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QUANTUM CLIFFORD ALGEBRA AND GENERATORS OF QUANTUM LORENTZ GROUP [1]

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Abstract

The quantum deformation of the classical Clifford algebra is constructed from the quantum Pauli matrices, which follows a tensor analysis of quantum $SU_q(2)$ group in a similar way as the ordinary Pauli matrices in $SU(2)$ analysis. From this the matrix form of the generators of quantum Lorentz group can be obtained directly. In the limit $q \longrightarrow 1$, they reduce to the classical Clifford algebra and ordinary generators of Lorentz group.

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1. Introduction

Motivated by the idea of quantum group theory[1], we investigate the q-deformation of the Clifford algebra from which the corresponding quantum orthogonal group generators in the spinor representation can be obtained immediately. After collecting the main results of classical Clifford algebra in this section we build, by a construction method, the quantum Pauli matrices and Dirac matrices in Sec.2 and Sec.3 respectively. The generators of the corresponding quantum group, $SO_{q^2}(3)$ and $SO_q(1,3)$, are obtained by the q-anti-symmetric part of the bilinear combination of Clifford matrices. All the classical relations have their q-deformed counterparts.

It is well known that a Clifford algebra is a set of matrices $\{\Gamma^A, A = 1, 2, \cdots, N\}$, satisfying the anti-commutation relations

$$\{\Gamma^A,\ \Gamma^B\} = 2\eta^{AB} \tag{1.1}$$

with η^{AB} being the metric in the N-dim space. Given such a set of matrices, we can define

$$M^{AB} = \frac{i}{4}[\Gamma^A,\ \Gamma^B] \tag{1.2}$$

Then it can be easily shown that

$$[M^{AB},\ \Gamma^C] = i(\Gamma^A\eta^{BC} - \eta^{AC}\Gamma^B) \tag{1.3}$$

and

$$[M^{AB},\ M^{CD}] = i(M^{AD}\eta^{BC} - M^{AC}\eta^{BD} + \eta^{AD}M^{BC} - \eta^{AC}M^{BD}) \tag{1.4}$$

Eq.(1.4) implies that M^{AB} are the generators of $SO(N)$ (or $SO(p,q)$ when η^{AB} has signature (p,q)) in the spinor representation whereas Eq.(1.3) shows that Γ^C transform as the N-dim vector representation of $SO(N)$.

For the simplest example, $N = 3$, we have three 2×2 Pauli matrices

$$\Gamma^i = \sigma^i,\ \ i = 1, 2, 3$$

with

$$\{\sigma^i,\ \sigma^j\} = 2\delta^{ij},\ [\sigma^i,\ \sigma^j] = 2i\epsilon^{ijk}\sigma^k\ . \tag{1.5}$$

Introducing

$$M^{ij} \equiv \frac{1}{4i}[\sigma^i,\ \sigma^j] = \frac{1}{2}\epsilon^{ijk}\sigma^k \tag{1.6}$$

then

$$[M^{ij},\ \sigma^l] = \frac{1}{2}\epsilon^{ijk}[\sigma^k,\ \sigma^l] = i(\delta^{il}\sigma^j - \sigma^i\delta^{jl}) \tag{1.7}$$

And the three generators $\{\frac{\sigma^k}{2}\}$ of $SO(3)$ in the 2-dim spinor space satisfy the following commutation relations

$$[\frac{1}{2}\sigma^i,\ \frac{1}{2}\sigma^j] = i\epsilon^{ijk}\frac{1}{2}\sigma^k\ . \tag{1.8}$$

For the next example, $N = 4$, we have four 4×4 Dirac matrices

$$\Gamma^\mu = \gamma^\mu,\ \mu = 0,1,2,3$$

$$\{\gamma^\mu,\ \gamma^\nu\} = 2\eta^{\mu\nu}\ ,\ \eta^{\mu\nu} = diag(1,-1,-1,-1) \tag{1.9}$$

Then

$$M^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\ \gamma^\nu] = \frac{1}{2}\Sigma^{\mu\nu} \tag{1.10}$$

are generators of Lorentz group $SO(1,3)$ in the spinor representation. Besides the above four γ's, we can introduce the fifth

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.11}$$

with the properties

$$\gamma_5^2 = 1,\ \{\gamma_5,\ \gamma^\mu\} = 0\ ,\ [\gamma_5,\ \Sigma^{\mu\nu}] = 0\ . \tag{1.12}$$

As is well known, the Dirac matrices γ can be expressed as the direct product of the Pauli matrices. For example, in the chiral representation

$$\begin{aligned}
&\gamma^0 = \beta = -\sigma^1 \otimes I = \begin{pmatrix} 0, & -I \\ -I, & 0 \end{pmatrix} \\
&\vec{\gamma} = \beta\vec{\alpha} = i\sigma^2 \otimes \vec{\sigma} = \begin{pmatrix} 0, & -\vec{\sigma} \\ -\vec{\sigma}, & 0 \end{pmatrix},\quad \gamma_5 = \begin{pmatrix} I, & 0 \\ 0, & -I \end{pmatrix} \\
&\Sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k, & 0 \\ 0, & \sigma^k \end{pmatrix},\quad \Sigma^{0k} = i\begin{pmatrix} \sigma^k, & 0 \\ 0, & -\sigma^k \end{pmatrix}\ .
\end{aligned} \tag{1.13}$$

In a similar way, all the higher dim matrices Γ's can be written as the direct product of the Pauli matrices.

