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Differential Calculus on Quantum Lorentz Group*

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Abstract

In this paper, we discuss the bicovariant differential calculus on quatum Lorentz group, and provide corresponding de Rham complex and Maurer-Cartan formulae.

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§0. Introduction

Quantum groups and quantum Lie algebras as deformed Lie groups and Lie algebras have been studied extensively following the pioneer papers of Drinfeld, Jimbo, Faddeev et al[1]. It is also very interesting to study the quantum groups from geometrical point of view[2], i.e. to find its relation with non-commutative geometry[3]. Although the bicovariant differential calculi on quantum groups have been provided by Woronowicz[4], their concrete constructions still attract much attention. So several groups have been working on this question on different subjects, for example, the differential calculi on quantum planes[5], on quantum groups[6,7,8] as well as on quantum groups with multi-deformed parameters[9], one can also find many results in the review paper of Zumino[10].

The construction of quantum Lorentz group was first given in [11] by Podlés and Woronowicz, and then discussed by Drabant et al.[12], Carow-Watamura et al.[13] and others[14]. Following the works of [11,12], the main purpose of this letter is to discuss the bicovariant differential calculus on quantum Lorentz group based on the method given in [7,8,9]. The results we obtained include the bicovariant first order and high order differential calculus on quantum Lorentz group, the quantum Maurer-Cartan formulae, i.e. the deformed de Rham complex on quantum Lorentz group as following sequence of bimodules of quantum Lorentz group

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots. \tag{0.1}$$

Summation convention will be used in this paper.

§1. Quantum Lorentz group: Ω^0

It is well known that the quantum matrix group $SL_q(2)$ is generated by four elements which can be written into a 2×2 matrix $T = (t_{ij})_{i,j=1,2}$ and also satisfy the following relations,

$$RT_1T_2 = T_2T_1R, \quad Det_qT = t_{11}t_{22} - qt_{12}t_{21} = 1, \tag{1.1}$$

where

$$R = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \lambda = q - q^{-1},\ q \in \mathbb{R}.$$

For quantum Lorentz group $SL_q(2,\mathbb{C})$, we need to introduce t^*_{ij} $(i,j = 1,2)$, the conjugates of elements t_{ij} $(i,j = 1,2)$, and generators $t_{\bar{i}\bar{j}}$ $(i,j = 1,2)$ as

$$\hat{T} = (t_{\bar{i}\bar{j}})_{i,j=1,2}, \quad t_{\bar{i}\bar{j}} = (S(t_{ji}))^*,$$

where S is the antipode. With a few calculation, we have

$$R\hat{T}_1\hat{T}_2 = \hat{T}_2\hat{T}_1R, \qquad Det_q\hat{T} = t_{\bar{1}\bar{1}}t_{\bar{2}\bar{2}} - qt_{\bar{1}\bar{2}}t_{\bar{2}\bar{1}} = 1,$$

or

$$(R^+)^{-1}\hat{T}_1\hat{T}_2 = \hat{T}_2\hat{T}_1(R^+)^{-1}, \qquad Det_q\hat{T} = t_{\bar{1}\bar{1}}t_{\bar{2}\bar{2}} - qt_{\bar{1}\bar{2}}t_{\bar{2}\bar{1}} = 1. \tag{1.2}$$

The commutation relations between t_{ij} and $t_{\bar{i}\bar{j}}$ are defined as[11]

$$R\hat{T}_1T_2 = T_2\hat{T}_1R. \tag{1.3}$$

Then we can simplely rewrite (1.1), (1.2), (1.3) as

$$\mathcal{R}\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1\mathcal{R}, \tag{1.4}$$

where

$$\mathcal{T} = \begin{pmatrix} T & \\ & \hat{T} \end{pmatrix},$$

$\mathcal{R} = (\mathcal{R}^{ab}_{cd})_{a,b,c,d=1,2,\bar{1},\bar{2}}$,

$$\mathcal{R}^{ij}_{kl} = R^{ij}_{kl},\ \mathcal{R}^{i\bar{j}}_{k\bar{l}} = ((R^+)^{-1})^{ij}_{kl},\ \mathcal{R}^{\bar{i}j}_{\bar{k}l} = R^{ij}_{kl},\ \mathcal{R}^{\bar{i}\bar{j}}_{\bar{k}\bar{l}} = ((R^+)^{-1})^{ij}_{kl},\ i,j,k,l = 1,2,$$

and other elements of $\mathcal{R}$ are zeros. It can be checked that $\mathcal{R}$ satisfies the Yang-Baxter equation.

