$$

This result is rather sensitive to the number F of SU(5) families. For Λ=0.5 GeV, one obtains

$$\frac{m_d(10\ \mathrm{GeV})}{m_e(10\ \mathrm{GeV})} = \begin{cases} 3.6 & \text{for} \quad F = 3 \\ 5.3 & \phantom{\text{for}} \quad = 4 \\ 8.5 & \phantom{\text{for}} \quad = 5 \end{cases} . \tag{9.31}$$

The present experimental situation implies that F is at least equal to 3 (though the t quark has not been discovered yet). In this case, one obtains at μ = 10 GeV, from the known values of the e,μ,τ masses :

$$\begin{cases} m_b \simeq 6 \text{ GeV} \\ m_s \simeq 0.4 \text{ GeV} \\ m_d \sim 1.6 \text{ MeV} \end{cases}$$

This prediction, which was made[24] before the discovery of the γ , is in good agreement with what is thought to be the bottom quark mass. Conversely, if m_b is correctly evaluated from the γ data to be around 5 or 6 GeV, the SU(5) prediction indicates that there are only three SU(5) families. The top quark, then, would be the last quark to be ever discovered.

The modifications due to RGE of the exact SU(5) predictions for strange and down quark masses clearly go in the right direction. Quantitative analysis is however extremely hard to perform, due to the lack of an operative definition of what a light quark mass is.

10. THE PROTON LIFETIME

In this short section we just gather the various ideas and formulae, already presented in different parts of these lectures, which are relevant to the problem of nucleon stability. Quantitative results on the proton lifetime within SU(5) models will be reported, but we also have to point out that many of the properties exhibited by SU(5) are not specific to it and belong to most grand unified models.

Nucleon decay may occur only if the baryon number B is not conserved (disregarding neutron β-decay which does not occur in most nuclei), leading to mesons and leptons. At the end of section 7, we noted that B (and L) conservation is unavoidably violated in models where quarks and leptons are in the same irreducible representation. Such is the case in SU(5) where the right handed d-quarks, e^+, ν^c are in the rep.5, the left-handed u, d-quarks, e^+ in the representation 10, and their charge conjugates in the corresponding conjugate representations (with similar assignments for the other families). Whenever such a situation is realized, there are gauge bosons coupled to currents build from a quark and a lepton. In SU(5), these gauge bosons are the X and Y bosons, and the general couplings of Eqs. (7.11) and (7.12) led us to the explicit formula (7.18) which exhibits all the B,L violating interactions induced by gauge invariance. Besides the currents of the form $\bar{q}\,\gamma_\mu\,\ell$ which mix quarks and leptons, Eq. (7.18) also contains terms like $\overline{q^c}\gamma_\mu q'$, involving a quark and an antiquark, which allow for qq' or $q^c q'^c$ annihilation. Again this property is not specific to SU(5), but rather belong to any model in which at least some of the quarks *and* antiquarks are classified in the same irreducible representation (like the 10 of SU(5)). Therefore, in models where *both currents* $\bar{q}\,\gamma_\mu\,\ell$ and $\overline{q^c}\gamma_\mu q'$ are coupled to at least one of the gauge bosons G, there is the possibility of $\Delta B = 1$ processes generated by the elementary interaction

$$qq' \xrightarrow{G} q''^c \ell \qquad . \tag{10.1}$$

At present energies, the relevant symmetries are SU(3), SU(2), U(1) which all conserve separately B and L. However, if these symmetries are the symmetries remaining after SSB of a larger symmetry group g, then the process (10.1) may become possible, although its rate is considerably lowered by symmetry breaking. The situation is exactly the same as in the Salam-Weinberg model which leads to an effective 4 Fermion-interaction, weak at low energies as its strength is measured by G_F,

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\, m_W^2} \quad . \tag{10.2}$$

In SU(5), the strength of the B,L violating interactions is

$$\frac{G_{B,L}}{\sqrt{2}} = \frac{g^2}{8\, m_{X,Y}^2} \quad , \tag{10.3}$$

and, as we have seen in section 9, $m_{X,Y}$ is of order 10^{14} GeV. So, hopefully, the corresponding rate is extremely small.