2. Quantum Pauli Matrices

In a recent paper[2], quantum Pauli matrices are introduced in a spinor analysis of the quantum group $SU_q(2)$[2,3], parallel to the one in the classical $SU(2)$ group. It begins with the quantum $\check{R}$ matrix associated with $SU_q(2)$ or $SL_q(2,C)$ satisfying the Yang-Baxter equation (in the braid form)

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{2.1}$$

and the reduction equation

$$(\check{R} - q^{1/2})(\check{R} + q^{-3/2}) = 0\ . \tag{2.2}$$

The eigenvalue equation of $\check{R}$ can be written as

$$\begin{aligned}
&\check{R}^{\alpha\beta}{}_{\gamma\delta}t_m(q)^{\gamma\delta} = q^{1/2}t_m(q)^{\alpha\beta}\ ,\ \check{R}^{\alpha\beta}{}_{\gamma\delta}s(q)^{\gamma\delta} = -q^{-3/2}s(q)^{\alpha\beta}\ , \\
&\bar{t}^m(q)_{\alpha\beta}\check{R}^{\alpha\beta}{}_{\gamma\delta} = q^{1/2}\bar{t}^m(q)_{\gamma\delta}\ ,\ \bar{s}(q)_{\alpha\beta}\check{R}^{\alpha\beta}{}_{\gamma\delta} = -q^{-3/2}\bar{s}_{\gamma\delta}\ ,
\end{aligned} \tag{2.3}$$

where $\alpha,\ \beta, \cdots = 1,2$ and $m = +,3,-$. For conventional form of $\check{R}$[4]

$$(\check{R}) = \check{R}^{\alpha\beta}{}_{\gamma\delta} = q^{-1/2}\begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix}, \tag{2.4}$$

we can choose

$$t_+(q) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ ,\ t_3(q) = \begin{pmatrix} 0 & -q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}[2]^{-1/2}\ ,\ t_-(q) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$s(q) = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}[2]^{-1/2} \tag{2.5}$$

with $[n] = (q^n - q^{-n})/(q - q^{-1})$, and $\bar{t}^m(q)_{\alpha\beta} = t_m(q)^{\alpha\beta}$, $s(q)_{\alpha\beta} = s(q)^{\alpha\beta}$. Collecting together $\bar{t}^m(q)_{\alpha\beta}$ and $\bar{s}(q)_{\alpha\beta}$ as a four-vector

$$\begin{aligned}
&\bar{t}^\mu(q) \equiv (\bar{t}^0(q), \bar{t}^m(q)) = (q^{-1}\bar{s}(q), \bar{t}^m(q)) \\
&t_\mu(q) \equiv (t_0(q),\ t_m(q)) = (q\ s(q), t_m(q))
\end{aligned} \tag{2.6}$$

we see that they satisfy the following orthogonal conditions

$$t_\mu(q)^{\alpha\beta}\ \bar{t}^\nu(q)_{\alpha\beta} = \delta_\mu{}^\nu \tag{2.7a}$$

and the completeness relation

$$t_\mu(q)^{\alpha\beta}\ \bar{t}^\mu(q)_{\gamma\delta} = \delta^\alpha{}_\gamma \delta^\beta{}_\delta \tag{2.7b}$$

The q-analogue of the Levi-Civita symbols $\epsilon(q)_{\alpha\beta}$ and $\epsilon(q)^{\alpha\beta}$ are related to the singlet eigenvectors

$$\epsilon(q)_{\alpha\beta} = -[2]^{1/2}\bar{s}(q)_{\alpha\beta}\ ,\quad \epsilon(q)^{\alpha\beta} = [2]^{1/2}s(q)^{\alpha\beta} \tag{2.8a}$$

and normalized in a way such that

$$\epsilon(q)_{\alpha\beta}\ \epsilon(q)^{\beta\gamma} = \delta_\alpha{}^\gamma\ . \tag{2.8b}$$

Then two conjugate sets of the quantum Pauli matrices are introduced[2]

$$\sigma_\mu(q)^\alpha{}_\beta = [2]^{1/2}t_\mu(q)^{\alpha\gamma}\epsilon(q)_{\gamma\beta}\ ,\ \bar{\sigma}^\mu(q)^\beta{}_\alpha = [2]^{1/2}\epsilon(q)^{\beta\gamma}\bar{t}^\mu(q)_{\alpha\gamma} \tag{2.9a}$$

with the properties

$$\sigma_\mu(q)^\alpha{}_\beta\bar{\sigma}^\nu(q)^\beta{}_\alpha = [2]\delta_\mu{}^\nu\ ,\ \sigma_\mu(q)^\alpha{}_\beta\bar{\sigma}^\mu(q)^\gamma{}_\delta = [2]\delta^\alpha{}_\delta\delta^\gamma{}_\beta\ . \tag{2.9b}$$

Explicitly

$$\begin{aligned}
\sigma_+(q) &= [2]^{1/2}\begin{pmatrix} & -q^{-1/2} \\ 0 & \end{pmatrix}\ ,\quad \sigma_3(q) = \begin{pmatrix} -q & \\ & q^{-1} \end{pmatrix}\ ,\\
\sigma_-(q) &= [2]^{1/2}\begin{pmatrix} & 0 \\ -q^{1/2} & \end{pmatrix}\ ,\quad \sigma_0(q) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} q\ ,\\
\bar{\sigma}^+(q) &= [2]^{1/2}\begin{pmatrix} & 0 \\ -q^{1/2} & \end{pmatrix}\ ,\quad \bar{\sigma}^3(q) = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\ ,\\
\bar{\sigma}^-(q) &= [2]^{1/2}\begin{pmatrix} & -q^{-1/2} \\ 0 & \end{pmatrix}\ ,\quad \bar{\sigma}^0(q) = \begin{pmatrix} q^{-1} & \\ & -q \end{pmatrix} q^{-1}\ .
\end{aligned} \tag{2.10}$$

