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Covariant Differential Calculus on Quantum Minkowski Space and The q-Analogue of Dirac Equation [1]

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Abstract

The covariant differential calculus on the quantum Minkowski space is presented with the help of the generalized Wess-Zumino method and the quantum Pauli matrices and quantum Dirac matrices are constructed parallel to those in the classical case. Combining these two aspects a q-analogue of Dirac equation follows directly.

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I. Introduction

In recent years, the concept of quantum group has extensively entered the physical literatures. This is due to its close connection to the Yang-Baxter equation which plays a deep role in various physically interesting models and theories. The mathematical structure of the quantum group has been progressively clarified by Drinfeld [1], Jimbo[2] and Reshetikhin *et al* [3]. The connection with noncommutative differential calculus was elaborated by Woronovicz[4]. A general construction of quantum group as linear transformations upon the quantum plane was suggested by Manin[5]. And the differential calculus on the quantum plane was developed by Wess and Zumino[6].

On the other hand, it is also interesting in considering the deformation of our physical space-time. It seems that the four-dimensional smooth manifold is a good model of our space-time only in the macro-scale. In the sub-macro level, the description of the space-time may need new tools. In this direction the idea that the symmetry of the space-time in the sub-macro scale is described by a quantum group is very attractive. The first step was put forward by Podles and Woronovicz[7], who constructed a one-parameter deformation of the Lorentz group and discussed various aspects about its properties. The tensor representation of the quantum group $SL_q(2,C)$ was investigated by Carow-Watamura et al [8], and the quantum Minkowski space M_q was defined, upon which the quantum Lorentz group $SO_q(3,1)$ is acting.

In a recent paper[9], with the help of the quantum Pauli matrices we introduced and the projection operator method, the covariant spinor analysis of $SU_q(2)$ was developed. These methods are taken over in Section 1 and 2 to discuss the $SL_q(2,C)$ and quantum Minkowski space M_q without any difficulty. Then by using the approach proposed by Wess and Zumino [6] and generalized in recent papers [10,11], we discuss the differential calculus on M_q in Section 3. Combining these two results we construct the q-deformation of the Dirac equation in Section 4 in a manifest way, which is, step by step, parallel to the one in the classical case and hopefully more familiar to most of physicists.

The quantum group $SL_q(2,C)$ is characterized by the $\check{R}$ matrix satisfying the Yang-Baxter equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{1}$$

and the characteristic equation

$$(\check{R} - q)(\check{R} + q^{-1}) = 0 \,. \tag{2}$$

For convenience we will take the standard form of $\check{R} = PR$ as given in Ref[3]

$$(\check{R}) = \check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} . \tag{3}$$

The left-acting as well as the right-acting eigenvalue equation can be written as [9] (summation over repeated indices is understood unless otherwise stated)

$$\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} t_m(q)^{\gamma\delta} = q\, t_m(q)^{\alpha\beta}, \quad \check{R}(q)^{\alpha\beta}{}_{\gamma\delta} s(q)^{\gamma\delta} = -q^{-1} s(q)^{\alpha\beta},$$

$$\bar{t}^m(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = q\, \bar{t}^m(q)_{\gamma\delta}, \quad \bar{s}(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = -q^{-1}\bar{s}(q)_{\gamma\delta} . \tag{4}$$

with $m = +, 3, -$ and $\alpha,\ \beta = 1, 2$. Now since the matrix $\check{R}$ is symmetric, the components of $\bar{t}^m(q)$ and $\bar{s}^m(q)$ may be taken the same as those of their left-acting counterparts $t_m(q)$ and $s_m(q)$:

$$t_+(q)^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \bar{t}^+(q)_{\alpha\beta}, \quad t_3(q)^{\alpha\beta} = \begin{pmatrix} 0 & q^{\frac{1}{2}} \\ q^{-\frac{1}{2}} & 0 \end{pmatrix} [2]^{-\frac{1}{2}} = \bar{t}^3(q)_{\alpha\beta},$$

$$t_-(q)^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \bar{t}^-(q)_{\alpha\beta}, \quad s(q)^{\alpha\beta} = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix} [2]^{-\frac{1}{2}} = \bar{s}(q)_{\alpha\beta}; \tag{5}$$

where the q-number is defined as $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. The q-analogues of the Levi-Civita symbols $\epsilon(q)_{\alpha\beta}$ and $\epsilon(q)^{\alpha\beta}$ are related to the singlet eigenvectors

$$\epsilon(q)_{\alpha\beta} = -[2]^{\frac{1}{2}}\bar{s}(q)_{\alpha\beta} , \quad \epsilon(q)^{\alpha\beta} = [2]^{\frac{1}{2}} s(q)^{\alpha\beta} . \tag{6}$$

