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Solving of the Yang-Baxter Equation (I) [1]

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Abstract

The quantum Yang-Baxter equation is a system of algebraic equations. By using Wu Elimination, it is solved for some special cases. A series solutions of YBE including multiparameters are obtained.

1. The Yang-Baxter equation

The quantum Yang-Baxter equation (YBE) [1,9] plays an important role in many fields of mathematics and theoretical physics such as knot theory, quantum groups and statistical mechanics, integrable systems [3,4].

We set the matrix R and the permutation matrix P to be

$$R = \begin{pmatrix} x_{15} & x_3 & x_1 & x_{11} \\ x_5 & x_{14} & x_{10} & x_2 \\ x_7 & x_9 & x_{13} & x_4 \\ x_{12} & x_8 & x_6 & x_{16} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where x_i are complex variables.

Let $\check{R} = RP$. Then the Yang-Baxter equation is

$$M = (m_{ij}) = \check{R}_{12}\check{R}_{23}\check{R}_{12} - \check{R}_{23}\check{R}_{12}\check{R}_{23} = 0, \qquad \text{(YBE)}$$

where $\check{R}_{12} = \check{R} \otimes I$ and $\check{R}_{23} = I \otimes \check{R}$.

The YBE is a system of algebraic equations consisting of 64 cubic polynomial equations. The Wu Elimination, established by Wu Wen-tsün , provides a perfect method to solve such a system [6,7,8]. By using Wu Elimination, some multiparameter solutions of YBE are presented in [2]. In this paper we study the YBE for a special case . We set

$$x_{14} = x_{13}, \; x_{10} = x_9, \; x_8 = x_6, \; x_7 = x_5, \; x_4 = x_2, \; x_3 = x_1. \tag{1.1}$$

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The matrix R becomes

$$\begin{pmatrix} x_{15} & x_1 & x_1 & x_{11} \\ x_5 & x_{13} & x_9 & x_2 \\ x_5 & x_9 & x_{13} & x_2 \\ x_{12} & x_6 & x_6 & x_{16} \end{pmatrix}.$$

In this case, there are 10 unknowns $x_{16}, x_{15}, x_{13}, x_{12}, x_{11}, x_9, x_6, x_5, x_2, x_1$ in YBE. And the elements m_{ij}, $i = 1,3,6,8; j = 1,3,6,8$ of matrix M are zeros automaticaly. Thus, YBE consists of 48 cubic polynomial equations. It is easy to check that the following relations between the polynomials m_{ij} are valid

$$\begin{aligned}
&m_{12} = -m_{15}, \; m_{17} = -m_{14}, \; m_{21} = -m_{51}, \; m_{41} = -m_{71}, \; m_{28} = -m_{58}, \\
&m_{48} = -m_{78}, \; m_{82} = -m_{85}, \; m_{84} = -m_{87}, \; m_{22} = -m_{55}, \; m_{77} = -m_{44}, \\
&m_{54} = -m_{27}, \; m_{45} = -m_{72}, \; m_{26} = -m_{56}, \; m_{43} = -m_{73}, \; m_{43} = -m_{37}, \\
&m_{62} = -m_{65}, \; m_{23} = m_{76} = -m_{53} = -m_{46}, \; m_{32} = m_{67} = -m_{35} = -m_{64}, \\
&m_{25} = m_{74} = -m_{52} = -m_{47}, \; m_{23} - m_{32} - m_{25} = 0, \\
&m_{48} + m_{24} + m_{12} = 0, \; m_{84} + m_{42} + m_{21} = 0, \; m_{15} + m_{57} + m_{78} = 0, \\
&m_{51} + m_{25} + m_{87} = 0, \; m_{27} - m_{37} - m_{78} = 0, \; m_{72} - m_{62} - m_{21} = 0.
\end{aligned}$$

Therefore, in order to solve the YBE with the assumation (1.1), it is sufficient to solve a system of 16 polynomial equations. These polynomials are