As in the classical cases, there exists close relation between $SU(2)$ and $SO(3)$, and between $SL(2,C)$ and $SO(3,1)$. Starting from the basic $\check{R}$ matrix as in (2.1), (2.2) and (2.4), we can construct two fusion $\check{R}$ matrices as follows[2]

$$\check{\mathbf{R}}^{mn}{}_{kl} \equiv \bar{t}^m{}_{12}\bar{t}^n{}_{34}(\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23})t_k{}^{12}t_l{}^{34} \tag{2.11}$$

and[5]

$$\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} \equiv \bar{t}^\mu{}_{12}\bar{t}^\nu{}_{34}(\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}^{-1})t_\kappa{}^{12}t_\lambda{}^{34}\ . \tag{2.12}$$

It can be directly shown that they satisfy their own Yang-Baxter equations similar to that of $\check{R}$ itself in (2.1):

$$\check{\mathbf{R}}^{mn}{}_{m'n'}\check{\mathbf{R}}^{n'k}{}_{r's}\check{\mathbf{R}}^{m'r'}{}_{lr} = \check{\mathbf{R}}^{nk}{}_{n'k'}\check{\mathbf{R}}^{mn'}{}_{lr'}\check{\mathbf{R}}^{r'k'}{}_{rs}\ , \tag{2.13}$$

$$\check{\mathcal{R}}^{\mu\nu}{}_{\mu'\nu'}\check{\mathcal{R}}^{\nu'\kappa}{}_{\rho'\sigma}\check{\mathcal{R}}^{\mu'\rho'}{}_{\lambda\rho} = \check{\mathcal{R}}^{\nu\kappa}{}_{\nu'\kappa'}\check{\mathcal{R}}^{\mu\nu'}{}_{\lambda\rho'}\check{\mathcal{R}}^{\rho'\kappa'}{}_{\rho\sigma}\ . \tag{2.14}$$

A straightforward calculation gives

$$\check{\mathbf{R}}^{mn}{}_{kl} = \begin{pmatrix}
q^2 & & & & & & & & \\
 & \Delta & & & 1 & & & & \\
 & & (1-q^{-2})\Delta & & & -q^{-1}\Delta & & q^{-2} & \\
 & 1 & & & 0 & & & & \\
 & & -q^{-1}\Delta & & & 1 & & & \\
 & & & & & & \Delta & & 1 \\
 & & q^{-2} & & & & & 0 & \\
 & & & & & & 1 & & 0 \\
 & & & & & & & & q^2
\end{pmatrix} \tag{2.15}$$

where $\Delta = q^2 - q^{-2}$, the index pair (m,n) or (k,l) are ordered as $(++)$, $(+3)$, $(+-)$, $(3+)$, (33), $(3-)$, $(-+)$, (-3), $(--)$. In comparison with the standard form in Ref[4], we recognize that the $\check{\mathbf{R}}$ matrix in Eq(2.15) is indeed the $\check{R}$ matrix corresponding to $SO_{q^2}(3)$. This builds the homomorphism between $SU_q(2)$ and $SO_{q^2}(3)$. So we denote this $\check{\mathbf{R}}$ matrix as $\check{\mathbf{R}}(q^2)$ which has three different eigenvalues, say $\lambda_2(q^2) = q^2$ corresponding to a quintet, $\lambda_2(q^2) = -q^{-2}$ to a triplet and $\lambda_2(q^2) = q^{-4}$ a singlet. Then the left and right eigenvectors can be written as

$$\begin{aligned}
&\bar{\mathbf{w}}^M(q^2)_{ij}\check{\mathbf{R}}(q^2)^{ij}{}_{kl} = \lambda_2(q^2)\bar{\mathbf{w}}^M(q^2)_{kl}\ ,\ \check{\mathbf{R}}(q^2)^{ij}{}_{kl}\mathbf{w}_M(q^2)^{kl} = \lambda_2(q^2)\mathbf{w}_M(q^2)^{ij}\ ,\\
&\bar{\mathbf{u}}^m(q^2)_{ij}\check{\mathbf{R}}(q^2)^{ij}{}_{kl} = \lambda_1(q^2)\bar{\mathbf{u}}^m(q^2)_{kl}\ ,\ \check{\mathbf{R}}(q^2)^{ij}{}_{kl}\mathbf{u}_m(q^2)^{kl} = \lambda_1(q^2)\mathbf{u}_m(q^2)^{ij}\ ,\\
&\bar{\mathbf{v}}(q^2)_{ij}\check{\mathbf{R}}(q^2)^{ij}{}_{kl} = \lambda_0(q^2)\bar{\mathbf{v}}(q^2)_{kl}\ ,\ \check{\mathbf{R}}(q^2)^{ij}{}_{kl}\mathbf{v}(q^2)^{kl} = \lambda_0(q^2)\mathbf{v}(q^2)^{ij}\ ,
\end{aligned} \tag{2.16}$$

More explicitly

$$\mathbf{v}(q^2)^{ij} = \begin{pmatrix} \mathbf{v}^{+-} \\ \mathbf{v}^{33} \\ \mathbf{v}^{-+} \end{pmatrix} = \frac{1}{\sqrt{q^2+1+q^{-2}}}\begin{pmatrix} q^{-1} \\ 1 \\ q \end{pmatrix}\ , \tag{2.17a}$$

$$
\begin{aligned}
\mathbf{u}_+(q^2)^{ij} &= \begin{pmatrix} \mathbf{u}_+{}^{+3} \\ \mathbf{u}_+{}^{3+} \end{pmatrix} = \frac{1}{\sqrt{q^2+q^{-2}}} \begin{pmatrix} -q^{-1} \\ q \end{pmatrix} , \\
\mathbf{u}_0(q^2)^{ij} &= \begin{pmatrix} \mathbf{u}_3{}^{+-} \\ \mathbf{u}_3{}^{33} \\ \mathbf{u}_3{}^{-+} \end{pmatrix} = \frac{1}{\sqrt{q^2+q^{-2}}} \begin{pmatrix} 1 \\ q-q^{-1} \\ 1 \end{pmatrix} , \\
\mathbf{u}_-(q^2)^{ij} &= \begin{pmatrix} \mathbf{u}_-{}^{3-} \\ \mathbf{u}_-{}^{-3} \end{pmatrix} = \frac{1}{\sqrt{q^2+q^{-2}}} \begin{pmatrix} -q^{-1} \\ q \end{pmatrix} ,
\end{aligned}
\tag{2.17b}
$$