They are normalized in the way such that

$$\epsilon(q)_{\alpha\beta}\epsilon(q)^{\beta\gamma} = \delta_\alpha{}^\gamma . \tag{7}$$

For later use we group $t(q)$ and $s(q)$ to matrix-valued four-vectors

$$t_\mu(q) = (t_0(q), t_m(q)), \quad t_0(q)^{\alpha\beta} = q\, s(q)^{\alpha\beta} ,$$

$$\bar{t}^\mu(q) = \left(\bar{t}^0(q), \bar{t}^m(q)\right), \quad \bar{t}^0(q)_{\alpha\beta} = q^{-1}\bar{s}(q)_{\alpha\beta} . \tag{8}$$

It is easy to see that $t_\mu(q)$ and $\bar{t}^\mu(q)$ satisfy the following orthonormality condition

$$t_\mu(q)^{\alpha\beta}\bar{t}^\nu(q)_{\alpha\beta} = \delta_\mu{}^\nu \tag{9a}$$

and the completeness relation

$$t_\mu(q)^{\alpha\beta}\bar{t}^\mu(q)_{\gamma\delta} = E^{\alpha\beta}{}_{\gamma\delta} = \delta^\alpha{}_\gamma\delta^\beta{}_\delta . \tag{9b}$$

In discussing the quantum group $SU_q(2)$, two conjugate sets of the quantum Pauli matrices are introduced [9]

$$\tau_\mu(q)^\alpha{}_\beta = t_\mu(q)^{\alpha\gamma}\epsilon(q)_{\gamma\beta}, \quad \bar{\tau}^\mu(q)^\beta{}_\alpha = \epsilon(q)^{\beta\gamma}\bar{t}^\mu(q)_{\alpha\gamma} \tag{10}$$

which also satisfy the orthonormality and completeness conditions

$$\tau_\mu(q)^\alpha{}_\beta\bar{\tau}^\nu(q)^\beta{}_\alpha = \delta_\mu{}^\nu , \quad \tau_\mu(q)^\alpha{}_\beta\bar{\tau}^\mu(q)^\gamma{}_\delta = E^{\alpha\gamma}{}_{\delta\beta} = \delta^\alpha{}_\delta\delta^\gamma{}_\beta . \tag{11}$$

In discussing $SL_q(2, C)$ case we must consider the basic (undotted) spinor u^α as well as its conjugate (dotted spinor) $(u^\alpha)^* = \bar{u}_{\dot\alpha}$. (We mainly follow the notation in Ref[12] for classical spinors with a little modification)

$$u^\alpha \longrightarrow M(q)^\alpha{}_\beta u^\beta, \quad \bar{u}_{\dot\alpha} \longrightarrow \bar{u}_{\dot\beta}M^+(q)^{\dot\beta}{}_{\dot\alpha} \tag{12}$$

where M is a 2×2 matrix with its entries taking values on a non-commutative algebra, and M^+ its hermitian (transpose complex conjugate). Further defining $\bar{M}(q) = M^+(q)^{-1}$, and considering the fact that the Levi-Civita symbols are invariant under $SL_q(2, C)$ transformations [7,8,9] (unimodularity of M and $\bar{M}$)

$$\epsilon(q)_{\gamma\delta} = \epsilon(q)_{\alpha\beta}M^\alpha{}_\gamma M^\beta{}_\delta , \quad \epsilon(q)^{\alpha\beta} = M^\alpha{}_\gamma M^\beta{}_\delta\epsilon(q)^{\gamma\delta} ,$$

$$\epsilon(q)_{\dot\gamma\dot\delta} = \epsilon(q)_{\dot\alpha\dot\beta}\bar{M}^{\dot\alpha}{}_{\dot\gamma}\bar{M}^{\dot\beta}{}_{\dot\delta} , \quad \epsilon(q)^{\dot\alpha\dot\beta} = \bar{M}^{\dot\alpha}{}_{\dot\gamma}\bar{M}^{\dot\beta}{}_{\dot\delta}\epsilon(q)^{\dot\gamma\dot\delta} , \tag{13}$$

we can define the undotted spinor with lower index and dotted spinor with upper index

$$u_\beta(q) = u^\alpha(q)\epsilon(q)_{\alpha\beta} , \quad \bar{u}^{\dot\beta}(q) = \bar{u}_{\dot\alpha}(q)\epsilon(q)^{\dot\alpha\dot\beta} \tag{14}$$

which transform under Lorentz group as

$$u_\alpha \longrightarrow u_\beta M^{-1}(q)^\beta{}_\alpha , \quad \bar{u}^{\dot\alpha} \longrightarrow \bar{M}(q)^{\dot\alpha}{}_{\dot\beta}\bar{u}^{\dot\beta} . \tag{15}$$