In the broken situation, there is in general other interactions leading to B,L violation, which are mediated by some of the Higgs bosons which couple to fermions. In SU(5), broken by an adjoint and a fundamental representation of Higgs bosons, only the latter couple to fermions (section 8b). We recall that in the standard model, among the 10 real Higgs fields {h}, 3 are "eaten" to give masses to the $W^{\pm}$, Z_o bosons, and 1 is the low-mass physical Salam-Weinberg boson which does not lead to B,L violating processes (it is colourless). The 6 other ones (colour triplets, SU(2) singlets) induce B,L violation via the interaction described by Eq. (8.18) for one generation, and Eq. (8.24) for the general case of any number of generations. The effective strength of the four fermion interaction mediated by the Yukawa coupling G (or $\tilde{G}$) is

$$G^h_{B,L} \sim \frac{G^2}{m_h^2} \quad .$$

The coupling G is related to the fermion masses (Eqs. (8.20), (8.21)), so that one has

$$G \sim m_f/\omega \simeq \frac{g\, m_f}{m_W}$$

where m_f is a typical fermion mass and ω the vacuum expectation value of the h Higgs field. On the other hand, the mass m_h of the 6 Higgs bosons under consideration, generated by the couplings to the Higgs H of the adjoint is of order $m_{X,Y}$ ($g\, m_{X,Y}$ if the h-H coupling is generated by radiative corrections[24]). Therefore one has

$$G^h_{B,L} \sim \frac{m_f^2}{m_W^2} \times G_{B,L}\, \frac{1}{\rho} \quad , \tag{10.4}$$

with ρ at most of order 1 and presumably of order g^2. Higgs contributions to nucleon decay are found to be small, especially because it involves only light fermions.

In order to compute the proton and neutron decay rates in SU(5), various steps have to be achieved. First one has to take into account generation mixing. The X,Y interactions of Eq. (7.18) are replaced by the general one of Eq. (8.31) in section 8c. As argued there, the mixing matrix $\tilde{U}$ is phenomenologically close to unity and S, a diagonal unitary matrix, has no visible effect at lowest order. There is no coupling dd-gauge boson, so that in a proton or a neutron, one may have uu or ud annihilations only. There are two kinds of elementary graphs to be considered (Fig. 10.1).

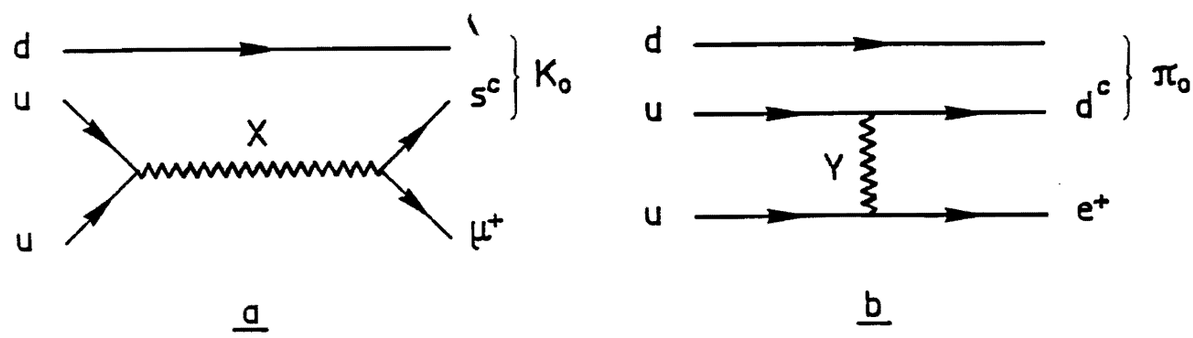


Fig. 10.1 - a) an example of annihilation graph. Even for $\tilde{U} = 1$, one may get second generation fermions in the final state.

b) exchange graph. In the limit $\tilde{U} = 1$, only first generation products are allowed.

Graphs of type a (annihilation) allow a change of generation, leading either to a strange quark accompanied by μ^+ or ν_μ, or to fermions of the first generation. For the exchange graph of Fig.10b, transitions from one generation to the other is Cabibbo suppressed. Once all graphs of type a or b have been listed, one has to compute the corresponding matrix element, which is easy from Eq. (8.31). One then uses some wave function in order to describe the nucleon in terms of the u and d quarks. Since, due to the high value of $m_{X,Y}$, the interaction is a contact interaction, the decay rate is proportional to the wave function at the origin squared $|\psi_N(0)|^2$. Finally, one needs some interpretation of the final states containing quarks and

leptons. One may compute the inclusive decay rate by summing over all possible qq'^c final states for a given outgoing lepton. This gives the total nucleon width for each lepton type and the inverse life time. From the above considerations, one expects the life time to be of the form

$$\tau \sim \frac{(m_{X,Y})^4}{g^4 |\psi(0)|^2 \, m_N^2} \tag{10.5}$$

up to Clebsch-Gordan coefficients given by the interaction Lagrangian (8.31), and numerical constants coming from phase space.