$$
\begin{aligned}
&\mathbf{w}_2(q^2)^{ij} = \mathbf{w}_2(q^2)^{++} = 1 \\
&\mathbf{w}_1(q^2)^{ij} = \begin{pmatrix} \mathbf{w}_1{}^{+3} \\ \mathbf{w}_1{}^{3+} \end{pmatrix} = \frac{1}{\sqrt{q^2+q^{-2}}} \begin{pmatrix} q \\ q^{-1} \end{pmatrix} , \\
&\mathbf{w}_0(q^2)^{ij} = \begin{pmatrix} \mathbf{w}_0{}^{+-} \\ \mathbf{w}_0{}^{33} \\ \mathbf{w}_0{}^{-+} \end{pmatrix} = \frac{1}{\sqrt{(q^2+q^{-2})(q^2+1+q^{-1})}} \begin{pmatrix} q^2 \\ -q-q^{-1} \\ q^{-1} \end{pmatrix} , \\
&\mathbf{w}_{\bar{1}}(q^2)^{ij} = \begin{pmatrix} \mathbf{w}_{\bar{1}}{}^{3-} \\ \mathbf{w}_{\bar{1}}{}^{-3} \end{pmatrix} = \frac{1}{\sqrt{q^2+q^{-2}}} \begin{pmatrix} q \\ q^{-1} \end{pmatrix} , \\
&\mathbf{w}_{\bar{2}}(q^2)^{ij} = \mathbf{w}_{\bar{2}}(q^2)^{--} = 1
\end{aligned}
\tag{2.17c}
$$

with all other components vanishing. The right-acting eigenvectors $\bar{\mathbf{v}}(q^2)_{ij}$, $\bar{\mathbf{u}}^m(q^2)_{ij}$ and $\bar{\mathbf{w}}^M(q^2)_{ij}$ have the same components as their left-acting counterparts $\mathbf{v}(q^2)^{ij}$, $\mathbf{u}_m(q^2)^{ij}$ and $\mathbf{w}_M(q^2)^{ij}$. All these eigenvectors are normalized in such a way that

$$
\begin{aligned}
\mathbf{C}_A(q^2)^{ij}\bar{\mathbf{C}}^B(q^2)_{ij} &= \delta_A{}^B , \\
\mathbf{C}_A(q^2)^{ij}\bar{\mathbf{C}}^A(q^2)_{kl} &= \delta^i{}_k\delta^j{}_l,
\end{aligned}
\tag{2.18}
$$

where $\mathbf{C}_A(q^2)^{ij} = (\mathbf{v}(q^2)^{ij}, \mathbf{u}_m(q^2)^{ij}, \mathbf{w}_M(q^2)^{ij})$ and $\mathbf{C}^B(q^2)_{ij} = (\bar{\mathbf{v}}(q^2)_{ij}, \bar{\mathbf{u}}^m(q^2)_{ij}, \bar{\mathbf{w}}^M(q^2)_{ij})$.

Now we can define the corresponding projection operators

$$
\begin{aligned}
\mathbf{Q}^{(0)}(q^2)^{ij}{}_{kl} &= \mathbf{v}(q^2)^{ij}\bar{\mathbf{v}}(q^2)_{kl} \\
\mathbf{Q}^{(1)}(q^2)^{ij}{}_{kl} &= \mathbf{u}_m(q^2)^{ij}\bar{\mathbf{u}}^m(q^2)_{kl} \\
\mathbf{Q}^{(2)}(q^2)^{ij}{}_{kl} &= \mathbf{w}_M(q^2)^{ij}\bar{\mathbf{w}}^M(q^2)_{kl}
\end{aligned}
\tag{2.19}
$$

with

$$
\mathbf{Q}^{(r)}(q^2)\mathbf{Q}^{(s)}(q^2) = \delta^{rs}\mathbf{Q}^{(0)}(q^2) , \quad \mathbf{Q}^{(0)}(q^2) + \mathbf{Q}^{(1)}(q^2) + \mathbf{Q}^{(2)}(q^2) = E . \tag{2.20}
$$

And then

$$
\check{\mathbf{R}}(q^2)^{ij}{}_{kl} = \lambda_s(q^2)\mathbf{Q}^{(s)}(q^2)^{ij}{}_{kl} , \quad \check{\mathbf{R}}^{-1}(q^2)^{ij}{}_{kl} = \lambda_s^{-1}(q^2)\mathbf{Q}^{(s)}(q^2)^{ij}{}_{kl} . \tag{2.21}
$$

As for the Levi-Civita symbols in the spinor space (2.8), we introduce

$$
\mathbf{g}(q^2)_{ij} = \sqrt{q^2+1+q^{-2}}\bar{\mathbf{v}}(q^2)_{ij} , \quad \mathbf{g}(q^2)^{ij} = \sqrt{q^2+1+q^{-2}}\mathbf{v}(q^2)^{ij} , \tag{2.22a}
$$

then

$$
\mathbf{g}(q^2)_{ij}\mathbf{g}(q^2)^{jk} = \delta_i{}^k . \tag{2.22b}
$$

It has been shown in Ref[2,6] that $\mathbf{g}(q^2)_{ij}$ and $\mathbf{g}(q^2)^{ij}$ play the role of metric in the q-deformed Euclidean space, which can be used to lower or raise the vector index, e.g.,