Definition 1.1

$$Fun(SL_q(2,\mathbb{C})) := \mathbb{C}[t_{ij}, t_{\bar{i}\bar{j}}]/\{\mathcal{R}\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1\mathcal{R},\ Det_qT = 1,\ Det_q\hat{T} = 1\}.$$

$SL_q(2,\mathbb{C})$ can be understood as the direct sum of two copies of $SL_q(2)$[11,12], so the comultiplication Δ, the counite ε and the antipode S on $SL_q(2,\mathbb{C})$ can be naturally induced from those operators on $SL_q(2)$. Therefore, $Fun(SL_q(2,\mathbb{C}))$ is a Hopf algebra. For simplicity we denote $Fun(SL_q(2,\mathbb{C}))$ by Ω^0 as the first bimodule of the de Rham complex in (0.1).

Now we can introduce two sets of linear functionals on Ω^0 by

$$\mathcal{L}^+ = \begin{pmatrix} L^+ & \\ & \hat{L}^+ \end{pmatrix}, \qquad \mathcal{L}^- = \begin{pmatrix} L^- & \\ & \hat{L}^- \end{pmatrix},$$

where $L^\pm = (l^\pm_{ij})_{i,j=1,2}$, $\hat{L}^\pm = (l^\pm_{\bar{i}\bar{j}})_{i,j=1,2}$.

Definition 1.2

$$< l^\pm_{ab}, t_{cd} >= (\mathcal{R}^\pm)^{ac}_{bd}, \quad a,b,c,d = 1,2,\bar{1},\bar{2}$$

where $\mathcal{R}^+ = P\mathcal{R}P$, $\mathcal{R}^- = \mathcal{R}^{-1}$ and P is the permutation matrix[1][12].

Definition 1.2 can be also write down as

$$\begin{array}{lll}
< l^+_{ij}, t_{kl} >= (R^+)^{ik}_{jl}, & \text{i.e.} & < L^+, T >= R^+, \\
< l^+_{i\bar{j}}, t_{k\bar{l}} >= (R^+)^{ik}_{jl}, & \text{i.e.} & < L^+, \hat{T} >= R^+, \\
< l^+_{\bar{i}j}, t_{kl} >= (R^-)^{ik}_{jl}, & \text{i.e.} & < \hat{L}^+, T >= R^-, \\
< l^+_{\bar{i}\bar{j}}, t_{k\bar{l}} >= (R^-)^{ik}_{jl}, & \text{i.e.} & < \hat{L}^+, \hat{T} >= R^-, \\
< l^-_{ij}, t_{kl} >= (R^-)^{ik}_{jl}, & \text{i.e.} & < L^-, T >= R^-, \\
< l^-_{i\bar{j}}, t_{k\bar{l}} >= (R^+)^{ik}_{jl}, & \text{i.e.} & < L^-, \hat{T} >= R^+, \\
< l^-_{\bar{i}j}, t_{kl} >= (R^-)^{ik}_{jl}, & \text{i.e.} & < \hat{L}^-, T >= R^-, \\
< l^-_{\bar{i}\bar{j}}, t_{k\bar{l}} >= (R^+)^{ik}_{jl}, & \text{i.e.} & < \hat{L}^-, \hat{T} >= R^+,
\end{array}$$

where $i,j,k,l = 1,2$. From the above, we see l^-_{ij} and $l^-_{\bar{i}\bar{j}}$ have the same definition, so we may assume $l^-_{ij} = l^-_{\bar{i}\bar{j}}$.

By general theory of quantum groups, we know

$$\mathcal{R}^+\mathcal{L}^\pm_1\mathcal{L}^\pm_2 = \mathcal{L}^\pm_2\mathcal{L}^\pm_1\mathcal{R}^+, \qquad \mathcal{R}^+\mathcal{L}^+_1\mathcal{L}^-_2 = \mathcal{L}^-_2\mathcal{L}^+_1\mathcal{R}^+. \tag{1.5}$$

(1.5) can also be written as

$$\begin{array}{ll}
R^+L^\pm_1L^\pm_2 = L^\pm_2L^\pm_1R^+, & R^+L^\pm_1\hat{L}^\pm_2 = \hat{L}^\pm_2L^\pm_1R^+, \\
R^-\hat{L}^\pm_1L^\pm_2 = L^\pm_2\hat{L}^\pm_1R^-, & R^-\hat{L}^\pm_1\hat{L}^\pm_2 = \hat{L}^\pm_2\hat{L}^\pm_1R^-, \\
R^+L^+_1L^-_2 = L^-_2L^+_1R^+, & R^+L^+_1\hat{L}^-_2 = \hat{L}^-_2L^+_1R^+, \\
R^-\hat{L}^+_1L^-_2 = L^-_2\hat{L}^+_1R^-, & R^-\hat{L}^+_1\hat{L}^-_2 = \hat{L}^-_2\hat{L}^+_1R^-,
\end{array}$$

Denote $Fun^*_0(SL_q(2,\mathbb{C}))$ the associate algebra generated by $L^\pm_{ij}$ and $L^\pm_{\bar{i}\bar{j}}$ $(i,j = 1,2)$. The comultiplication Δ^*, the counite ε^* and the antipode S^* can also be defined on it. It is clear that $Fun^*_0(SL_q(2,\mathbb{C}))$ is a Hopf subalgebra of $Fun^*(SL_q(2,\mathbb{C}))$, the dual of Ω^0.