$$\begin{aligned}
m_{12} = \; & x_{15}x_{11}x_5 - x_{15}x_9x_1 + x_{13}x_{11}x_5 - x_{13}x_9x_1 + x_{12}x_{11}x_2 \\
& + x_{12}x_{11}x_1 - x_{11}x_9x_6 - x_{11}x_9x_5 - x_6x_1^2 + x_5x_2x_1, \\
m_{21} = \; & x_{15}x_{12}x_1 - x_{15}x_9x_5 + x_{13}x_{12}x_1 - x_{13}x_9x_5 + x_{12}x_{11}x_6 \\
& + x_{12}x_{11}x_5 - x_{12}x_9x_2 - x_{12}x_9x_1 + x_6x_5x_1 - x_5^2x_2, \\
m_{17} = \; & x_{16}x_{11}x_9 + x_{16}x_1^2 + x_{15}^2x_{11} - x_{15}x_{11}x_9 + 2x_{15}x_2x_1 - x_{15}x_1^2 \\
& - x_{13}^2x_{11} - 2x_{13}x_1^2 - x_{11}x_6x_2 - x_{11}x_6x_1 + x_{11}x_5x_2 + x_{11}x_5x_1, \\
m_{71} = \; & x_{16}x_{12}x_9 + x_{16}x_5^2 + x_{15}^2x_{12} - x_{15}x_{12}x_9 + 2x_{15}x_6x_5 - x_{15}x_5^2 \\
& - x_{13}^2x_{12} - 2x_{13}x_5^2 - x_{12}x_6x_2 + x_{12}x_6x_1 - x_{12}x_5x_2 + x_{12}x_5x_1, \\
m_{28} = \; & x_{16}^2x_{11} - x_{16}x_{11}x_9 - x_{16}x_2^2 + 2x_{16}x_2x_1 + x_{15}x_{11}x_9 + x_{15}x_2^2 \\
& - x_{13}^2x_{11} - 2x_{13}x_2^2 + x_{11}x_6x_2 + x_{11}x_6x_1 - x_{11}x_5x_2 - x_{11}x_5x_1, \\
m_{82} = \; & x_{16}^2x_{12} - x_{16}x_{12}x_9 - x_{16}x_6^2 + 2x_{16}x_6x_5 + x_{15}x_{12}x_9 + x_{15}x_6^2 \\
& - x_{13}^2x_{12} - 2x_{13}x_6^2 + x_{12}x_6x_2 - x_{12}x_6x_1 + x_{12}x_5x_2 - x_{12}x_5x_1, \\
m_{78} = \; & x_{16}x_{11}x_6 - x_{16}x_9x_2 + x_{13}x_{11}x_6 - x_{13}x_9x_2 + x_{12}x_{11}x_2 + x_{12}x_{11}x_1 \\
& - x_{11}x_9x_6 - x_{11}x_9x_5 + x_6x_2x_1 - x_5x_2^2, \\
m_{87} = \; & x_{16}x_{12}x_2 - x_{16}x_9x_6 + x_{13}x_{12}x_2 - x_{13}x_9x_6 + x_{12}x_{11}x_6 + x_{12}x_{11}x_5 \\
& - x_{12}x_9x_2 - x_{12}x_9x_1 - x_6^2x_1 + x_6x_5x_2, \\
m_{23} = \; & x_{13}x_{12}x_{11} - x_{13}x_9^2 + x_{13}x_6x_1 - x_{13}x_5x_2 + x_{12}x_1^2 + x_{11}x_6^2 \\
& - x_9x_6x_2 - x_9x_5x_1, \\
m_{32} = \; & x_{13}x_{12}x_{11} - x_{13}x_9^2 - x_{13}x_6x_1 + x_{13}x_5x_2 + x_{12}x_2^2 + x_{11}x_5^2 \\
& - x_9x_6x_2 - x_9x_5x_1,
\end{aligned}$$

$$
\begin{aligned}
m_{27} = \; & x_{16}x_{13}x_1 + x_{16}x_{11}x_6 + x_{15}x_{13}x_2 + x_{15}x_{11}x_5 - x_{13}^2x_2 - x_{13}^2x_1 \\
& -x_{13}x_9x_2 - x_{13}x_9x_1 + x_{12}x_{11}x_2 + x_{12}x_{11}x_1 - x_{11}x_9x_6 - x_{11}x_9x_5 \\
& -x_6x_2^2 + x_6x_2x_1 + x_5x_2x_1 - x_5x_1^2, \\
m_{72} = \; & x_{16}x_{13}x_5 + x_{16}x_{12}x_2 + x_{15}x_{13}x_6 + x_{15}x_{12}x_1 - x_{13}^2x_6 - x_{13}^2x_5 \\
& -x_{13}x_9x_6 - x_{13}x_9x_5 + x_{12}x_{11}x_6 + x_{12}x_{11}x_5 - x_{12}x_9x_2 - x_{12}x_9x_1 \\
& -x_6^2x_2 + x_6x_5x_2 + x_6x_5x_1 - x_5^2x_1, \\
m_{26} = \; & x_{16}x_{13}x_1 + x_{16}x_{11}x_6 + x_{15}x_{13}x_2 + x_{15}x_9x_1 - x_{13}^2x_2 - x_{13}^2x_1 \\
& -x_{13}x_{11}x_5 - x_{13}x_9x_2 - x_6x_2^2 + x_6x_2x_1 + x_6x_1^2 - x_5x_1^2, \\
m_{73} = \; & x_{16}x_{13}x_5 + x_{16}x_9x_6 + x_{15}x_{13}x_6 + x_{15}x_{12}x_1 - x_{13}^2x_6 - x_{13}^2x_5 \\
& -x_{13}x_{12}x_2 - x_{13}x_9x_5 - x_6^2x_2 + x_6^2x_1 + x_6x_5x_1 - x_5^2x_1, \\
m_{22} = \; & x_{16}x_{12}x_{11} + x_{16}x_5x_1 + x_{15}^2x_9 - x_{15}x_9^2 + x_{15}x_6x_1 + x_{15}x_5x_2 \\
& -x_{15}x_5x_1 - x_{13}^2x_9 - 2x_{13}x_5x_1 + x_{12}x_2x_1 + x_{11}x_6x_5 - x_9x_6x_2 \\
& -x_9x_6x_1 - x_9x_5x_2 + x_9x_5x_1, \\
m_{77} = \; & x_{16}^2x_9 - x_{16}x_9^2 - x_{16}x_6x_2 + x_{16}x_6x_1 + x_{16}x_5x_2 + x_{15}x_{12}x_{11} \\
& +x_{15}x_6x_2 - x_{13}^2x_9 - 2x_{13}x_6x_2 + x_{12}x_2x_1 + x_{11}x_6x_5 + x_9x_6x_2 \\
& -x_9x_6x_1 - x_9x_5x_2 - x_9x_5x_1.
\end{aligned}
$$