It is clear from Eq. (10.5) that the most important piece for computing τ is the value of $m_{X,Y}$. We refer to section 9 for a discussion of what the problems are, and to Refs.[33,34] for a complete study of the uncertainties involved. The value of g^4 is $(4\pi\,\alpha_{GU})^2$ at a scale of order $m_{X,Y}$, by definition of the unification point ($\alpha_{GU} \sim \frac{1}{40}$). It has to be further renormalized by radiative corrections and proper use of the renormalization group[24]. The presently admitted result found for τ_p is [34]

$$\tau_p = 8 \times 10^{32\pm2} \text{ years}$$

in the standard SU(5) model.

Since it happens that the range in between 10^{30} (the present experimental lower bound) and $\sim 10^{33}$ years seems to be experimentally accessible, the last question of interest is that of the branching ratios for the various allowed decay modes. This subject has been recently studied by many people[37,33]. It is generally agreed that the most favoured channel is $\pi_o e^+$ as it contains the lightest particles and because the corresponding exchange graph is Cabibbo allowed. There is more variety in the predicted branching ratios for other experimentally interesting channels such as $e^+\rho^o$, $e^+\eta$, $e^+\omega$. The channels involving neutrinos are generally found to be suppressed.

Among the properties found in SU(5) for $\Delta B = 1$ processes, all of which have not been quoted here, some are more general. For example, one may show by inspection of general effective Lagrangians[38] violating B and L conservation that B-L conservation should hold at a very high degree of accuracy, that $\Delta S = \Delta B$ transitions are forbidden, or that relations due to the weak isospin structure of the Lagrangians exist between some inclusive rates involving given chirality

leptons. On another hand, it must be possible to distinguish between various unified models by more specific experimental investigations, such as the ν_e/e^+ ratio or μ^+ polarization in nucleon decays. Also it has been noted that, although the simplest Higgs mechanism leads to suppressions of channels like $p \to k^o e^+$ or $\pi^o \mu^+$ (Cabibbo suppression), it would not be so in other schemes involving at least 2 Higgs multiplets. So the measurement of ratios of "Cabibbo forbidden" to "Cabibbo allowed" decay rates may help in distinghishing between various possibilities.[39] This is especially worthwhile as the Higgs pattern is intimately connected to the very important problems of fermion masses and flavour mixings.

We do not want to try and give any general conclusion about the topics covered in these lectures. Let us just emphasize the central role of proton decay in all grand unified theories based on SSB of simple Lie group symmetries like SU(5), O(10) and E_6. Alternatives to the theories based on spontaneous symmetry breaking have been proposed. In the so-called *dynamical* symmetry breaking schemes, a new mass scale in the range 1 to 100 TeV should show up, and specific predictions about a new, rich, spectroscopy below 1 TeV and about CP violation have been made[40].

Finally we recall that it may be that only when gravity is included a new fundamental insight into particle physics will be obtained[41].

- APPENDIX A -

YANG-MILLS AND GROUP THEORY

This appendix intends to remind the reader with a few results and useful concepts in gauge field[11] and Lie group theory[20]. Suppose we have a field theory with fermions (and scalars). The Lagrangian L is taken to be invariant under some compact(*) Lie group of transformations G. The various fields belong to (generally reducible) representations U of G. U may be chosen unitary (any continuous, finite dimension, representation of a compact Lie group is equivalent to a unitary one). L is thus invariant under the field transformation

$$\psi(x) \to U_g \, \psi(x)$$

with

$$U_g^{\dagger} = U_g^{-1} = U_{g^{-1}} \quad . \tag{A.1}$$

For the time being, U is independent of the point x. One says that L is invariant under a *global* transformation. In the vicinity of the identity e, any element g of G can be written as