The quantum matrices M and $\bar{M}$ satisfy the following Yang-Baxter relations

$$\check{R}(q)^{\alpha\beta}{}_{\alpha'\beta'}M(q)^{\alpha'}{}_\gamma M(q)^{\beta'}{}_\delta = M(q)^\alpha{}_{\alpha'}M(q)^\beta{}_{\beta'}\check{R}(q)^{\alpha'\beta'}{}_{\gamma\delta} ,$$

$$\check{R}_{12}M_1M_2 = M_1M_2\check{R}_{12} ,$$

$$\check{R}(q)^{\dot\alpha\dot\beta}{}_{\dot\alpha'\dot\beta'}\bar{M}(q)^{\dot\alpha'}{}_{\dot\gamma}\bar{M}(q)^{\dot\beta'}{}_{\dot\delta} = \bar{M}(q)^{\dot\alpha}{}_{\dot\alpha'}\bar{M}(q)^{\dot\beta}{}_{\dot\beta'}\check{R}(q)^{\dot\alpha'\dot\beta'}{}_{\dot\gamma\dot\delta} ,$$

$$\check{R}_{12}\bar{M}_1\bar{M}_2 = \bar{M}_1\bar{M}_2\check{R}_{12} \ . \tag{16}$$

And also a relation between M and $\bar{M}$ must be added [7,8]

$$\check{R}(q)^{\dot{\alpha}\beta}{}_{\alpha\dot{\beta}}M(q)^{\alpha}{}_{\gamma}\bar{M}(q)^{\dot{\beta}}{}_{\dot{\delta}} = \bar{M}(q)^{\dot{\alpha}}{}_{\gamma}M(q)^{\beta}{}_{\delta}\check{R}(q)^{\dot{\gamma}\delta}{}_{\gamma\dot{\delta}} \ ,$$

$$\check{R}_{12}M_1\bar{M}_2 = \bar{M}_1M_2\check{R}_{12} \ , \tag{17}$$

where all the $\check{R}$ matrices with undotted or dotted indices have the same matrix form. Besides the $t_\mu(q)^{\alpha\beta}$ and $\bar{t}^\mu(q)_{\alpha\beta}$ we also use their dotted and mixed counterparts $t_\mu(q)^{\dot{\alpha}\dot{\beta}}$ and $\bar{t}^\mu(q)_{\dot{\alpha}\dot{\beta}}$; $t_\mu(q)^{\alpha\dot{\beta}}$ and $\bar{t}^\mu(q)_{\alpha\dot{\beta}}$; and $t_\mu(q)^{\dot{\alpha}\beta}$ and $\bar{t}^\mu(q)_{\dot{\alpha}\beta}$. In a similar way the quantum Pauli matrices with dotted and mixed indices, $\tau_\mu(q)^{\alpha}{}_{\dot{\beta}}$, $\bar{\tau}^\mu(q)^{\dot{\alpha}}{}_{\beta}$ etc. are also introduced.

As in the classical case, a quantity $X^{\alpha}{}_{\dot{\alpha}}$, transforms like $u^\alpha\bar{w}_{\dot{\alpha}}$, is said to belong to the representation $(\frac{1}{2},\frac{1}{2})$, which transform as

$$X^{\alpha}{}_{\dot{\alpha}} \longrightarrow M(q)^{\alpha}{}_{\beta}X^{\beta}{}_{\dot{\beta}}M^{+}(q)^{\dot{\beta}}{}_{\dot{\alpha}} \tag{18}$$

and which can be converted to a 4-vector with the help of matrices $\tau_\mu(q)$ and $\bar{\tau}^\mu(q)$,

$$X^{\alpha}{}_{\dot{\alpha}} = \tau_\mu(q)^{\alpha}{}_{\dot{\alpha}}x^\mu \ , \quad x^\mu = \bar{\tau}^\mu(q)^{\dot{\alpha}}{}_{\alpha}X^{\alpha}{}_{\dot{\alpha}} \ . \tag{19}$$