We denote the set of these polynomials by YS .

2. The decomposition of the zeros of YS

Let K be the complex field. We denote the polynomial set $\{p_1, p_2, ...p_n\}$ by PS where p_i are polynomials in the ring $K[x_1, x_2, ..., x_{16}]$. The system of $p_i = 0, i = 1, 2, ..., n$ is denoted by $PS = 0$. The set of all zeros of PS , i.e., the solution set of $PS = 0$, is denoted by $Zero(PS)$. The set of zeros of PS satisfying $I \neq 0$ is denoted by $Zero(PS/I)$.

The charactersitic set of a polynomial set plays a profound role in Wu Elimination. Since a characteristic set is in triangular form its zeros are well-obtained. Furthermore, the zeros of a polynomial set can be determined by some characteristic sets. The conclusions are

Theorem 1.(Wu Wen-tsun) If $CS = \{C_1, C_2, ..., C_r\}$ is the characteristic set of a polynomial set PS , then the zero set of PS has the following decomposition

$$Zero(PS) = Zero(CS/I) + \sum_i Zero(PS, I_i), \tag{2.1}$$

where I_i are the initials of C_i and I is the product of all I_i .

Theorem 2.(Wu Wen-tsun) The zero set of a polynomial set PS has the following decomposition

$$Zero(PS) = \sum_j Zero(CS_j/I_j), \tag{2.2}$$

in which CS_j are some characteristic sets.

Theorem 3.(Wu Wen-tsun) By eliminating, the characteristic set of a polynomial set can be obtained in a finit number of steps.

In our case, for the polynomial set YS which defined in preceeding section, we do first the following two eliminating procedures.

Procedure 1. p(1)=m_{12}; p(2)=m_{21}; p(3)=m_{78}; p(4)=m_{87}; p(5)=p(1)-p(3); p(6)=p(2)-p(4); p(7)=Rem(p(5),p(6)); p(8)=$lcof(p(5), x_{15})$); p(9)=$lcof(p(6), x_{15})$); p(10)=p(8)p(6)-p(9)p(5); p(11)=$x_6x_1 - x_5x_2$; p(12)= p(7)/p(11); p(13)= p(10)/p(11).

Procedure 2. p(1)=m_{12}; p(2)=m_{21}; p(3)=m_{78}; p(4)=m_{87}; p(15)=Rem(p(1),p(2)); p(16)=Rem(p(3),p(4)); p(17)=Rem(p(15),p(16)); p(18)=$x_6x_1-x_5x_2$; p(19)=p(17)/p(18); p(20)=$x_{11}(x_2 + x_1)$; p(21)=p(19)/p(20).

We obtain three polynomials $p(7) = p(11) * p(12)$, $p(10) = p(11) * p(13)$ and $p(17) = p(11) * p(20) * p(21)$, in which

$$
\begin{aligned}
p(12) = \; & x_{15}x_{12}x_{11} - x_{15}x_9^2 + x_{13}x_{12}x_{11} - x_{13}x_9^2 + x_{12}x_2^2 + x_{12}x_2x_1 + x_{11}x_6^2 \\
& +x_{11}x_6x_5 - 2x_9x_6x_2 - x_9x_6x_1 - x_9x_5x_2, \\
p(13) = \; & x_{16}x_{12}x_{11} - x_{16}x_9^2 + x_{13}x_{12}x_{11} - x_{13}x_9^2 + x_{12}x_2x_1 + x_{12}x_1^2 \\
& +x_{11}x_6x_5 + x_{11}x_5^2 - x_9x_6x_1 - x_9x_5x_2 - 2x_9x_5x_1, \\
p(21) = \; & x_{12}x_{11}^2x_6 + x_{12}x_{11}^2x_5 - x_{12}x_2^2x_1 - x_{12}x_2x_1^2 - x_{11}x_9^2x_6 - x_{11}x_9^2x_5 \\
& -x_{11}x_6^2x_1 - x_{11}x_5^2x_2 + 2x_9x_6x_2x_1 + 2x_9x_5x_2x_1.
\end{aligned}
$$

Therefore, we have

Proposition. The zero set of YS , i.e., the set of solutions of the system $YS = 0$ can be decomposed into two parts: (I) the zeros of $\{YS, p(11)\}$ and (II) the zeros of $\{YS, p(12), p(13), p(21)\}$.