$$g \simeq e - i \, \delta\alpha_a \, t^a \quad . \tag{A.2}$$

The $\delta\alpha_a$'s are the infinitesimal parameters of the group, the t^a's its generators. By real linear combinations of the generators, one generates the Algebra $\mathcal{a}$ for the group. The inner product of two elements of $\mathcal{a}$ is -i times their commutator. The structure constants of $\mathcal{a}$ are defined by the relation

$$[t^a, t^b] = i \, f^{ab}{}_c \, t^c \quad . \tag{A.3}$$

The structure constants are real. The appearance of a factor i in Eq. (A.3) comes from the choice made in (A.2), which is frequent in Physics. Were the generators be multiplied by -i, there would be no i at all in the algebraic manipulations, but they would appear in the relations between observables and generators.

(*) A set M is compact if any infinite subset of M contains a sequence which converges to an element of M. For example, any closed region of finite extension of $\mathbb{R}^n$ is compact. A Lie group is compact if its parameters have finite domains of variation.

To each representation of the group it corresponds a representation of its algebra. Let T^a be the matrix representing t^a in the space of representation which the vector ψ belongs to. The infinitesimal transformation of Ψ reads

$$\psi_j \to \psi_j - i \sum_a \delta\alpha_a \, T^a_{jk} \, \Psi_k$$

or in matrix notation

$$\psi \to \psi - i \, \delta\alpha_a \, T^a \, \Psi \qquad . \tag{A.4}$$

Here $T = T^\dagger$ and since T is a representation of $\mathcal{a}$

$$[T^a, T^b] = i f^{ab}{}_c \, T^c \qquad .$$

An ideal I is an invariant sub-algebra of $\mathcal{a}$: for all elements t of $\mathcal{a}$,

$$[I,t] \subset i \, I \qquad .$$

In what follows, we restrict ourselves to the so-called *simple Lie algebras*, which by definition have no proper (i.e. different from {0} and from $\mathcal{a}$ itself) ideal. A wider class is that of the semi-simple algebras, which have no *abelian* ideal except {0}. An abelian algebra is an algebra all the elements of which commute. A group is simple (semi-simple) if its algebra is simple (semi-simple). By convention, U(1) and its corresponding one dimensional algebra is not simple.

Gauge fields are introduced to insure invariance under group transformations which now depend on the point x where they are performed :

$$\psi(x) \to U_g(x) \, \psi(x) \qquad .$$

The global symmetry is enlarged to a *local* symmetry. Under local transformations, the kinetic part of the free fermion (or scalar) lagrangian namely $i\bar{\psi}\partial\!\!\!/\Psi$ (or $(\partial_\mu\varphi)(\partial^\mu\varphi)^\dagger$), is no more invariant. The invariance is restored if the derivative ∂_μ is replaced by the *covariant derivative*

$$\partial^\mu \to D^\mu = \partial^\mu + i \, A^\mu_a(x) \, T^a \qquad . \tag{A.5}$$

The $A^\mu_a(x)$'s are the gauge fields. Their transformation properties are designed

to compensate the terms generated by ∂^μ acting on the x-dependent transformation. If one considers the gauge field as a member A^μ of the representation of a, defining

$$A^\mu \equiv \sum_a A^\mu_a \; T^a \qquad , \tag{A.6}$$

A^μ transforms according to

$$A^\mu(x) \overset{U}{\rightarrow} A^{\mu U}(x) = U(x)A^\mu(x)U^{-1}(x) - iU(x)(\partial^\mu U^{-1}(x)) \qquad . \tag{A.7}$$

Then one verifies that

$$D^\mu(A) \rightarrow D^\mu(A^U) = U \; D^\mu(A) \; U^{-1} \qquad ,$$

which insures that $\bar{\psi} \not{D} \Psi$ is actually invariant. Intrinsically, one defines the gauge field as a member of the algebra itself by

$$\underset{\sim}{A}^\mu(x) = A^\mu_a(x) \; t^a \qquad ,$$

so that $A^\mu(x)$ is a particular representation of $\underset{\sim}{A}^\mu(x)$. From (A.7), we obtain that under an infinitesimal transformation, A^μ transforms according to

$$A^\mu(x) \rightarrow A^\mu(x) - i \; \delta\alpha_a [T^a, A^\mu(x)] + \partial^\mu(\delta\alpha_a(x)) \; T^a \qquad . \tag{A.8}$$

For $\delta\alpha_a$ constant, the last term drops out, and the transformation of A^μ is that of *the adjoint representation* of the algebra.