$$
\sigma^m(q) = -\sigma_n(q)\mathbf{g}(q^2)^{nm} . \tag{2.23}
$$

(The minus sign here is introduced to meet the Minkowski metric, see (3.11).) For the triplet eigenvectors $\mathbf{u}_m(q^2)^{jk}$ and $\bar{\mathbf{u}}^m(q^2)_{jk}$, the quantum number m used to specified different eigenvectors takes the same values as the component indices j and k. All three indices m, j, k can be put on the same footing. We can easily show that

$$
\mathbf{u}^m(q^2)_{jk} = \mathbf{g}(q^2)_{jl}\mathbf{u}_k(q^2)^{lm} = \mathbf{u}_j(q^2)^{ml}\mathbf{g}(q^2)_{lk} , \tag{2.24a}
$$

and

$$
\mathbf{u}_m(q^2)^{jk} = \mathbf{g}(q^2)^{jl}\mathbf{u}^k(q^2)_{lm} = \mathbf{u}^j(q^2)_{ml}\mathbf{g}(q^2)^{lk} . \tag{2.24b}
$$

Now we define

$$
\mathbf{u}(q^2)_{jkl} = \mathbf{g}(q^2)_{jm}\bar{\mathbf{u}}^m(q^2)kl = \bar{\mathbf{u}}^m(q^2)^{jk}\mathbf{g}(q^2)_{ml} , \tag{2.25a}
$$

$$
\mathbf{u}(q^2)^{jkl} = \mathbf{g}(q^2)^{jm}\mathbf{u}_m(q^2)^{kl} = \mathbf{u}_m(q^2)^{jk}\mathbf{g}(q^2)^{ml} . \tag{2.25b}
$$

It follows at once

$$
\begin{aligned}
\mathbf{u}(q^2)^{jkl}\mathbf{u}(q^2)_{jkl} &= q^2+1+q^{-2} , \\
\mathbf{u}(q^2)^{jkl}\mathbf{u}(q^2)_{klm} &= \delta^j{}_m , \\
\mathbf{u}(q^2)^{jkl}\mathbf{u}(q^2)_{lmn} &= \mathbf{Q}^{(1)}(q^2)^{jk}{}_{mn} .
\end{aligned}
\tag{2.26}
$$

This means that $\mathbf{u}(q^2)_{jkl}$ and $\mathbf{u}(q^2)^{jkl}$ can be used as the q-deformed Levi-Civita symbols in the 3-dim q-Euclidean space, i.e.,

$$
\epsilon(q^2)_{jkl} = -i\sqrt{q^2+q^{-2}}\mathbf{u}(q^2)_{jkl} , \quad \epsilon(q^2)^{jkl} = i\sqrt{q^2+q^{-2}}\mathbf{u}(q^2)^{jkl} . \tag{2.27}
$$

The eigenvectors $\mathbf{v}(q^2)$, $\mathbf{u}_m(q^2)$, $\mathbf{w}_M(q^2)$ and then the $\check{\mathbf{R}}$ matrix have a lot of useful relations among which the most important ones are [6]

$$\check{\mathbf{R}}^{-1}(q^2)^{jl'}{}_{ki'} = \mathbf{g}(q^2)_{ki}\check{\mathbf{R}}(q^2)^{ij}{}_{i'j'}\mathbf{g}(q^2)^{j'l'} = \mathbf{g}(q^2)^{jl}\check{\mathbf{R}}(q^2)^{l'k'}{}_{lk}\mathbf{g}(q^2)_{k'i'} \ . \tag{2.28}$$

Now a straightforward calculation shows

$$\begin{aligned}
&\mathbf{Q}^{(2)}(q^{-2})^{ij}{}_{kl}\sigma^k\sigma^l = 0 \\
&\mathbf{Q}^{(0)}(q^{-2})^{ij}{}_{kl}\sigma^k\sigma^l = \mathbf{g}(q^{-2})^{ij} \\
&\mathbf{Q}^{(1)}(q^{-2})^{ij}{}_{kl}\sigma^k\sigma^l = -\sqrt{q^2+q^{-2}}\mathbf{u}_m(q^{-2})^{ij}\sigma^m
\end{aligned} \tag{2.29}$$

from which it follows by using (2.20) and (2.21) that

$$\sigma^i\sigma^j + q^{-2}\check{\mathbf{R}}(q^{-2})^{ij}{}_{kl}\sigma^k\sigma^l = (1+q^2)\mathbf{g}(q^{-2})^{ij} \ , \tag{2.30a}$$

$$\begin{aligned}
\sigma^i\sigma^j - q^{-4}\check{\mathbf{R}}(q^{-2})^{ij}{}_{kl}\sigma^k\sigma^l &= (1+q^2)\mathbf{Q}^{(1)}(q^{-2})^{ij}{}_{kl}\sigma^k\sigma^l \\
&= i(1+q^{-2})\epsilon(q^{-2})^{ijk}\mathbf{g}(q^{-2})_{km}\sigma^m \ .
\end{aligned} \tag{2.30b}$$

these two are indeed the quantum version of relations appearing in Eqs(1.5). Similar to Eq(1.6), we can define the q-deformation of the $SO_{q^2}(3)$ generators as

$$M^{ij} \equiv \frac{-i}{(1+q^{-2})}\mathbf{Q}^{(1)}(q^2)^{ij}{}_{kl}\sigma^k\sigma^l = \frac{1}{(1+q^{-2})}\epsilon(q^{-2})^{ijk}\mathbf{g}(q^{-2})_{km}\sigma^m \ .$$