In this paper we study only the first case (I),that is the zeros of $\{YS, p(11)\}$ and we present all the solutions of it with the hypothesis formulated below. The second case, that is the zeros of $\{YS, p(12), p(13), p(21)\}$ is studied in [5].

The calculusion is much simpler when some unknowns are zero. But the discussion is lengthier if we try to list it case by case. We will leave this discussion in another paper. In this paper we take the following agreement

Hypothesis: The all unknowns are not zeros.

By this hypothesis, the factors of unknowns power will be omited in next sections.

3. Some solutions of YBE

Now we study the zeros of $\{YS, x_6x_1 - x_5x_2\}$. Note that

$$m_{25} = m_{23} - m_{32} = 2x_{13}x_6x_1 - 2x_{13}x_5x_2 - x_{12}x_2^2 + x_{12}x_1^2 + x_{11}x_6^2 - x_{11}x_5^2.$$

The remander of m_{25} to $(x_6x_1 - x_5x_2)$ is $(x_2 - x_1) * (x_2 + x_1) * (x_{12}x_1^2 - x_{11}x_2^2)$. Therefore the zeros of $\{YS, x_6x_1 - x_5x_2\}$ can be decomposed into the zeros of three polynomial sets

$\{YS, x_6 - x_5, x_2 - x_1\}$, $\{YS, x_6 + x_5, x_2 + x_1\}$ *and* $\{YS, x_6x_1 - x_5x_2, x_{12}x_1^2 - x_{11}x_2^2\}$. According to Wu elimination, in order to get the zeros of these polynomial sets it is sufficient to obtain the characteristic sets of them. We do this case by case.

3. 1.The case I_a. We consider the polynomial set $YS_a = \{YS, A_1, A_2\}$ where $A_1 = x_6 - x_5, A_2 = x_2 - x_1$.

By eliminating, from the polynomials $m_{12}, m_{21}, m_{78}, m_{87}$, we obtaine a polynomial $A_3 * A_4$, where $A_3 = x_{12}x_{11} - x_9^2$, $A_4 = x_{12}x_1^2 - x_9x_5^2$. Thus the zeros of $YS_a = \{YS, A_1, A_2\}$ can be decomposed into the zeros of two polynomial sets $\{YS_a, A_3\}$ and $\{YS_a, A_4\}$

For the polynomial set $\{YS_a, A_3\}$, from the remander of m_{23} to A_3 we get two polynomials $A_5 = x_{12}x_1 - x_9x_5$, $A_6 = x_{11}x_5 - x_9x_1$. Thus the zeros of $\{YS_a, A_3\}$ equals the zeros of $\{YS_a, A_5, A_6\}$. The remanders of m_{27}, m_{72} to the basic set $\{A_5, A_6\}$ are factorized as $A_7 * A_8$, where $A_7 = x_{13} + x_9$, $A_8 = x_{16} + x_{15} - 2x_{13}$. The zeros of $\{YS_a, A_5, A_6\}$ can be decomposed into the zeros of $\{YS_a, A_5, A_6, A_7\}$ and $\{YS_a, A_5, A_6, A_8\}$.

From the remanders of $m_{17}, m_{71}, m_{28}, m_{82}, m_{22}, m_{77}$ to the basic set $\{A_8, A_5, A_6\}$ we obtain the polynomial $A_9 = x_{15} - x_{13}$. The characteristc set of $\{YS_a, A_5, A_6, A_8\}$ is

$$CS_1(I_a) = \{A_8, A_9, A_5, A_6, A_1, A_2\}. \tag{3.a.1}$$

By eliminating the remanders of m_{22}, m_{77} to the basic set $\{A_7, A_5, A_6\}$, we obtain four polynomials

$$\begin{aligned}
A_{10} &= x_{16}x_9^2 + x_{16}x_5x_1 + x_{15}^2x_9 - x_{15}x_9^2 + x_{15}x_5x_1 - x_9^3 + 2x_9x_5x_1,\\
A_{11} &= x_{15} + x_9,\\
A_{12} &= x_{15}x_9 - x_9^2 + 2x_5x_1,\\
A_{13} &= x_{15}^2 - 2x_{15}x_9 + x_9^2 + 4x_5x_1.
\end{aligned}$$

Therefore, the characteristc sets of $\{YS_a, A_5, A_6, A_7\}$ are

$$CS_2(I_a) = \{A_{10}, A_{11}, A_5, A_6, A_1, A_2\}, \tag{3.a.2}$$

$$CS_3(I_a) = \{A_{10}, A_{12}, A_5, A_6, A_1, A_2\}, \tag{3.a.3}$$

$$CS_4(I_a) = \{A_{10}, A_{13}, A_5, A_6, A_1, A_2\}. \tag{3.a.4}$$

The initial of A_{10} is $A_{14} = x_9^2 + x_5x_1$. The polynomial set $\{YS_a, A_5, A_6, A_7, A_{14}\}$ is obtained by adjointing the polynomial A_{14} into $\{YS_a, A_5, A_6, A_7\}$. And the characteristic set of it is

$$CS_5(I_a) = \{A_{16}, A_{15}, A_5, A_6, A_{14}, A_1, A_2\}, \tag{3.a.5}$$

where $A_{16} = x_{16}^2x_9 + 2x_{16}x_5x_1 + 3x_9x_5x_1$ and $A_{15} = x_{15}^2x_9 + 2x_{15}x_5x_1 + 3x_9x_5x_1$.