We will see later on that x^μ transforms as a Lorentz vector indeed. Raising the dotted index we see that $X^{\alpha\dot{\alpha}}(= X^{\alpha}{}_{\dot{\beta}}\epsilon(q)^{\dot{\beta}\dot{\alpha}})$ will transform as

$$X^{\alpha\dot{\beta}} \longrightarrow M(q)^{\alpha}{}_{\alpha'}\bar{M}(q)^{\dot{\beta}}{}_{\dot{\beta}'}X^{\alpha'\dot{\beta}'} = T^{(\alpha\dot{\beta})}{}_{(\alpha'\dot{\beta}')}X^{\alpha'\dot{\beta}'} \ . \tag{20}$$

It has been pointed out that [8,9] Yang-Baxter relation for $T_{(12)} = T^{(\alpha\dot{\beta})}_{(\alpha'\dot{\beta}')}$ can be obtained, with the $\check{R}$ matrix given by $\check{R}_{(12)(34)} = \check{R}_{(23)}\check{R}_{(12)}\check{R}_{(34)}\check{R}^{-1}_{(23)}$, *i.e.*,

$$\check{R}(q)^{(\alpha\dot{\beta})(\gamma\dot{\delta})}_{(\alpha'\dot{\beta}')(\gamma'\dot{\delta}')} = \check{R}(q)^{\dot{\beta}\gamma}{}_{\dot{\beta}'\gamma}\check{R}(q)^{\alpha\beta}{}_{\alpha'\beta'}\check{R}(q)^{\dot{\gamma}\dot{\delta}}{}_{\dot{\gamma}'\dot{\delta}'}\check{R}^{-1}(q)^{\beta'\dot{\gamma}'}{}_{\dot{\beta}'\gamma'} \ . \tag{21}$$

And by using Eq(1) repeatedly, we can show that $\check{R}_{(12)(34)}$ also satisfy a new Yang-Baxter equation

$$\check{R}_{(12)(34)}\check{R}_{(34)(56)}\check{R}_{(12)(34)} = \check{R}_{(34)(56)}\check{R}_{(12)(34)}\check{R}_{(34)(56)} \ . \tag{22}$$

2. $\check{R}$ matrix for quantum Lorentz group

The $\check{R}$ matrix thus obtained is really the $\check{R}$ matrix for quantum Lorentz group. To explore this fact more manifestly, we transit these equations from the bispinor form into the vector form by making use of the $t_\mu(q)$ matrices. Define

$$\Lambda^\mu{}_\nu(q) = \bar{t}^\mu(q)_{\alpha\dot{\beta}}M^{\alpha}{}_{\alpha'}\bar{M}^{\dot{\beta}}{}_{\dot{\beta}'}t_\nu(q)^{\alpha'\dot{\beta}'} = \bar{t}^\mu_{12}M_1\bar{M}_2t^{12}_\nu \tag{23}$$

and

$$\begin{aligned}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= q^{-1}\bar{t}^\mu(q)_{\alpha\dot{\beta}}\bar{t}^\nu(q)_{\gamma\dot{\delta}}\check{R}(q)^{(\alpha\dot{\beta})(\gamma\dot{\delta})}_{(\alpha'\dot{\beta}')(\gamma'\dot{\delta}')}t_\kappa(q)^{\alpha'\dot{\beta}'}t_\lambda(q)^{\gamma'\dot{\delta}'} \\ &= q^{-1}\bar{t}^\mu_{12}\bar{t}^\nu_{34}\check{R}_{(12)(34)}t^{12}_\kappa t^{34}_\lambda \ ,\end{aligned} \tag{24}$$

then we can show directly

$$\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda}\Lambda^\kappa{}_\rho\Lambda^\lambda{}_\sigma = \Lambda^\mu{}_\kappa\Lambda^\nu{}_\lambda\check{\mathcal{R}}(q)^{\kappa\lambda}{}_{\rho\sigma} \tag{25}$$

and

$$\check{\mathcal{R}}(q)^{\mu\nu}{}_{\mu'\nu'}\check{\mathcal{R}}(q)^{\nu'\kappa}{}_{\rho'\sigma}\check{\mathcal{R}}(q)^{\mu'\rho'}{}_{\lambda\rho} = \check{\mathcal{R}}(q)^{\nu\kappa}{}_{\nu'\kappa'}\check{\mathcal{R}}(q)^{\mu\nu'}{}_{\lambda\rho'}\check{\mathcal{R}}(q)^{\rho'\kappa'}{}_{\rho\sigma} \ . \tag{26}$$