The main results in this section have already appeared in [12], here we use a different convention used in this letter.

§2. The first order differential calculus: Ω^1

Assume $\mathcal{A}$ is an associative algebra with unit. The first order differential calculus on $\mathcal{A}$, which is denoted by (Γ, δ), consists of a bi-module Γ of $\mathcal{A}$ and a linear operator δ satisfying

(i) Leibnitz rule

$$\delta(xy) = (\delta x)y + x\delta y, \qquad \forall x, y \in \mathcal{A},$$

(ii) for arbitrary element ρ in Γ, there always exist some elements $x_k, y_k \in \mathcal{A}$ $(k = 1,2,\cdots,N)$ such that

$$\rho = \sum_{k=1}^{N} x_k\delta y_k.$$

Now we introduce the convolution "$*$" on Ω^0. For $f \in Fun^*(SL_q(2,\mathbb{C}))$, the convolution "$*$" from Ω^0 to Ω^0 is defined by

$$f*(x) = (id \otimes f)\Delta x, \ x \in \mathrm{Fun}(GL_q(n)), \tag{2.1}$$

where id is the identity operator on Ω^0. Furthermore, we introduce two sets of functionals on Ω^0 as follows:

$$\begin{aligned} \nabla_{ab} &:= \frac{1}{\lambda}(S^*(l^-_{ac})l^+_{cb} - \delta_{ab}\varepsilon), \\ \theta_{abcd} &:= S^*(l^-_{ca})l^+_{bd}, \end{aligned}$$

where $a,b,c,d = 1,2,\bar{1},\bar{2}$.

Proposition 2.1 For $\forall x,y \in \Omega^0$, $a,b,c,d,e,f = 1,2,\bar{1},\bar{2}$, we have

$$\begin{aligned} &\text{(i)} \quad \nabla_{ab}(1) = 0, \quad \theta_{abcd}(1) = \delta_{ac}\delta_{bd}, \\ &\text{(ii)} \quad \Delta^*\nabla_{ab} = \nabla_{ef} \otimes \theta_{efab} + \varepsilon \otimes \nabla_{ab}, \\ &\qquad\quad \Delta^*\theta_{abcd} = \theta_{abef} \otimes \theta_{efcd}, \\ &\text{(iii)} \quad \nabla_{ab} * (xy) = (\nabla_{ef} * x)(\theta_{efab} * y) + x(\nabla_{ab} * y), \\ &\qquad\quad \theta_{abcd} * (xy) = (\theta_{abef} * x)(\theta_{efcd} * y). \end{aligned}$$

The proof of Proposition 2.1 can be found in [8]. Let Ω^1 be the left module generated by eight generators $\omega^{ij}, \omega^{\bar{i}\bar{j}}$, $(i,j = 1,2)$, and define the right multiplication on Ω^1 by

$$\omega^{ab} \cdot x = (\theta_{abcd} * x)\omega^{cd}, \quad \forall x \in \Omega^0, \quad a,b,c,d = 1,2,\bar{1},\bar{2}.$$

Therefore, Ω^1 becomes a Ω^0-bimodule.

Definition 2.1

$$dx := (\nabla_{ab} * x)\omega^{ab}, \quad \forall x \in \Omega^0, a,b = 1,2,\bar{1},\bar{2}. \tag{2.2}$$

It is easy to check

$$d(xy) = (dx)y + xdy, \quad \forall x,y \in \Omega^0$$

To prove (Ω^1, d) is a differential calculus on Ω^0, we must show that each of $\omega^{ij}, \omega^{\bar{i}\bar{j}}$, $(i,j = 1,2)$ can be represented by

$$\sum_{k=1}^{N} x_k dy_k, \quad x_k, y_k \in \Omega^0.$$

We have for $i,j,k,l,u,v = 1,2$,

$$\begin{aligned} \nabla_{ij}(t_{\bar{k}\bar{l}}) &= \tfrac{1}{\lambda}(S^*(l^-_{iu})l^+_{uj} - \delta_{ij}\varepsilon)(t_{\bar{k}\bar{l}}) \\ &= \tfrac{1}{\lambda}[< S^*(l^-_{iu}), t_{\bar{k}\bar{v}} >< l^+_{uj}, t_{\bar{v}\bar{l}} > - \delta_{ij}\delta_{kl}] \\ &= \tfrac{1}{\lambda}[((R^+)^{-1})^{ik}_{uv}(R^+)^{uv}_{jl} - \delta_{ij}\delta_{kl}] \\ &= 0. \end{aligned}$$