For the polynomial set $\{YS_a, A_4\}$, the remanders of m_{17}, m_{71}, m_{28}, m_{82}, m_{27}, m_{72}, m_{22}, m_{77}, to the basic set $\{m_{87}, m_{21}, m_{23}, A_4\}$ are

$$A_{17} * A_{18} * A_{20}, \quad A_{17} * A_{18} * A_{21}, \quad A_{17} * A_{18} * A_{22},$$

in which

$$\begin{aligned}
A_{18} &= x_{11}x_5 - x_9x_1,\\
A_{17} &= x_{11}^2x_5 + x_{11}x_9x_1 - 2x_1^3,\\
A_{20} &= x_{11}^2x_5 - x_1^3,\\
A_{21} &= x_{11}^2x_5^2 + 2x_{11}x_9x_5x_1 + x_9^1x_1^2 - 4x_5x_1^3,\\
A_{22} &= x_{11}^2x_5^2 - 2x_{11}x_9x_5x_1 - x_9^2x_1^2 + 2x_5x_1^3.
\end{aligned}$$

By eliminating A_{20}, A_{21}, A_{22} we obtain following two polynomials $A_{23} = 3x_{11}x_5 + x_9x_1$, $A_{24} = x_9^2 - x_5x_1$. Therefore, the characteistic set of $\{YS_a, A_4\}$ are

$$CS_6(I_a) = \{m_{87}, m_{21}, m_{23}, A_4, A_{17}, A_1, A_2\}, \tag{3.a.6}$$

$$CS_7(I_a) = \{m_{87}, m_{21}, m_{23}, A_4, A_{18}, A_1, A_2\}, \tag{3.a.7}$$

$$CS_8(I_a) = \{m_{87}, m_{21}, m_{23}, A_4, A_{23}, A_{24}, A_1, A_2\}. \tag{3.a.8}$$

The initials of m_{87}, m_{21}, m_{23} are $A_{25} = x_{12}x_1 - x_9x_5$, $A_{26} = x_{12}x_{11} - x_9^2$. The polynomial set $\{YS_a, A_4, A_{25}, A_{26}\}$ is obtained by adjointing them into $\{YS_a, A_4\}$. By eliminating A_4, A_{25}, A_{26}, we obtain following polynomials

$$A_{27} = x_{12} - x_9, \quad A_{28} = x_{11} - x_9, and\ A_{29} = x_5 - x_1.$$

The remander of m_{72} to the basic set $\{A_{27}, A_{28}, A_{29}, A_2, A_1\}$ is $A_{30} * A_{31}$, where $A_{30} = x_{13} + x_9$, $A_{31} = x_{16} + x_{15} - 2x_{13}$. Thus teh zeros of $\{YS_a, A_4, A_{25}, A_{26}\}$ can be decomposed into the zeros of polynomial sets $\{YS_a, A_4, A_{25}, A_{26}, A_{30}\}$ and $\{YS_a, A_4, A_{25}, A_{26}, A_{31}\}$. Similarly to the above, we get the characteristic sets of $\{YS_a, A_4, A_{25}, A_{26}, A_{30}\}$. They are

$$CS_1((I_a) = \{A_{32}, A_{33}, A_{30}, A_{27}, A_{28}, A_{29}, A_1, A_2\}, \tag{3.a.9}$$

$$CS_{10}(I_a) = \{A_{32}, A_{34}, A_{30}, A_{27}, A_{28}, A_{29}, A_1, A_2\}, \tag{3.a.10}$$

$$CS_{11}(I_a) = \{A_{32}, A_{35}, A_{30}, A_{27}, A_{28}, A_{29}, A_1, A_2\}, \tag{3.a.11}$$

in which

$$\begin{aligned}
A_{32} &= x_{16}x_9^2 + x_{16}x_1^2 + x_{15}^2x_9 - x_{15}x_9^2 + x_{15}x_1^2 - x_9^3 + 2x_9x_1^2,\\
A_{33} &= x_{15} + x_9,\\
A_{34} &= x_{15}x_9 - x_9^2 + 2x_1^2,\\
A_{35} &= x_{15}^2 - 2x_{15}x_9 + x_9^2 + 4x_1^2.
\end{aligned}$$

And the characteristic sets of $\{YS_a, A_4, A_{25}, A_{26}, A_{31}\}$ are

$$CS_{12}(I_a) = \{A_{36}, A_{37}, A_{27}, A_{28}, A_{29}, A_1, A_2\}, \tag{3.a.12}$$

$$CS_{13}(I_a) = \{A_{36}, A_{38}, A_{27}, A_{28}, A_{29}, A_1, A_2\}, \tag{3.a.13}$$

where $A_{36} = x_{16} + x_{15} - 2x_9$, $A_{37} = x_{15} - x_{13}$, and $A_{38} = x_{13} - x_9$.