As for the classical case, these generators are indeed the Pauli matrices themselves. And (2.30b) gives the q-commutation relations among these generators

$$\frac{\sigma^i}{1+q^{-2}}\frac{\sigma^j}{1+q^{-2}} - q^{-4}\check{\mathbf{R}}(q^{-2})^{ij}{}_{kl}\frac{\sigma^k}{1+q^{-2}}\frac{\sigma^l}{1+q^{-2}} = i\epsilon(q^{-2})^{ijk}\mathbf{g}(q^{-2})_{km}\frac{\sigma^m}{1+q^{-2}} \ . \tag{2.31}$$

Also we have

$$M^{ij}\sigma^l - q^{-4}\check{\mathbf{R}}(q^{-2})^{jl}{}_{j'n}\check{\mathbf{R}}(q^{-2})^{ij'}{}_{km}\sigma^k M^{mn} = -i(q^2+q^{-2})\sigma^k\mathbf{Q}^{(1)}(q^{-2})^{ij}{}_{km}\mathbf{g}(q^{-2})^{ml} \ . \tag{2.32}$$

This is the q-version of Eq(1.7). Eqs(2.32) and (2.31) are equivalent.

3. Quantum Dirac Matrices

It has been pointed out, in Ref[5,7], that the fusion $\check{\mathcal{R}}$ matrix given at (2.12) and (2.14) is really the $\check{R}$ matrix for quantum Lorentz group $SO(3,1)$. The explicit

form of $\check{\mathcal{R}}$ was shown there. The 16×16 $\check{R}$ matrix has three different eigenvalues: the singlet eigenvalue $\lambda_0(q) = q^{-3}$, the sixfold one $\lambda_1(q) = -q^{-1}$ and the nonet $\lambda_2(q) = q$. The left-acting eigenvectors as well as the right-acting ones are written as [7]

$$\begin{aligned}
&\bar{v}(q)_{\mu\nu}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \lambda_0(q)\bar{v}(q)_{\kappa\lambda} \ , && \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda}v(q)^{\kappa\lambda} = \lambda_0(q)v(q)^{\mu\nu} \ , \\
&\bar{u}^{ms}(q)_{\mu\nu}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \lambda_1(q)\bar{u}^{ms}(q)_{\kappa\lambda} \ , && \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda}u_{ms}(q)^{\kappa\lambda} = \lambda_1(q)u_{ms}(q)^{\mu\nu} \ , \\
&\bar{w}^{mn}(q)_{\mu\nu}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \lambda_2(q)\bar{w}^{mn}(q)_{\kappa\lambda} \ , && \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda}w_{mn}(q)^{\kappa\lambda} = \lambda_2(q)w_{mn}(q)^{\mu\nu} \ ,
\end{aligned} \tag{3.1}$$

where for sextet eigenvectors $(ms) = (1+),(1-),(0+),(0-),(\bar{1}+),(\bar{1}-)$ and for nonet $(mn) = (2,0),(1,1),(1,\bar{1}),(0,2),(0,0),(0,\bar{2}),(\bar{1},1),(\bar{1},\bar{1})$ and $(\bar{2},0)$. The singlet eigenvectors are

$$\begin{aligned}
v(q)^{\mu\nu} &= (v^{+-}, v^{33}, v^{30}, v^{03}, v^{00}, v^{-+}) = (-q^{-1}, -1, 0, 0, 1, -q)[2]^{-1} \ , \\
\bar{v}(q)_{\mu\nu} &= (\bar{v}_{+-}, \bar{v}_{33}, \bar{v}_{30}, \bar{v}_{03}, \bar{v}_{00}, \bar{v}_{-+}) = (-q^{-1}, -1, 0, 0, 1, -q)[2]^{-1} \ .
\end{aligned} \tag{3.2}$$

These are scaled to give

$$g(q)_{\mu\nu} = [2]\bar{v}(q)_{\mu\nu} \ , \quad g(q)^{\mu\nu} = [2]v(q)^{\mu\nu} \ . \tag{3.3}$$

Then

$$g(q)_{\mu\nu}g(q)^{\nu\lambda} = \delta_\mu{}^\lambda = g(q)^{\lambda\nu}g(q)_{\nu\mu} \tag{3.4a}$$

and

$$g(q)_{\mu\nu} = g(q^{-1})_{\nu\mu} \ . \tag{3.4b}$$

And it has been pointed out that $g(q)_{\mu\nu}$ and $g(q)^{\mu\nu}$ are the q-deformed Lorentz metrics of quantum Minkowski space.

The sextet eigenvector $u_{ms}(q)^{\mu\nu}$ and $\bar{u}^{ms}(q)_{\mu\nu}$ were presented in Ref[7], which can be expressed as

$$\begin{aligned}
u_{m+}(q)^{ij} &= u_{m-}(q)^{ij} = \mathbf{u}(q^2)^{ij}\sqrt{q^2+q^{-2}}[2]^{-1} \\
&= -i\mathbf{g}(q^2)_{mk}\epsilon(q^2)^{kij}[2]^{-1} \\
u_{m+}(q)^{0k} &= u_{m-}(q)^{k0} = -q^{-1}[2]^{-1}\delta_m{}^k \\
u_{m+}(q)^{k0} &= u_{m-}(q)^{0k} = q[2]^{-1}\delta_m{}^k
\end{aligned} \tag{3.5a}$$

$$\begin{aligned}
\bar{u}^{m+}(q)_{ij} &= \bar{u}^{m-}(q)_{ij} = \bar{\mathbf{u}}_{ij}(q^2)\sqrt{q^2+q^{-2}}[2]^{-1} \\
&= i\mathbf{g}(q^2)^{mk}\epsilon(q^2)_{kij}[2]^{-1} \\
\bar{u}^{m+}(q)_{0k} &= \bar{u}^{m-}(q)_{k0} = -q[2]^{-1}\delta^m{}_k \\
\bar{u}^{m+}(q)_{k0} &= \bar{u}^{m-}(q)_{0k} = q^{-1}[2]^{-1}\delta^m{}_k \ .
\end{aligned} \tag{3.5b}$$