Similarly, we have $\nabla_{\bar{i}\bar{j}}(t_{kl}) = 0$. Therefore, we have for $i,j,k,u,v = 1,2$,

$$\begin{aligned} S(t_{ik})dt_{kj} &= \nabla_{uv}(t_{ij})\omega^{uv}, \\ S(t_{\bar{i}\bar{k}})dt_{\bar{k}\bar{j}} &= \nabla_{\bar{u}\bar{v}}(t_{\bar{i}\bar{j}})\omega^{\bar{u}\bar{v}}. \end{aligned}$$

Now we can treat the representation of ω^{ij} and $\omega^{\bar{i}\bar{j}}$, $(i,j = 1,2)$ seperately, therefore we have

$$\begin{aligned} \omega^1 &= [q^3(t_{22}dt_{11} - q^{-1}t_{12}dt_{21}) - (q^3 - q - 1)(-qt_{21}dt_{12} + t_{11}dt_{22})]/(q^3 - 1), \\ \omega^2 &= q(-qt_{21}dt_{11} + t_{11}dt_{21}), \\ \omega^3 &= q(t_{22}dt_{12} - q^{-1}t_{12}dt_{22}), \\ \omega^4 &= [q^2(t_{22}dt_{11} - q^{-1}t_{12}dt_{21}) + q^3(-qt_{21}dt_{12} + t_{11}dt_{22})]/(q^3 - 1), \\ \omega^{\bar{1}} &= [(t_{\bar{2}\bar{2}}dt_{\bar{1}\bar{1}} - q^{-1}t_{\bar{1}\bar{2}}dt_{\bar{2}\bar{1}}) + q(-qt_{\bar{2}\bar{1}}dt_{\bar{1}\bar{2}} + t_{\bar{1}\bar{1}}dt_{\bar{2}\bar{2}})]/(q^3 - 1), \\ \omega^{\bar{2}} &= -q^{-1}(-qt_{\bar{2}\bar{1}}dt_{\bar{1}\bar{1}} + t_{\bar{1}\bar{1}}dt_{\bar{2}\bar{1}}), \\ \omega^{\bar{3}} &= -q^{-1}(t_{\bar{2}\bar{2}}dt_{\bar{1}\bar{2}} - q^{-1}t_{\bar{1}\bar{2}}dt_{\bar{2}\bar{2}}), \\ \omega^{\bar{4}} &= [(q^3 + q^2 - 1)(t_{\bar{2}\bar{2}}dt_{\bar{1}\bar{1}} - q^{-1}t_{\bar{1}\bar{2}}dt_{\bar{2}\bar{1}}) + (-qt_{\bar{2}\bar{1}}dt_{\bar{1}\bar{2}} + t_{\bar{1}\bar{1}}dt_{\bar{2}\bar{2}})]/(q^3 - 1). \end{aligned}$$

By result of [8], the differential calculus provided above on quantum Lorentz group is bicovariant.

Let $\tilde{\mathcal{L}} = S^*(\mathcal{L}^-)\mathcal{L}^+$, i.e. $\mathrm{L} = S^*(L^-)L^+$, $\hat{\mathrm{L}} = S^*(\hat{L}^-)\hat{L}^+$,

$$\tilde{\mathcal{L}} = \begin{pmatrix} \mathrm{L} & \\ & \hat{\mathrm{L}} \end{pmatrix},$$

we have

$$\mathcal{R}\tilde{\mathcal{L}}_1\mathcal{R}^+\tilde{\mathcal{L}}_2 = \tilde{\mathcal{L}}_2\mathcal{R}\tilde{\mathcal{L}}_1\mathcal{R}^+. \tag{2.3}$$

We can also write (2.3) as

$$\begin{aligned} R\mathrm{L}_1R^+\mathrm{L}_2 &= \mathrm{L}_2R\mathrm{L}_1R^+, & (R^+)^{-1}\mathrm{L}_1R^+\hat{\mathrm{L}}_2 &= \hat{\mathrm{L}}_2(R^+)^{-1}\mathrm{L}_1R^+, \\ R\hat{\mathrm{L}}_1R^-\mathrm{L}_2 &= \mathrm{L}_2R\hat{\mathrm{L}}_1R^-, & (R^+)^{-1}\hat{\mathrm{L}}_1R^-\hat{\mathrm{L}}_2 &= \hat{\mathrm{L}}_2(R^+)^{-1}\hat{\mathrm{L}}_1R^-. \end{aligned}$$