The initial of A_{32} is $A_{40} = x_9^2 + x_1^2$. The polynomial set $\{YS_a, A_4, A_{25}, A_{26}, A_{30}, A_{40}\}$ is obtained by adjointing A_{40} into $\{YS_a, A_4, A_{25}, A_{26}, A_{30}\}$. The characteristic set of it is

$$CS_{14}(I_a) = \{A_{41}, A_{42}, A_{30}, A_{27}, A_{28}, A_{29}, A_{40}, A_1, A_2\}, \qquad (3.a.14)$$

where $A_{41} = x_{16}^2 x_9 + 2x_{16}x_1^2 + 3x_9x_1^2$, $A_{42} = x_{15}^2 x_9 + 2x_{15}x_1^2 + 3x_9x_1^2$.

3.2. The case I_b. We provide the characteristic set of the polynomial set $YS_b = \{YS, B_1, B_2\}$ where $B_1 = x_6 + x_5$, $B_2 = x_2 + x_1$.

If we set $x_2 = -x_1, x_6 = -x_5$, then we have $m_{12} = B_4 * B_6$, $m_{21} = B_4 * B_5$, $m_{78} = -B_3 * B_6$, $m_{87} = -B_3 * B_5$, where

$$B_3 = x_{16} + x_{13},\ B_4 = x_{15} + x_{13},\ B_5 = x_{12}x_1 - x_9x_5,\ B_6 = x_{11}x_5 - x_9x_1.$$

The zeros of YS_b cab be decomposed into the zeros of polynomial sets $\{YS_b, B_3, B_4,\}$ and $\{YS_b, B_5, B_6.\}$

In the polynomial set YS_b there is only one polynomial which is not zero if we set $x_{16} = -x_{15}, x_{15} = -x_{13}$ and $x_6 = -x_5, x_2 = -x_1$. It is $B_7 = x_{13}(x_{12}x_{11} - x_9^2) + x_{12}x_1^2 + x_{11}x_5^2 - 2x_9x_5x_1$. Therefore the characterristic set of $\{YS_b, B_3, B_4,\}$ is

$$CS_1(I_b) = \{B_3, B_4, B_7, B_1, B_2\}. \qquad (3.b.1)$$

The initial of B_7 is $B_8 = x_{12}x_{11} - x_9^2$. For the polynomial set $\{YS_b, B_3, B_4, B_8\}$, the characteristic set is

$$CS_2(I_b) = \{B_3, B_4, B_8, B_9, B_1, B_2\}, \qquad (3.b.2)$$

in which $B_8 = x_{12}x_1 - x_9x_5$, $B_9 = x_{11}x_5 - x_9x_1$.

For the polynomial set $\{YS_b, B_5, B_6\}$, the remanders of m_{27}, m_{72} to the basic set $\{B_5, B_6\}$ are the same polynomial $B_{10} * B_{11}$, where $B_{10} = x_{16} - x_{13}$, $B_{11} = x_{13} - x_9$. Thus, the zeros of $\{YS_b, B_5, B_6\}$, can be decomposed as the zeros of polynomial sets $\{YS_b, B_5, B_6, B_{10}\}$, and $\{YS_b, B_5, B_6, B_{11}\}$.

The remanders of $m_{12}, m_{21}, m_{28}, m_{82}, m_{22}, m_{77}$ to the basic set $\{B_{10}, B_5, B_6, B_1, B_2\}$, can be all factorized as $B_{12} * B_{13}$, where $B_{12} = x_{15}x_9 - x_{13}x_9 - 2x_5x_1$, $B_{13} = x_{15} + x_{13}$. Therefore, two characteristic sets of $\{YS_b, B_5, B_6, B_{10}\}$, are obtained. They are

$$CS_3(I_b) = \{B_{10}, B_{12}, B_5, B_6, B_1, B_2\}, \qquad (3.b.3)$$

$$CS_4(I_b) = \{B_{10}, B_{13}, B_5, B_6, B_1, B_2\}. \qquad (3.b.4)$$

The remanders of $m_{12}, m_{21}, m_{28}, m_{82}, m_{22}, m_{77}$ to the basic set $\{B_{11}, B_5, B_6, B_1, B_2\}$ are only two polynomials of x_{16}. By eliminating them, we get two characteristic set of $\{YS_b, B_5, B_6, B_{11}\}$. They are

$$CS_5(I_b) = \{B_{14}, B_{15}, B_5, B_6, B_1, B_2\}, \qquad (3.b.5)$$

$$CS_6(I_b) = \{B_{14}, B_{16}, B_5, B_6, B_1, B_2\}, \qquad (3.b.6)$$

in which

$$B_{14} = x_{16}(x_9^2 + x_5x_1) + x_{15}^2x_9 - x_{15}x_9^2 - 3x_{15}x_5x_1 - x_9^3 - 2x_9x_5x_1,$$
$$B_{15} = x_{15}x_9 - x_9^2 - 2x_5x_1, \qquad B_{16} = x_{15} + x_9.$$