The $\check{\mathcal{R}}(q)$ obtained in Eq(24) has block diagonal form. To write it down more explicitly we order the Lorentz index $\mu = (+,3,0,-)$ and define a "charge" for each index: $c(+) = +1$, $c(-) = -1$, $c(3) = c(0) = 0$. The $\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda}$ matrix is "charge" conservative, *i.e.*, $c(\mu) + c(\nu) = c(\kappa) + c(\lambda) = m$, and breaks into the block form according to the total "charge " m:

$$\check{\mathcal{R}}(q) = S_1 \oplus S_4 \oplus S_6 \oplus \bar{S}_4 \oplus \bar{S}_1 \tag{27}$$

where $S_n(\bar{S}_n)$ is an $n \times n$ submatrix. The double "charge" sectors comprise two single block:

$$S_1 \equiv \check{\mathcal{R}}(q)^{++}{}_{++} = \bar{S}_1 \equiv \check{\mathcal{R}}(q)^{--}{}_{--} = 1 \ . \tag{28}$$

The single "charge" sectors are two 4×4 matrices

$$S_4 = \bar{S}_4 = \begin{pmatrix} -dq & dq & 2 & dq \\ -dq & dq & dq^{-1} & h \\ 2 & -dq^{-1} & d(q+2q^{-1}) & -dq^{-1} \\ -dq & h & dq^{-1} & dq \end{pmatrix}[2]^{-1} \tag{29}$$

where $d = q - q^{-1}$ and $h = q^2 + q^{-2}$, and the index pair (μ,ν) taking the values of $(+3,+0,3+,0+)$ for the positively "charged" S_4 and $(3-,0-,-3,-0)$ for the

negatively "charged" $\bar{S}_4$. The neutral sector is a 6×6 matrix

$$S_6 = \begin{pmatrix} 0 & d & -d & -d & d & [2]q^{-1} \\ d & 4-h & d^2 & d^2 & dq^{-1} & -d(2+q^{-2}) \\ d & -d^2 & dq & h & 0 & -d \\ d & -d^2 & h & dq & 0 & -d \\ d & dq^{-1} & 0 & 0 & h & dq^{-2} \\ [2]q^{-1} & -d(2+q^{-2}) & d & d & dq^{-2} & d^2q^{-1}[2] \end{pmatrix} [2]^{-1} \tag{30}$$

and the index pair $(\mu,\nu) = (+-, 33, 30, 03, 00, -+)$. Throughout this paper we will adopt the order of the index pair given here as the standard order for the submatrices. The total 16×16 $\check{\mathcal{R}}(q)$ matrix has three different eigenvalues: the single eigenvalue $\lambda_0(q) = q^{-3}$, the sixfold one $\lambda_1(q) = -q^{-1}$ and the nonet $\lambda_2(q) = q$. The distribution of the eigenvalues coincides with that of $SO_q(4)$ [3], corresponding to the decomposition in the product representation $4 \otimes 4 = 1 \oplus 6 \oplus 9$, 6 being antisymmetric while $1 \oplus 9$ symmetric in the classical case. Corresponding to the single eigenvalue $\lambda_0(q) = q^{-3}$, we have the left-acting eigenvector $v(q)^{\mu\nu}$ and right-acting one $\bar{v}(q)_{\mu\nu}$

$$\begin{aligned} v(q)^{\mu\nu} &= (-q, -1, 0, 0, 1, -q^{-1})[2]^{-1} , \\ \bar{v}(q)_{\mu\nu} &= (-q, -1, 0, 0, 1, -q^{-1})[2]^{-1} \end{aligned} \tag{31}$$

where $(\mu\nu) = (+-, 33, 30, 03, 00, -+)$. We will scale them to give the metric $g(q)_{\mu\nu} = [2]\bar{v}(q)_{\mu\nu}$, $g(q)^{\mu\nu} = [2]v(q)^{\mu\nu}$. We can easily see that

$$g(q)_{\mu\nu}g(q)^{\nu\lambda} = \delta^{\lambda}{}_{\mu} = g(q)^{\lambda\nu}g(q)_{\nu\mu} \tag{32}$$

but

$$g(q)_{\mu\nu} = g(q^{-1})_{\nu\mu} \neq g(q)_{\nu\mu} . \tag{33}$$

It can be directly checked that $g_{\mu\nu}$ and $g^{\mu\nu}$ are invariants, *i.e.*,

$$g(q)_{\mu\nu}\Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda} = g(q)_{\kappa\lambda} , \quad \Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda}g(q)^{\kappa\lambda} = g(q)^{\mu\nu} . \tag{34}$$

Therefore $\Lambda^{\mu}{}_{\nu}$ is the q-deformation of the Lorentz transformation and $g(q)_{\mu\nu}$ the q-deformation of the Lorentz metric.