With each element t^a of the algebra, the adjoint representation associates the linear transformation of t^b

$$t^b \rightarrow -i[t^a, t^b] \qquad .$$

The adjoint representation of a simple algebra is irreducible (there is no subspace of the representation space which is invariant under all group transformations). Note that the matrix T^a associated with t^a in the adjoint representation verifies by definition

$$t^b \rightarrow (T^a)^b{}_c \; t^c = -i[t^a, t^b]$$

Hence $(T^a)^b{}_c = f^{ab}{}_c$.

We finally have to add the pure gauge field term in the Lagrangian. To our disposal, we have the gauge invariant quantity $Tr(F^{\mu\nu}F_{\mu\nu})$, where

$$F^{\mu\nu} = -i[D^\mu,D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu - i[A^\mu,A^\nu] \quad .$$

A mass term $Tr(A_\mu A^\mu)$ for the gauge fields would spoil the gauge invariance. In taking traces, we have to specify which matrix representation A^μ of $\underset{\sim}{A}^\mu$ we choose. A usual choice is

$$A^\mu = A^\mu_a T^a \quad ,$$

T^a being the matrix representing t^a in the adjoint representation. This particular choice has no consequence since, for any simple group, if in the adjoint representation we have

$$Tr_{adj}\,(T^a\,T^b) = g^{ab} \quad ,$$

then in any irreducible representation R

$$Tr_R\,(T^a\,T^b) = C_2(R)\,g^{ab} \quad ,$$

with $C_2(R) > 0$. The only exception is the trivial representation which associates 0 to all elements of $\mathcal{a}$. Since the gauge group is supposed to be compact and semi-simple, one can always find a basis for the algebra for which the matrix g is unity.

$$Tr_{adj}(T^a\,T^b) = \delta^{ab} \quad .$$

One can show that in this basis the structure constants are totally antisymmetric.

The complete Lagrangian (ommitting scalars) finally reads :

$$L = -\frac{1}{4g^2}\,F^{\mu\nu}_a\,F_{\mu\nu,a} + i\bar{\psi}(\not{\partial}+i\not{A})\psi - \bar{\psi}\,M\,\Psi \quad ,$$

where g is the coupling constant and M the fermion mass matrix. When the algebra is simple, one cannot separate it into pieces transforming independently. Hence there is only one coupling constant when we have exact gauge symmetry under a simple group.

The above results can be generalized to groups which are (locally) direct products of simple groups and (eventually) of U(1) groups. This is in fact the most general case : we want the form $Tr(F^{\mu\nu} F_{\mu\nu})$ to be non degenerate (all gauge fields present in the kinetic part), and positive definite (all kinetic terms of the same sign). This requirement implies that the gauge group is a direct product of a semi-simple group and of an abelian group. The algebra is the direct sum of the corresponding algebras (which commute between themselves). Since any semi-simple Lie Algebra is a direct sum of simple algebras, the algebra of a general gauge group is a direct sum of simple algebras and eventually of U(1) algebras(*). The most general Lagrangian to be considered thus is :

$$L = \sum_k \left[- \frac{1}{4g_k^2} \left(\sum_{a_k} F^{\mu\nu}_{a_k} F_{\mu\nu a_k} \right) \right] +$$

$$+ i \bar{\psi} (\not\partial + i \sum_k \not{A}_k) \Psi - \bar{\psi} M \Psi \quad .$$

There are as many coupling constants as there are factor groups. Under the field rescaling $A_k \to g_k A_k$, L gets the more familiar form (now including eventual complex scalars) :

$$L = \sum_k - \frac{1}{4} \sum_{a_k} F^{\mu\nu}_{a_k} F_{\mu\nu, a_k} + i \bar{\psi}(\not\partial + i \sum_k g_k \not{A}_k)\Psi$$

$$- \bar{\psi} M \psi + \left| (\partial^\mu + i \sum_k g_k A^\mu_k) \Phi \right|^2 - \Phi^\dagger M^2 \Phi \quad . \qquad \text{(A.9)}$$

As shown above, if we restrict ourselves to simple Lie groups (such as SU(5)), the theory depends on one coupling constant only, and it is unified in this sense. The charges of the various particles are moreover seriously constrained. For example, if some representation appears several times (several families), each generator has the same representative for all these replications, and in particular the charge operators are the same. Finally the charge operators in different irreducible representations are also related.