The initial of B_{14} is $B_{17} = x_9^2 + x_5x_1$. The polynomial set $\{YS_b, B_5, B_6, B_{11}, B_{17}\}$ is obtained by adjointing B_{17} into $\{YS_b, B_5, B_6, B_{11}\}$. The characteristic set of $\{YS_b, B_5, B_6, B_{11}, B_{17}\}$ is

$$CS_7(I_b) = \{B_{18}, B_{19}, B_5, B_6, B_{17}, B_1, B_2\}, \qquad (3.b.7)$$

in which

$$\begin{aligned} B_{19} &= x_{15}^2x_9 - 2x_{15}x_5x_1 - x_9x_5x_1, \\ B_{18} &= x_{16}^2x_9 - 2x_{16}x_5x_1 - x_9x_5x_1. \end{aligned}$$

3.3. The case I_c. We consider the polynomial set $YS_c = \{YS, C_1, C_2\}$ where $C_1 = x_{12}x_1^2 - x_{11}x_5^2$, $C_2 = x_6x_1 - x_5x_2$. The remanders of $m_{12}, m_{21}, m_{78}, m_{87}, m_{26}, m_{73}$ to the basic set $\{C_1, C_2\}$ are $C_3 * C_4, C_3 * C_5, C_3 * C_6$, where

$$\begin{aligned} C_3 &= x_{11}x_5 - x_9x_1, \\ C_4 &= x_{16}x_2x_1 + x_{13}x_2x_1 + x_{11}x_5x_2 + x_{11}x_5x_1, \\ C_5 &= x_{15}x_1^2 + x_{13}x_1^2 + x_{11}x_5x_2 + x_{11}x_5x_1, \\ C_6 &= x_{13}x_{11}x_5 + x_{13}x_9x_1 + x_5x_2^2 + x_5x_1^2. \end{aligned}$$

Thus the zeros of YS_c cab be decomposed into the zeros of polynomial sets $\{YS_c, C_3\}$ and $\{YS_c, C_4, C_5, C_6\}$.

For $\{YS_c, C_3\}$, there are three remanders $c(31), c(32), c(33)$ of $m_{17}, m_{71}, m_{28}, m_{82}, m_{22}, m_{77}, m_{27}, m_{72}$ to the basic set $\{C_1, C_3, C_2\}$. By eliminating, from them we get two polynomials $C_7 * C_8, C_7 * C_9$, where

$$C_7 = x_{15}x_9 - x_{13}x_9 + x_5x_2 - x_5x_1,$$
$$C_9 = x_{15}x_{13}x_1 + x_{15}x_9x_2 + x_{13}^2x_1 - x_{13}x_9x_1 - x_9^2x_2 - x_9^2x_1 + x_5x_2^2 - x_5x_1^2.$$

Thus we get the first characteristic set of polynomial set $\{YS_c, C_3\}$

$$CS_1(I_c) = \{c(33), C_7, C_1, C_3, C_2\}, \qquad (3.c.1)$$

in which

$$\begin{aligned} c(33) = \ & x_{16}x_{13}x_1^2 + x_{16}x_9x_2x_1 + x_{15}x_{13}x_2x_1 + x_{15}x_9x_1^2 - x_{13}^2x_2x_1 - x_{13}^2x_1^2 \\ & -x_{13}x_9x_2x_1 - x_{13}x_9x_1^2 - x_5x_2^3 + x_5x_2^2x_1 + x_5x_2x_1^2 - x_5x_1^3. \end{aligned}$$

The remander $Rem(C_8, C_9)$ can be factorized as $C_{10} * C_{11} * C_{12}$ where

$$\begin{aligned} C_{10} &= x_{13}^2x_1 + 2x_{13}x_9x_1 + x_9^2x_1 + x_5x_2^2 - 2x_5x_2x_1 + x_5x_1^2, \\ C_{11} &= x_{13} - x_9, \qquad C_{12} = 2x_{13}x_2x_1 + x_9x_2^2 + x_9x_1^2. \end{aligned}$$