Now from the definition of $\Lambda^{\mu}{}_{\nu}$ and the completeness condition (9b) we see immediately that

$$M^{\alpha}{}_{\beta}\bar{M}^{\dot{\beta}}{}_{\dot{\alpha}}t_{\nu}(q)^{\beta\dot{\alpha}} = t_{\mu}(q)^{\alpha\dot{\beta}}\Lambda^{\mu}{}_{\nu}(q) \tag{35}$$

and then

$$M^{\alpha}{}_{\beta}\tau_{\nu}(q)^{\beta}{}_{\dot{\beta}}M^{+\dot{\beta}}{}_{\dot{\alpha}} = \tau_{\mu}(q)^{\alpha}{}_{\dot{\alpha}}\Lambda^{\mu}{}_{\nu}(q) \tag{36}$$

from (10) and (13). This implies that under the Lorentz transformation (18)

$$x^{\mu} \longrightarrow \Lambda^{\mu}{}_{\nu}(q)x^{\nu} \tag{37}$$

as expected. Notice that, we can use the metric to raise or lower the Lorentz index as in the classical case. But, now, since the metric is not symmetric, we must take care of the order. The convention we adopted is

$$\tau^{\mu}(q)^{\alpha}{}_{\dot{\alpha}} = \tau_{\nu}(q)^{\alpha}{}_{\dot{\alpha}}g^{\nu\mu}(q) , \quad \bar{\tau}_{\mu}(q)^{\dot{\alpha}}{}_{\alpha} = g_{\mu\nu}(q)\bar{\tau}^{\nu}(q)^{\dot{\alpha}}{}_{\alpha} \tag{38a}$$

and

$$x_{\mu} = g(q)_{\mu\nu}x^{\nu} , \quad \partial_{\mu} = g(q)_{\mu\nu}\partial^{\nu} . \tag{38b}$$

The eigenvectors u_{ms} associated to the sixfold eigenvalue $\lambda_1(q) = -q^{-1}$ are

$$\left.\begin{aligned} u_{1+}(q)^{\mu\nu} &= (q, q, -q^{-1}, -q^{-1})[2]^{-1} \\ u_{1-}(q)^{\mu\nu} &= (q, -q^{-1}, -q^{-1}, q)[2]^{-1} \end{aligned}\right\} \; in\ S_4\ sector$$

$$\left.\begin{aligned} u_{0+}(q)^{\mu\nu} &= (-1, d, q, -q^{-1}, 0, 1)[2]^{-1} \\ u_{0-}(q)^{\mu\nu} &= (-1, d, -q^{-1}, q, 0, 1)[2]^{-1} \end{aligned}\right\} \; in\ S_6\ sector \tag{39}$$

$$\left.\begin{aligned} u_{\bar{1}+}(q)^{\mu\nu} &= (q, -q^{-1}, -q^{-1}, q)[2]^{-1} \\ u_{\bar{1}-}(q)^{\mu\nu} &= (q, q, -q^{-1}, -q^{-1})[2]^{-1} \end{aligned}\right\} \; in\ \bar{S}_4\ sector$$

with m the total "charge" and $s = \pm 1$. Correspondingly the right-acting vectors can be identified as

$$\begin{aligned} \bar{u}^{1+}(q)_{\mu\nu} &= (q, q^{-1}, -q^{-1}, -q)[2]^{-1} , & \bar{u}^{1-}(q)_{\mu\nu} &= (q, -q, -q^{-1}, q^{-1})[2]^{-1} , \\ \bar{u}^{0+}(q)_{\mu\nu} &= (-1, d, q^{-1}, -q, 0, 1)[2]^{-1} , & \bar{u}^{0-}(q)_{\mu\nu} &= (-1, d, -q, q^{-1}, 0, 1)[2]^{-1} , \\ \bar{u}^{\bar{1}+}(q)_{\mu\nu} &= (q, -q, -q^{-1}, q^{-1})[2]^{-1} , & \bar{u}^{\bar{1}-}(q)_{\mu\nu} &= (q, q^{-1}, -q^{-1}, -q)[2]^{-1} . \end{aligned} \tag{40}$$

Also we have nonet eigenvectors corresponding to $\lambda_2(q) = q$

$$\begin{aligned} \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda}w_{mn}(q)^{\kappa\lambda} &= \lambda_2(q)w_{mn}(q)^{\mu\nu} , \\ \bar{w}^{mn}(q)_{\mu\nu}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= \lambda_2(q)\bar{w}^{mn}(q)_{\kappa\lambda} \end{aligned} \tag{41}$$

where $(m,n) = (2,0)$, $(1,1)$, $(1,\bar{1})$, $(0,2)$, $(0,0)$, $(0,\bar{2})$, $(\bar{1},1)$, $(\bar{1},\bar{1})$ and $(\bar{2},0)$, and m is the total "charge". We will not list them out for simplicity.