The projection operators corresponding to different eigenvalues are defined as

$$
\begin{aligned}
Q^{(0)}(q)^{\mu\nu}{}_{\kappa\lambda} &= v(q)^{\mu\nu}\bar{v}(q)_{\kappa\lambda} \ , \\
Q^{(1)}(q)^{\mu\nu}{}_{\kappa\lambda} &= u_{ms}(q)^{\mu\nu}\bar{u}_{ms}(q)_{\kappa\lambda} \ , \\
Q^{(2)}(q)^{\mu\nu}{}_{\kappa\lambda} &= w_{mn}(q)^{\mu\nu}\bar{w}_{mn}(q)_{\kappa\lambda} \ ,
\end{aligned}
\tag{3.6}
$$

which satisfy the relations

$$
Q^{(i)}Q^{(j)} = \delta^{ij}Q^{(j)} \ , \quad Q^{(0)} + Q^{(1)} + Q^{(2)} = E \ . \tag{3.7}
$$

And then

$$
\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \lambda_i(q)Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda} \ , \quad \check{\mathcal{R}}^{-1}(q)^{\mu\nu}{}_{\kappa\lambda} = \lambda_i^{-1}(q)Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda} \ . \tag{3.8}
$$

The projection operators $Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda}$ as well as the $\check{\mathcal{R}}$ matrix have a lot of useful symmetries, among which the most important ones are

$$
\begin{aligned}
&Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda} = Q^{(i)}(q^{-1})^{\nu\mu}{}_{\lambda\kappa} \ , \quad \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \check{\mathcal{R}}^{-1}(q^{-1})^{\nu\mu}{}_{\lambda\kappa} \ , \\
&g(q)_{\rho\mu}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \check{\mathcal{R}}^{-1}(q)^{\nu\sigma}{}_{\rho\kappa}g(q)_{\sigma\lambda} \ , \quad g(q)^{\sigma\kappa}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} = \check{\mathcal{R}}^{-1}(q)^{\sigma\mu}{}_{\lambda\rho}g(q)_{\rho\nu} \ .
\end{aligned}
\tag{3.9}
$$

The quantum Dirac matrices are obtained by a construction method in terms of the quantum Pauli matrices, very similar to the classical case

$$
\gamma^\mu = \beta\alpha^\mu \ , \quad \alpha^\mu = \begin{pmatrix} \sigma^\mu & \\ & \hat{\sigma}^\mu \end{pmatrix} \ , \quad \beta = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \tag{3.10}
$$

where

$$
\sigma^\mu(q) = \sigma_\nu(q)g(q)^{\nu\mu} \tag{3.11}
$$

$\sigma_\nu(q)$ are defined as in Eq(2.9), and $\hat{\sigma}^\mu$ are related to the conjugate set $\bar{\sigma}^\mu$ in (2.9) to ensure the adequate transformation property. [7] More explicitly,

$$
\gamma^0 = \begin{pmatrix} 0 & -q^{-1} \\ -q & 0 \end{pmatrix} \ , \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \ . \tag{3.12}
$$

A direct calculation shows that

$$
\begin{aligned}
&Q^{(0)}(q^{-1})^{\mu\nu}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = g(q^{-1})^{\mu\nu} \ , Q^{(2)}(q^{-1})^{\mu\nu}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = 0 \ , \\
&iQ^{(1)}(q^{-1})^{\mu\nu}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = \Sigma(q)^{\mu\nu} = \begin{pmatrix} \bar{\sigma}^{\mu\nu}(q) & 0 \\ 0 & \sigma^{\mu\nu}(q) \end{pmatrix}
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
\bar{\sigma}^{\mu\nu}(q) &= iQ^{(1)}(q^{-1})^{\mu\nu}_{\kappa\lambda}\hat{\sigma}^\kappa\sigma^\lambda = i[2]\sigma^m(q)u_{m-}(q^{-1})^{\mu\nu} \ , \\
\sigma^{\mu\nu}(q) &= iQ^{(1)}(q^{-1})^{\mu\nu}_{\kappa\lambda}\sigma^\kappa\hat{\sigma}^\lambda = i[2]\sigma^m(q)u_{m+}(q^{-1})^{\mu\nu} \ .
\end{aligned}
\tag{3.14}
$$

By using (3.5a) we see that

$$
\begin{aligned}
\Sigma(q)^{ij} &= \sigma^m(q)g(q^{-2})_{mk}\epsilon(q^{-2})^{kij}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \ , \\
\Sigma(q)^{0k} &= i\sigma^k(q)\begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} \ , \\
\Sigma(q)^{k0} &= i\sigma^k(q)\begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix} \ .
\end{aligned}
\tag{3.15}
$$

From (3.13) we can obtain

$$
\gamma^\mu\gamma^\nu + q^{-1}\check{\mathcal{R}}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = (1+q^2)g(q^{-1})^{\mu\nu} \ , \tag{3.16}
$$

$$
\gamma^\mu\gamma^\nu - q^{-3}\check{\mathcal{R}}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = (1+q^{-2})Q^{(1)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = -i(1+q^{-2})\Sigma(q)^{\mu\nu} \ . \tag{3.17}
$$

Eq(3.16) is the Clifford definition relation for quantum Dirac matrices γ^μ. A similar relation has also been proposed by Zumino in a different consideration.[8] The generators of quantum Lorentz group can be defined as

$$
M^{\mu\nu} = \frac{i}{1+q^{-2}}Q^{(1)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda = \frac{1}{1+q^{-2}}\Sigma(q)^{\mu\nu} \ . \tag{3.18}
$$