(*) In the U(1) case, the algebra and all its irreducible representations are isomorphic to $\mathbb{R}$ (except the trivial representation by 0). Since all commutators vanish, the adjoint representation is the trivial one and cannot be used to define $Tr(F^{\mu\nu}F_{\mu\nu})$. In any representation, $\underset{\sim}{A}^\mu$ is represented by $A^\mu(x)Y/2$, where Y is the U(1) hypercharge (Y=0 in the adjoint). The Lagrangian is

$$\begin{cases} - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi} (\not\partial + i \not{A} Y/2)\psi \\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad . \end{cases}$$

A further important point is that moreover the charges are commensurate. It seems a priori very easy to build a candidate for the charge which has non commensurate eigenvalues : such are the eigenvalues of the SU(3) generator $\lambda_3+\lambda_8$ in the fundamental representation :

$$\lambda_3+\lambda_8 = \begin{pmatrix} 1+\frac{1}{\sqrt{3}} & & \\ & -1+\frac{1}{\sqrt{3}} & \\ & & -\frac{2}{\sqrt{3}} \end{pmatrix} .$$

Let us explain why the charge operator Q cannot be of this kind if the theory is based on a compact Lie group g. We consider a simplified example in which the group generator associated with Q is, say, the one parameter group

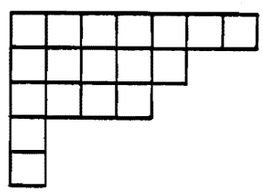
$$G = e^{-i\alpha Q} = e^{-i\alpha\begin{pmatrix}1 & 0\\ 0 & e\end{pmatrix}} \tag{A.10}$$

G has two non commensurate eigenvalues 1 and e. It is non compact since α varies from $-\infty$ to $+\infty$ (as opposed to the case $Q = \begin{pmatrix}p & 0\\ 0 & q\end{pmatrix}$, p,q integers, where the maximum length of the variation domain is $2\pi\, p\, q$, finite). Moreover, because 1 and e are non commensurate, any point of the square $x \in [0,2\pi]$, $y \in [0,2\pi]$ can be arbitrarily closely approached by a point $x_o = \alpha(\text{mod}.2\pi)$, $y_o = e\alpha\ (\text{mod}.2\pi)$. One says that the set $\{x_o, y_o\}$ is dense in the square$[0,2\pi]^2$. Accordingly, any transformation of the 2 parameter group $[U(1)]^2$ can be approached by an infinite sequence of elements of the 1 parameter group G. But if G is a subgroup of the unifying group g, the above sequence has a limit which belongs to g because g is compact (definition of compactness - see first note of this section). Hence in our example, g should contain two unbroken U(1) generators, that is two say two photons. This situation is neither that of the real world, nor the one we describe by the successive breakings of SU(5) : $U_{e.m.}(1)$ is the only abelian component left unbroken. Note that compactness of g was essential for the argument : charge is not quantized in the Salam Weinberg model because SU(2) $\otimes$ U(1) is not compact.

- APPENDIX B -

YOUNG TABLEAUX FOR SU(N)

All irreducible representations of SU(N) [42] can be obtained as properly symmetrized powers of the fundamental representation. This is why the irreducible representations of the permutation group of N objects, and hence Young Tableaux are relevant to the study of SU(N) representations. A Young tableau is composed of boxes, each box corresponding to one fundamental representation Ψ_α. These boxes are arranged in rows in non increasing length order.



A Young tableau

For SU(N), there is no tableau containing columns with more than N boxes. The tableaux made of only N box columns represent a singlet. Any N box column in a tableau can be ommitted, unless nothing is left, in which case the tableau represents a singlet (we call it a unit tableau).

The conjugate representation q^α of the SU(N) fundamental q_α is built from the totally antisymmetric product of N-1 fundamentals $q^{(1)}, q^{(2)}, \ldots q^{(N-1)}$:

$$q^\alpha \sim \varepsilon^{\alpha\beta_1\ldots\beta_{N-1}} q^{(1)}_{\beta_1} q^{(2)}_{\beta_2} \ldots q^{(N-1)}_{\beta_{N-1}} \quad .$$

It is represented by a N-1 box column. More generally, each Young tableau corresponds to a given symmetry pattern with respect to box permutations (in fact to one irreducible representation of the permutation group), obtained as follows :

- one first symmetrizes each row
- one then antisymmetrizes each column (this destroys row symmetry, unless all rows have the same length).