All these eigenvectors are chosen in such a way that the orthonormality conditions

$$C_A(q)^{\mu\nu}\bar{C}^B(q)_{\mu\nu} = \delta_A{}^B , \quad C_A(q)^{\mu\nu}\bar{C}^A(q)_{\kappa\lambda} = \delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\lambda} \tag{42}$$

are satisfied , where $C_A(q)$ ia an abbreviation of $(v(q), u_{ms}(q), w_{mn}(q))$ and $\bar{C}^B(q) = (\bar{v}(q), \bar{u}^{ms}(q), \bar{w}^{mn}(q))$.

Now we can define the projection operators projecting onto subspaces with different eigenvalues $\lambda_i(q)$:

$$\begin{aligned} Q^{(0)}(q)^{\mu\nu}{}_{\kappa\lambda} &= v(q)^{\mu\nu}\bar{v}(q)_{\kappa\lambda} , \\ Q^{(1)}(q)^{\mu\nu}{}_{\kappa\lambda} &= u_{ms}(q)^{\mu\nu}\bar{u}^{ms}(q)_{\kappa\lambda} , \\ Q^{(2)}(q)^{\mu\nu}{}_{\kappa\lambda} &= w_{mn}(q)^{\mu\nu}\bar{w}^{mn}(q)_{\kappa\lambda} , \end{aligned} \tag{43}$$

which satisfy the relations

$$Q^{(i)}Q^{(j)} = \delta^{ij}Q^{(i)} , \quad Q^{(0)} + Q^{(1)} + Q^{(2)} = E. \tag{44}$$

Then $\check{\mathcal{R}}$ matrix and other relevant matrices can be expressed as linear combination of Q's, e.g.,

$$\begin{aligned} \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= \lambda_i(q)Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda} , \\ \check{\mathcal{R}}^{-1}(q)^{\mu\nu}{}_{\kappa\lambda} &= \lambda_i^{-1}(q)Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda} . \end{aligned} \tag{45}$$

The projection operators $Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda}$ as well as the $\check{\mathcal{R}}$ matrix have a lot of useful symmetries, among which the most important ones are

$$\begin{aligned} Q^{(i)}(q)^{\mu\nu}{}_{\kappa\lambda} &= Q^{(i)}(q^{-1})^{\nu\mu}{}_{\lambda\kappa} , \\ \check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= \check{\mathcal{R}}^{-1}(q^{-1})^{\nu\mu}{}_{\lambda\kappa} , \\ g(q)_{\rho\mu}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= \check{\mathcal{R}}^{-1}(q)^{\nu\sigma}{}_{\rho\kappa}g(q)_{\sigma\lambda} , \\ g(q)^{\sigma\kappa}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= \check{\mathcal{R}}^{-1}(q)^{\sigma\mu}{}_{\lambda\rho}g(q)^{\rho\nu} . \end{aligned} \tag{46}$$

3. Covariant Differential Calculus on M_q

The covariant differential calculus on the quantum plane was first discussed by Wess and Zumino [6] for the A type plane in which the corresponding $\check{R}$ matrix has two different eigenvalues, and then generalized to more general cases where the quantum orthogonal plane (B,D type) [10,11] or quantum symplectic plane(C type) [10] is considered and the associated $\check{R}$ matrix has three or more different eigenvalues [10]. We only sketch the main procedure here. On a quantum plane,

the coordinates x^μ, their differentials $dx^\mu = \xi^\mu$ and derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ are all non-commutative. Suppose the cross commutation relations among these quantities are expressed as[6]