Since

$$\mathrm{L}_{ij} = (\lambda\nabla_{ij} + \delta_{ij}\varepsilon), \qquad \hat{\mathrm{L}}_{\bar{i}\bar{j}} = (\lambda\nabla_{\bar{i}\bar{j}} + \delta_{ij}\varepsilon),$$

by the results of [7], we have the commutation relations of derivatives ∇_{ab} $(a,b = 1,2,\bar{1},\bar{2})$ on quantum Lorentz group

$$\nabla_{cc'}\nabla_{dd'} - \nabla_{aa'}\nabla_{bb'}\mathbf{R}^{aa'bb'}_{cc'dd'} = \nabla_{ee'}\mathbf{F}^{ee'}_{cc'dd'}, \tag{2.4}$$

where

$$\begin{aligned} \mathbf{R}^{aa',bb'}_{cc',dd'} &= < \theta_{aa'dd'}, T_{cb}S(T_{b'c'}) >, \\ \mathbf{F}^{aa'}_{bb',cc'} &= \lambda < \nabla_{cc'}, T_{ca}S(T_{a'c'}) >, \end{aligned}$$

$a,a',b,b',c,c',d,d',e,e' = 1,2,\bar{1},\bar{2}$, and $\mathbf{R}$ is a 256×256 matrix satisfying Y-B equation. (2.4) give the commutation relations of ∇_{ab} $(a,b = 1,2,\bar{1},\bar{2})$ which are shown in Table 1.

Table 1

(i)
$$\begin{aligned}
&\nabla_2\nabla_1 - q^2\nabla_1\nabla_2 = (q^2-1)\nabla_2,\\
&\nabla_3\nabla_1 - q^{-2}\nabla_1\nabla_2 = (q^{-2}-1)\nabla_3,\\
&\nabla_4\nabla_1 - \nabla_1\nabla_4 = 0,\\
&\nabla_3\nabla_2 - \nabla_2\nabla_3 + (1-q^{-2})\nabla_4\nabla_1 - (1-q^{-2})\nabla_1\nabla_1 = (1-q^{-2})(\nabla_1-\nabla_4),\\
&\nabla_4\nabla_2 - \nabla_2\nabla_4 + (q^{-2}-1)\nabla_1\nabla_2 = (1-q^{-2})\nabla_2,\\
&\nabla_4\nabla_3 - \nabla_3\nabla_4 + q^{-2}(1-q^{-2})\nabla_1\nabla_3 = -q^{-2}(1-q^{-2})\nabla_3,
\end{aligned}$$

(ii)
$$\begin{aligned}
&\nabla_{\bar 2}\nabla_{\bar 1} - \nabla_{\bar 1}\nabla_{\bar 2} + (q^2-1)\nabla_{\bar 2}\nabla_{\bar 4} = -(q^2-1)\nabla_{\bar 2},\\
&\nabla_{\bar 3}\nabla_{\bar 1} - \nabla_{\bar 1}\nabla_{\bar 3} - (q^2-1)\nabla_{\bar 4}\nabla_{\bar 3} = (q^2-1)\nabla_{\bar 3},\\
&\nabla_{\bar 4}\nabla_{\bar 1} - \nabla_{\bar 1}\nabla_{\bar 4} = 0,\\
&\nabla_{\bar 3}\nabla_{\bar 2} - \nabla_{\bar 2}\nabla_{\bar 3} + (q^2-1)\nabla_{\bar 1}\nabla_{\bar 4} - (q^2-1)\nabla_{\bar 4}\nabla_{\bar 4} = -(q^2-1)(\nabla_{\bar 1}-\nabla_{\bar 4}),\\
&\nabla_{\bar 4}\nabla_{\bar 2} - q^{-2}\nabla_{\bar 2}\nabla_{\bar 4} = (q^{-2}-1)\nabla_{\bar 2},\\
&\nabla_{\bar 4}\nabla_{\bar 3} - q^2\nabla_{\bar 3}\nabla_{\bar 4} = (q^2-1)\nabla_{\bar 3},
\end{aligned}$$