Examples

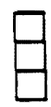
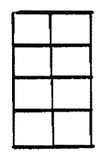
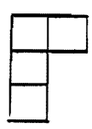
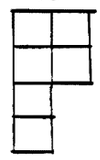
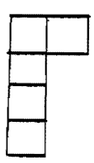
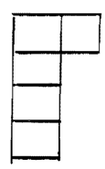
means ψ_α

means $\psi_\alpha\varphi_\beta + \psi_\beta\,\varphi_\alpha$

means $(\psi_\alpha\varphi_\beta + \psi_\beta\varphi_\alpha)\,\varphi_\gamma - (\psi_\gamma\varphi_\beta + \psi_\beta\varphi_\gamma)\,\varphi_\alpha$

The Young tableau $t(\bar{R})$ for the conjugate representation $\bar{R}$ of some representation R is obtained from the complement to a unit tableau of t(R).

<u>*Examples for SU(5)*</u> (unit t = any number of 5 box columns)

t(R)	$t(\bar{R})$
	
R=10	$\bar{R}=\overline{10}$
	
15	$\overline{15}$
	
45	$\overline{45}$
	
24	$\overline{24}=24$

When necessary (last two examples), we have to rotate upside down the tableau obtained by complement of t(R) to a unit tableau, in order to obtain a proper Young tableau (rows in non increasing length order). The representations are labelled by their dimensions in the above examples. The dimension of a representation of SU(N) given by its Young tableau can actually be computed directly from the tableau, as explained now. A simple recipe is :

(i) make the product of the factors indicated below for each box of the considered tableau

N	N+1	N+2	N+3	....
N−1	N	N+1	N+2	....
N−2	N−1	N	N+1	....
⋮	⋮	⋮	⋮	

(ii) divide the result obtained in (i) by the product on all boxes of factors r_j. For the box j, $r_j = \{1 +$ the number of boxes to the right of box j + the number of boxes below it$\}$.

Examples

$$\frac{\begin{array}{|c|c|}\hline N & N+1 \\ \hline \end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}} = \frac{N(N+1)}{2}$$

$$\frac{\begin{array}{|c|c|}\hline N & N+1 \\ \hline \end{array}\ \begin{array}{|c|}\hline N-1 \\ \hline \end{array}}{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline \end{array}\ \begin{array}{|c|}\hline 1 \\ \hline \end{array}} = \frac{(N-1)N(N+1)}{3}$$

$$\frac{\begin{array}{|c|c|}\hline N & N+1 \\ \hline \end{array}\ \begin{array}{|c|c|}\hline N-1 & N \\ \hline \end{array}}{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array}\ \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}} = \frac{(N-1)N^2(N+1)}{12}$$

The nature and multiplicities of the irreducible representations entering the decomposition of the product $R_1 \otimes R_2$ of two irreducible representations can also be found from Young tableau considerations. For each operation we give the example of the product

$$P = \begin{array}{|c|}\hline \ \ \\ \hline \ \ \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \ \ \\ \hline \ \ \\ \hline \end{array}$$

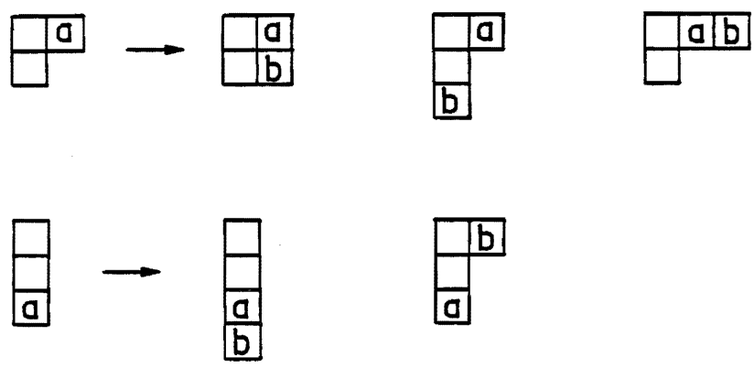
(i) One first labels each row of the R_2 tableau : all boxes of the first row get label a, those of the second one get label b, etc...

$$P = \begin{array}{|c|}\hline \ \ \\ \hline \ \ \\ \hline \end{array} \otimes \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}$$

(ii) One enlarges the Young tableau of R_1 in all possible ways by adding one box labelled a to R_1 on the right or below each of its own boxes

(iii) One excludes the tableaux which are not Young tableaux (rows not in non increasing order or columns with more than N boxes). In the example, one throws the last tableau away.