Then a tedious calculation gives

$$
M^{\mu\nu}\gamma^\lambda - q^{-2}\check{\mathcal{R}}(q^{-1})^{\nu\lambda}{}_{\nu'\sigma}\check{\mathcal{R}}(q^{-1})^{\mu\nu'}{}_{\kappa\rho}\gamma^\kappa M^{\rho\sigma} = i(1+q^2)Q^{(1)}(q^{-1})^{\mu\nu}{}_{\rho\sigma}\gamma^\rho g(q^{-1})^{\sigma\lambda} \tag{3.19}
$$

and

$$
M^{\mu\nu}M^{\kappa\lambda} - \check{\mathcal{R}}(q^{-1})^{(\mu\nu)(\kappa\lambda)}_{(\mu'\nu')(\kappa'\lambda')}M^{\mu'\nu'}M^{\kappa'\lambda'} = f^{(\mu\nu)(\kappa\lambda)}_{\rho\sigma}M^{\rho\sigma} \tag{3.20}
$$

where

$$
\check{\mathcal{R}}(q^{-1})^{(\mu\nu)(\kappa\lambda)}_{(\mu'\nu')(\kappa'\lambda')} \equiv q^{-4}\check{\mathcal{R}}(q^{-1})^{\nu\kappa}{}_{\nu''\sigma}\check{\mathcal{R}}(q^{-1})^{\mu\nu''}{}_{\mu'\rho}\check{\mathcal{R}}(q^{-1})^{\sigma\lambda}{}_{\kappa''\lambda'}\check{\mathcal{R}}(q^{-1})^{\rho\kappa''}{}_{\nu'\kappa'} \tag{3.21a}
$$

which is a 256×256 matrix satisfying Yang-Baxter equation, and

$$
f^{(\mu\nu)(\kappa\lambda)}_{\rho\sigma} = i[2]^2Q^{(1)}(q^{-1})^{\mu\nu}{}_{\rho\nu'}g(q^{-1})^{\nu'\kappa'}Q^{(1)}(q^{-1})^{\kappa\lambda}{}_{\kappa'\sigma} \tag{3.21b}
$$

It is easily seen that Eqs(3.16), (3.18), (3.19) and (3.20) are just the quantum version of the classical relations.

4. Brief Discussion

In two previous sections, we have established the quantum Pauli matrices and quantum Dirac matrices, as the special case of quantum Clifford matrices in 3-dim and 4-dim. The generators of the corresponding quantum groups, the quantum rotation group $SO_{q^2}(3)$ and the quantum Lorentz group $SO_q(1,3)$, are obtained by extracting the q-anti-symmetric combination from the bilinear form of two Clifford matrices. The common features are:

(i) Every main classical equation has its quantum counterpart, as least for the 3-dim and 4-dim cases.

(ii) The ordinary commutation relations in the classical case are changed into $\check{R}$ commutation relations, just as proposed by Woronowicz,[1] with adequately given $\check{R}$ matrices.

(iii) In the limit $q \longrightarrow 1$, all these quantum relations reduce back to their classical counterparts.

For arbitrary N, the Clifford definition relation can be written as

$$\Gamma^A\Gamma^B - \lambda_1^{-1}\check{R}^{AB}{}_{CD}\Gamma^C\Gamma^D = (1-\lambda_1^{-1}\lambda_0)g^{AB} \tag{4.1}$$

where $\check{R}^{AB}{}_{CD}$, being the $\check{R}$ matrix characterizing $SO_q(N)$ (or $SO_q(p-N-p)$) group, has three different eigenvalues: $\lambda_0(q) = q^{1-N}$ corresponding to the singlet, $\lambda_1(q) = -q^{-1}$ to the q-anti-symmetric multiplet with $\dim\frac{N(N-1)}{2}$ and $\lambda_2(q) = q$ to the q-traceless-symmetric multiplet with $\dim\frac{N(N+1)}{2} - 1$. The projection operator to the subspaces with different eigenvalues λ_i are denoted by $Q^{(i)}$, and g^{AB} is the eigenvector of $\check{R}^{AB}{}_{CD}$ corresponding to λ_0. The definition given at (4.1) coincides with that in Ref[8]. Then the generators of quantum orthogonal group are taken as

$$M^{AB} = \frac{\mp i}{1-\lambda_1\lambda_0^{-1}}Q^{(1)AB}{}_{CD}\Gamma^C\Gamma^D \tag{4.2}$$

with $-$ sign for Euclidean case $SO_q(N)$ and $+$ sign for pseudo-Euclidean case $SO_q(1,N-1)$ (since the time-favor metric convention is adopted). The vector properties of the Γ^C is now expressed by

$$M^{AB}\Gamma^C - \lambda_1^{-2}\check{R}^{BC}{}_{DB'}\check{R}^{AD}{}_{C'A'}\Gamma^{C'}M^{A'B'} = \pm i\lambda_0\lambda_1^{-1}(1+\lambda_1^{-2})\Gamma^{A'}Q^{(1)AB}{}_{A'B'}g^{B'C}. \tag{4.3}$$

But the commutation relations among M^{AB} take a little bit complicated form for higher dimensions.

$$M^{AB}M^{CD} - \check{\mathcal{R}}^{(AB)(CD)}{}_{A'B')(C'D'}M^{A'B'}M^{C'D'} = f^{(AB)(CD)}_{(FG)}(M^{FG}+\xi g^{FG}), \tag{4.4}$$

where

$$\check{\mathcal{R}}^{(12)(34)} = \lambda_1^{-4}\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}, \tag{4.5}$$

and

$$f^{(AB)(CD)}_{(FG)} = \pm i\lambda_0\lambda_1^{-1}(1+\lambda_1^{-2})Q^{(1)AB}{}_{FB'}g^{B'C'}Q^{(1)CD}{}_{C'G}, \tag{4.6}$$

$$\xi = \mp i\frac{1+\lambda_1^{-3}\lambda_0}{(1+\lambda_1^{-2})(1-\lambda_1\lambda_0^{-1})}. \tag{4.7}$$

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