(iii)
$$\begin{aligned}
&\nabla_{\bar 1}\nabla_1 = \nabla_1\nabla_{\bar 1} + q^2(q^2-1)\nabla_3\nabla_{\bar 2},\\
&\nabla_{\bar 1}\nabla_2 = -(q^2-1)\nabla_1\nabla_{\bar 2} + \nabla_2\nabla_{\bar 1} + (q^2-1)\nabla_4\nabla_{\bar 2},\\
&\nabla_{\bar 1}\nabla_3 = \nabla_3\nabla_{\bar 1},\\
&\nabla_{\bar 1}\nabla_4 = -(q^2-1)\nabla_3\nabla_{\bar 2} + \nabla_4\nabla_{\bar 1},\\
&\nabla_{\bar 2}\nabla_1 = \nabla_1\nabla_{\bar 2},\\
&\nabla_{\bar 2}\nabla_2 = q^{-2}\nabla_2\nabla_{\bar 2},\\
&\nabla_{\bar 2}\nabla_3 = q^2\nabla_3\nabla_{\bar 2},\\
&\nabla_{\bar 2}\nabla_4 = \nabla_4\nabla_{\bar 2},\\
&\nabla_{\bar 3}\nabla_1 = \nabla_1\nabla_{\bar 3} - (q^2-1)\nabla_3\nabla_{\bar 1} + (q^2-1)\nabla_3\nabla_{\bar 4},\\
&\nabla_{\bar 3}\nabla_2 = (q^2-1)\nabla_1\nabla_{\bar 1} - (q^2-1)\nabla_1\nabla_{\bar 4} + q^2\nabla_2\nabla_{\bar 3}\\
&\qquad +(q^1-1)(1-q^{-2})\nabla_3\nabla_{\bar 2} - (q^2-1)\nabla_4\nabla_{\bar 1} + (q^2-1)\nabla_4\nabla_{\bar 4},\\
&\nabla_{\bar 3}\nabla_3 = q^{-2}\nabla_3\nabla_{\bar 3},\\
&\nabla_{\bar 3}\nabla_4 = (1-q^{-2})\nabla_3\nabla_{\bar 1} - (1-q^{-2})\nabla_3\nabla_{\bar 4} + \nabla_4\nabla_{\bar 3},\\
&\nabla_{\bar 1}\nabla_1 = \nabla_1\nabla_{\bar 1} - (q^2-1)\nabla_3\nabla_{\bar 2},\\
&\nabla_{\bar 4}\nabla_2 = (1-q^{-2})\nabla_1\nabla_{\bar 2} + \nabla_2\nabla_{\bar 4} - (1-q^{-2})\nabla_4\nabla_{\bar 2},\\
&\nabla_{\bar 1}\nabla_3 = \nabla_3\nabla_{\bar 1},\\
&\nabla_{\bar 4}\nabla_4 = (1-q^{-2})\nabla_3\nabla_{\bar 2} + \nabla_4\nabla_{\bar 4}.
\end{aligned}$$

For simplicity, in Table 1 and following two tables we write "$1,2,3,4,\bar 1,\bar 2,\bar 3,\bar 4$" instead of "$11,12,21,22,\bar 1\bar 1,\bar 1\bar 2,\bar 2\bar 1,\bar 2\bar 2$" respectively.

By result of [7], we also have the deformed Jacobian identities satified by ∇_{ab} and θ_{abcd} together with (2.4)

$$\begin{aligned}
&\nabla_{aa'}\theta_{bb'cc'} = \theta_{bb'dd'}\nabla_{ee'}\mathbf{R}^{dd'ee'}_{aa'cc'},\\
&\mathbf{R}^{aa'bb'}_{ee'ff'}\theta_{ee'cc'}\theta_{ff'dd'} = \theta_{aa'ee'}\theta_{bb'ff'}\mathbf{R}^{ee'ff'}_{cc'dd'},\\
&\mathbf{F}^{aa'}_{dd'ee'}\theta_{dd'bb'}\theta_{ee'cc'} + \theta_{aa'bb'}\nabla_{cc'} = \nabla_{dd'}\theta_{aa'ee'}\mathbf{R}^{dd'ee'}_{bb'cc'} + \theta_{aa'dd'}\mathbf{F}^{dd'}_{bb'cc'}.
\end{aligned}$$

§3. High order differential calculus: $\Omega^\wedge$

As Woronowitz pointed out in [4], the high order differential calculus $\Omega^\wedge$ is constructed as follows

$$\Omega^\wedge = \Omega^\otimes/N, \quad \Omega^k = \Omega^{\otimes k}/N_k,$$

where

$$\Omega^\otimes = \oplus_{n=0}^\infty \Omega^{\otimes n}, \quad \Omega^\otimes = \oplus_{n=0}^k \Omega^{\otimes n}, \quad \Omega^{\otimes 0} = \Omega^0, \quad \Omega^{\otimes 1} = \Omega^1,$$

N and N_k are the two-sided ideal generated by $ker(1-\sigma)$ in $\Omega^\otimes$ and $\Omega^{\otimes k}$ respectively, σ is an automorphism on $\Omega^1\otimes\Omega^1$, compatible with left and right group actions, and also satisfies the braid relation.