(iv) One starts again by adding the b boxes to the allowed tableaux, again excluding illegal tableaux



The other possible additions of b do not lead to Young tableaux.

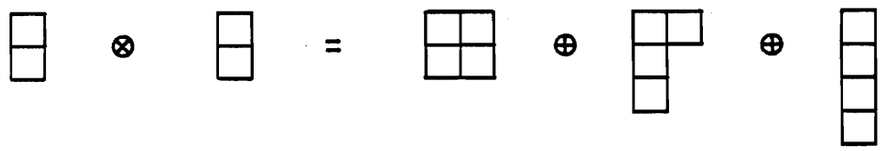
One goes on with boxes c etc..., and stops when all the boxes of R_2 have been used.

(v) Among the tableaux which are left, one further reject those
- which have several a's, or b's, ... in the same column (no such case in the example)
- which are not a "lattice permutation". A tableau is a lattice permutation under the following condition. One starts counting the numbers N(a), N(b),... of boxes a,b,... occurring in the tableaus obtained, from *right to left* and from *up to down*. At each stage of the counting these numbers must be found in non increasing order :

$$N(a) \geq N(b) \geq \ \ldots$$

In our example, this rule leads us to reject the 3rd and 5th tableaux obtained

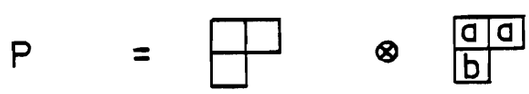
after the step (iv). The result for the product of two antisymmetric tensor representations of SU(N) is thus



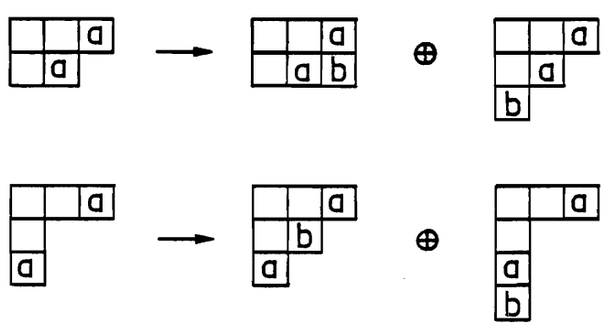
The equality of the dimensions on both sides of the equation can easily be checked. The first non trivial application is to SU(3) (for SU(2) = unit tableau), in which case the last tableau does not exist (4 box column) and the second one can be reduced to (a 3 box column can be ommitted). The result is then to be read : $\bar{3} \otimes \bar{3} = \bar{6} + 3$.

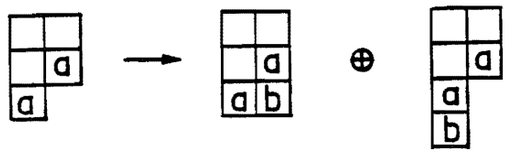
Once all the steps from (i) to (v) have been made, one has all the irreducible representations of SU(N) contained in $R_1 \otimes R_2$, with furthermore their right multiplicities, obtained as follows. A given representation occurring in the decomposition has a multiplicity equal to the number of times its Young tableau appears with not all labels a,b,... at the same place. If a tableau appears several times with all the labels at the same place, then it is counted only once.

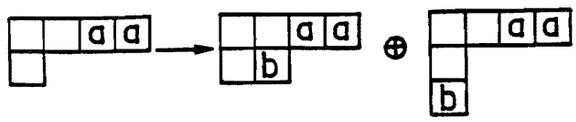
We end up with a second example, where only the allowed tableaux are kept. We compute

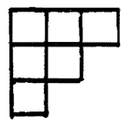


We first add the a boxes, and obtain 4 distinguishable tableaux to which we add the b box







We see that the representation  appears twice, all others appear only once. Application to SU(3), for which is the adjoint representation, of dimension 8, allows one to recover the well known result

$$8 \otimes 8 = 1 + 8 + 8 + 10 + \overline{10} + 27 \quad .$$

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