Thus we get the characteristic sets of $\{YS_c, C_3\}$

$$CS_2(I_c) = \{c(33), C_9, C_{10}, C_1, C_3, C_2\}, \tag{3.c.2}$$

$$CS_3(I_c) = \{c(33), C_9, C_{11}, C_1, C_3, C_2\}, \tag{3.c.3}$$

$$CS_4(I_c) = \{c(33), C_9, C_{12}, C_1, C_3, C_2, \}. \tag{3.c.4}$$

The inatial of $c(33)$ is $C_{13} = x_{13}x_1 + x_9x_2$. The characteristic set of the polynomial set $\{YS_c, C_3, C_{13}\}$ is

$$CS_5(I_c) = \{C_{14}, C_{15}, C_{13}, C_1, C_3, C_2\}, \tag{3.c.5}$$

in which

$$\begin{aligned} C_{14} = \ & x_{16}x_9^2x_1^2 + x_{16}x_5x_1^3 + x_{15}^2x_9x_1^2 - x_{15}x_9^2x_1^2 + 2x_{15}x_5x_2x_1^2 - x_{15}x_5x_1^3 \\ & -x_9^3x_2^2 - x_9x_5x_2^2x_1 + 2x_9x_5x_2x_1^2 + x_9x_5x_1^3, \\ C_{15} = \ & x_{15}x_9x_2^2x_1 - x_{15}x_9x_1^3 + x_9^2x_2^3 - x_9^2x_2x_1^2 + x_5x_2^3x_1 - x_5x_2^2x_1^2 - x_5x_2x_1^3 + x_5x_1^4. \end{aligned}$$

And the initials of C_{14}, C_{15} are $C_{16} = x_9^2 + x_5x_1$, $C_{17} = x_2^2 - x_1^2$. C_{17} is omited for it can be introduced to case I_a or I_b. Similarly to above, the characteristic set of $\{YS_c, C_3, C_{13}, C_{16}\}$ is

$$CS_6(I_c) = \{C_{18}, C_{19}, C_{13}, C_1, C_3, C_2\}, \tag{3.c.6}$$

in which

$$\begin{aligned} C_{18} = \ & x_{16}^2x_1^2 + x_{16}x_9x_2^2 - 2x_{16}x_9x_2x_1 - x_{16}x_9x_1^2 + 2x_5x_2^3 + x_5x_2^2x_1, \\ C_{19} = \ & x_{15}x_9 - x_5x_1. \end{aligned}$$

By eliminating, the zero of polynomial set $\{YS_c, C_4, C_5, C_6\}$ can be decomposed into the zeros of $\{YS_c, C_4, C_5, C_6, C(61)\}$ and $\{YS_c, C_4, C_5, C_6, C(62), C(63), C(64), C(65)\}$ in which

$$\begin{aligned} C(61) = \ & x_{11}^2x_5 + x_{11}x_9x_1 - 2x_2x_1^2, \\ C(62) = \ & x_{11}^2x_5x_2^2 + x_{11}^2x_5x_2x_1 + x_{11}x_9x_2x_1^2 - x_{11}x_9x_1^3 - x_2^2x_1^3 - x_1^5, \\ C(63) = \ & -(x_{11}^2x_5x_2 + x_{11}^2x_5x_1 - x_{11}x_9x_2^2 + x_{11}x_9x_2x_1 - x_2^4 - x_2^2x_1^2), \\ C(64) = \ & x_{11}^2x_5^2x_2 + 2x_{11}x_9x_5x_2x_1 + x_9^2x_2x_1^2 - x_5x_2^4 - 2x_5x_2^2x_1^2 - x_5x_1^4, \\ C(65) = \ & -(x_{11}^2x_5^2x_1 - x_{11}x_9x_5x_2^2 - x_{11}x_9x_5x_2x_1 - x_9^2x_2x_1^2 + x_5x_2^2x_1^2 + x_5x_1^4). \end{aligned}$$

The characteristic set of $\{YS_c, C_4, C_5, C_6, C(61)\}$ is

$$CS_7(I_c) = \{C_4, C_5, C_6, C(61), C_1, C_2\}. \tag{3.c.7}$$

The characteristic sets of $\{YS_c, C_4, C_5, C_6, C(62), C(63), C(64), C(65)\}$ are

$$CS_8(I_c) = \{C_4, C_5, C_6, C_{20}, C_{21}, C_1, C_2\}, \tag{3.c.8}$$

$$CS_9(I_c) = \{C_4, C_5, C_6, C_1, C_{24}, C_2, C_{23}\}, \tag{3.c.9}$$

in which

$$\begin{aligned} C_{20} &= x_{11}x_9 + x_2^2 + x_2x_1 + x_1^2, \\ C_{21} &= x_9^2x_2x_1^2 - x_5x_2^4 - 2x_5x_2^3x_1 - 3x_5x_2^2x_1^2 - 2x_5x_2x_1^3 - x_5x_1^4, \\ C_{23} &= x_2^2 + x_1^2, \qquad C_{24} = x_{11}x_5 + x_9x_1. \end{aligned}$$

Summing up the study above, the main result of this paper is

Theorem I. Assume the unknowns x_i are not zeros. All zeros of polynomial set $\{YS, x_6x_1 - x_5x_2\}$, i.e., the first part of all solutions of the YBE with the condition (1.1), are provided by the characteristic sets in formulae (3.a.1–14), (3.b.1–7) and (3.c.1–9).

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