By the result of [8], the automorphism σ on $\Omega^1\otimes\Omega^1$ can be constructed as

$$\begin{aligned}
&\sigma(X_{aa'bb'}\omega^{aa'}\otimes\omega^{bb'}) = X_{aa'bb'}\mathbf{R}^{aa'bb'}_{cc'dd'}(\omega^{cc'}\otimes\omega^{dd'}),\\
&\forall X_{aa'bb'}\in\Omega^0,\ a,a',b,b' = 1,2,\bar 1,\bar 2.
\end{aligned}$$

Therefore, we obtain $\Omega^\wedge$, the external algebra of quantum Lorentz group and Ω^k, the bimodules in (0.1) of de Rham complex of quantum Lorentz group. By some calculations, we obtain the minimal polynomial of $\mathbf{R}$ as following,

$$(t-1)(t+1)(t+q^2)(t+q^{-2}),$$

so that N is generated by following elements

$$[(\mathbf{R}+1)(\mathbf{R}+q^2)(\mathbf{R}+q^{-2})]^{aa'bb'}_{cc'dd'}(\omega^{cc'}\otimes\omega^{dd'}), \qquad a,a',b,b',c,c',d,d' = 1,2,\bar 1,\bar 2.$$

Therefore, we obtain an equivalent relation on $\Omega^1\otimes\Omega^1$ which are shown in Table 2.

Table 2

$$
\begin{aligned}
\text{(i)}\quad & \omega^2\wedge\omega^2=\omega^3\wedge\omega^3=\omega^4\wedge\omega^4=0,\\
& \omega^3\wedge\omega^2=-\omega^2\wedge\omega^3,\quad \omega^4\wedge\omega^2=-\omega^2\wedge\omega^4,\\
& \omega^4\wedge\omega^3=-\omega^3\wedge\omega^4,\\
& \omega^2\wedge\omega^1=-q^{-2}\omega^1\wedge\omega^2-q^{-2}(q^{-2}-1)\omega^2\wedge\omega^4,\\
& \omega^3\wedge\omega^1=-q^2\omega^1\wedge\omega^3+(q^{-2}-1)\omega^3\wedge\omega^4,\\
& \omega^4\wedge\omega^1=-\omega^1\wedge\omega^4+(q^{-2}-1)\omega^2\wedge\omega^3,\\
& \omega^1\wedge\omega^1=(1-q^{-2})\omega^2\wedge\omega^3,\\
\text{(ii)}\quad & \omega^{\bar 1}\wedge\omega^{\bar 1}=\omega^{\bar 2}\wedge\omega^{\bar 2}=\omega^{\bar 3}\wedge\omega^{\bar 3}=0,\\
& \omega^{\bar 2}\wedge\omega^{\bar 1}=-\omega^{\bar 1}\wedge\omega^{\bar 2},\quad \omega^{\bar 3}\wedge\omega^{\bar 1}=-\omega^{\bar 1}\wedge\omega^{\bar 3},\\
& \omega^{\bar 3}\wedge\omega^{\bar 2}=-\omega^{\bar 2}\wedge\omega^{\bar 3},\\
& \omega^{\bar 4}\wedge\omega^{\bar 1}=-\omega^{\bar 1}\wedge\omega^{\bar 4}-(q^2-1)\omega^{\bar 2}\wedge\omega^{\bar 3},\\
& \omega^{\bar 4}\wedge\omega^{\bar 2}=-q^2\omega^{\bar 2}\wedge\omega^{\bar 4}-q^2(q^2-1)\omega^{\bar 1}\wedge\omega^{\bar 2},\\
& \omega^{\bar 4}\wedge\omega^{\bar 3}=-q^{-2}\omega^{\bar 3}\wedge\omega^{\bar 4}+(q^2-1)\omega^{\bar 1}\wedge\omega^{\bar 3},\\
& \omega^{\bar 4}\wedge\omega^{\bar 4}=(q^2-1)\omega^{\bar 2}\wedge\omega^{\bar 3},\\
\text{(iii)}\quad & \omega^{\bar 1}\wedge\omega^1=-\omega^1\wedge\omega^{\bar 1}+(1-q^{-2})\omega^2\wedge\omega^{\bar 3},\\
& \omega^{\bar 1}\wedge\omega^2=-\omega^2\wedge\omega^{\bar 1},\\
& \omega^{\bar 1}\wedge\omega^3=-\omega^3\wedge\omega^{\bar 1}-(q^2-1)\omega^1\wedge\omega^{\bar 3}+(1-q^{-2})\omega^4\wedge\omega^{\bar 3},\\
& \omega^{\bar 1}\wedge\omega^4=-\omega^4\wedge\omega^{\bar 1}-(1-q^{-2})\omega^2\wedge\omega^{\bar 3},\\
& \omega^{\bar 2}\wedge\omega^1=-\omega^1\wedge\omega^{\bar 2}-(q^2-1)\omega^2\wedge\omega^{\bar 1}+(1-q^{-2})\omega^2\wedge\omega^{\bar 4},\\
& \omega^{\bar 2}\wedge\omega^2=-q^2\omega^2\wedge\omega^{\bar 2},\\
& \omega^{\bar 2}\wedge\omega^3=(q^2-1)\omega^1\wedge\omega^{\bar 1}-(1-q^{-2})\omega^1\wedge\omega^{\bar 4}-(1-q^{-4})(q^2-1)\omega^2\wedge\omega^{\bar 3}\\
& \qquad -q^{-2}\omega^3\wedge\omega^{\bar 2}+(q^{-2}-1)\omega^4\wedge\omega^{\bar 1}+q^{-4}(q^2-1)\omega^4\wedge\omega^{\bar 4},\\
& \omega^{\bar 2}\wedge\omega^4=(q^2-1)\omega^2\wedge\omega^{\bar 1}+(q^{-2}-1)\omega^2\wedge\omega^{\bar 4}-\omega^4\wedge\omega^{\bar 2},\\
& \omega^{\bar 3}\wedge\omega^1=-\omega^1\wedge\omega^{\bar 3},\\
& \omega^{\bar 3}\wedge\omega^2=-q^{-2}\omega^2\wedge\omega^{\bar 3},\\
& \omega^{\bar 3}\wedge\omega^3=-q^2\omega^3\wedge\omega^{\bar 3},\\
& \omega^{\bar 3}\wedge\omega^4=-\omega^1\wedge\omega^{\bar 3},\\
& \omega^{\bar 4}\wedge\omega^1=-\omega^1\wedge\omega^{\bar 4}+(q^{-2}-1)\omega^2\wedge\omega^{\bar 3},\\
& \omega^{\bar 4}\wedge\omega^2=-\omega^2\wedge\omega^{\bar 4},\\
& \omega^{\bar 4}\wedge\omega^3=(q^2-1)\omega^1\wedge\omega^{\bar 3}-\omega^3\wedge\omega^{\bar 4}+(q^{-2}-1)\omega^4\wedge\omega^{\bar 3},\\
& \omega^{\bar 4}\wedge\omega^4=(1-q^{-2})\omega^2\wedge\omega^{\bar 3}-\omega^4\wedge\omega^{\bar 4}.
\end{aligned}
$$

We can also obtain quantum Maurer-Cartan formulae which are shown in Table 3.

Table 3

$$
\begin{aligned}
d\omega^1 &= q^{-1}\omega^2\wedge\omega^3,\\
d\omega^2 &= q^{-1}\omega^1\wedge\omega^2+q^{-3}\omega^2\wedge\omega^4,\\
d\omega^3 &= -q\omega^1\wedge\omega^3-q^{-1}\omega^3\wedge\omega^4,\\
d\omega^4 &= -q^{-1}\omega^2\wedge\omega^3,\\
d\omega^{\bar 1} &= -q\omega^{\bar 2}\wedge\omega^{\bar 3},\\
d\omega^{\bar 2} &= -q^3\omega^{\bar 1}\wedge\omega^{\bar 2}-q\omega^{\bar 2}\wedge\omega^{\bar 4},\\
d\omega^{\bar 3} &= q\omega^{\bar 1}\wedge\omega^{\bar 3}+q^{-1}\omega^{\bar 3}\wedge\omega^{\bar 4},\\
d\omega^{\bar 4} &= q\omega^2\wedge\omega^3.
\end{aligned}
$$

Based on the quantum Maurer-Cartan formulae, it is possible to construct the de Rham cohomology on quantum Lorentz group which will appear in a seperate paper.

Remark After the present paper was finished, we received a very nice preprint titled "Vector Fields on Complex Quantum Groups" by C. Chryssomalakos, B. Drabant, M. Schlieker, W. Weich and B. Zumino. Although the vector fields related to bicovariant differential calculus on quantum groups including quantum Lorentz group is discussed in this paper, the main results of our present paper on quantum de Rham complex of quantum Lorentz group are still useful.

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