$$\begin{aligned} (E_{12} - B_{12})x_1x_2 = 0 : &\quad (E - B)^{\mu\nu}{}_{\kappa\lambda}x^\kappa x^\lambda = 0 , \\ \partial_2\partial_1(E_{12} - B_{12}) = 0 : &\quad \partial_\nu\partial_\mu(E - B)^{\mu\nu}{}_{\kappa\lambda} = 0 , \\ (E_{12} + D_{12})\xi_1\xi_2 = 0 : &\quad (E + D)^{\mu\nu}{}_{\kappa\lambda}\xi^\kappa\xi^\lambda = 0 , \\ x_1\xi_2 = C_{12}\xi_1x_2 : &\quad x^\mu\xi^\nu = C^{\mu\nu}{}_{\kappa\lambda}\xi^\kappa x^\lambda , \\ &\quad \partial_\kappa x^\mu = \delta_\kappa{}^\mu + C^{\mu\nu}{}_{\kappa\lambda}x^\lambda\partial_\nu , \\ &\quad \partial_\kappa\xi^\mu = D^{\mu\nu}{}_{\kappa\lambda}\xi^\lambda\partial_\nu , \end{aligned} \tag{47}$$

where B, C, D are numerical matrices to be determined later. The consistency conditions can be summarized in the following relations

$$\begin{aligned} (E_{12} - B_{12})(E_{12} + C_{12}) = 0, \quad D = C^{-1} , \\ (E_{12} - B_{12})C_{23}C_{12} = C_{23}C_{12}(E_{23} - B_{23}), \\ D_{23}C_{12}C_{23} = C_{12}C_{23}D_{12} . \end{aligned} \tag{48}$$

For the case where the $\check{\mathcal{R}}$ matrix has three different eigenvalues, the solutions to these consistency conditions are given by [10]

$$(E - B) = bQ^{(1)}(q). \quad C = -\lambda_1(q)^{-1}\check{\mathcal{R}}(q) \tag{49}$$

(Alternatively $C = -\lambda_1(q)\check{\mathcal{R}}^{-1}(q)$) with b a free parameter, which can be chosen as 1 for simplicity. Then the first three relations in Eq(47) are given by

$$Q^{(1)}(q)_{12}x_1x_2 = \partial_2\partial_1Q^{(1)}(q)_{12} = 0 . \quad Q^{(0)}(q)_{12}\xi_1\xi_2 = Q^{(2)}(q)_{12}\xi_1\xi_2 = 0 , \tag{50}$$

which can be rewritten in the simple form explicitly:

$$\begin{aligned} x^+x^0 = x^0x^+, \ x^3x^0 = x^0x^3, \ x^-x^0 = x^0x^- , \\ qx^+x^3 - q^{-1}x^3x^+ = (q - q^{-1})x^0x^+ , \\ x^+x^- - x^-x^+ = (q - q^{-1})x^3(x^3 - x^0) , \\ qx^3x^- - q^{-1}x^-x^3 = (q - q^{-1})x^-x^0 ; \end{aligned} \tag{51}$$

$$
\begin{aligned}
&\partial_+\partial_0 = \partial_0\partial_+ \ , \quad \partial_3\partial_0 = \partial_0\partial_3 \ , \quad \partial_0\partial_- = \partial_-\partial_0 \ , \\
&q^{-1}\partial_+\partial_3 - q\partial_3\partial_+ = (q-q^{-1})\partial_0\partial_+ \ , \\
&\partial_+\partial_- - \partial_-\partial_+ = (q^{-1}-q)\partial_3(\partial_3+\partial_0) \ , \\
&q^{-1}\partial_3\partial_- - q\partial_-\partial_3 = (q-q^{-1})\partial_-\partial_0 \ ;
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&\xi^+\xi^+ = 0 \ , \quad \xi^0\xi^0 = 0 \ , \quad \xi^-\xi^- = 0 \ , \\
&\xi^+\xi^- + \xi^-\xi^+ = 0 \ , \quad \xi^3\xi^0 + \xi^0\xi^3 = \xi^3\xi^3 \ , \\
&\xi^3\xi^3 - \xi^0\xi^0 = (q-q^{-1})\xi^-\xi^+ \ , \quad q\xi^3\xi^+ + q^{-1}\xi^+\xi^3 = 0 \ , \\
&q\xi^-\xi^3 + q^{-1}\xi^3\xi^- = 0 \ , \quad \xi^3\xi^+ + \xi^+\xi^3 = \xi^0\xi^+ + \xi^+\xi^0 \ , \\
&\xi^-\xi^3 + \xi^3\xi^- = \xi^-\xi^0 + \xi^0\xi^- \ .
\end{aligned} \tag{53}
$$

The next three relations in Eq(47) are much complicated. By raising the index of the derivatives $\partial^\mu = g^{\mu\nu}(q)\partial_\nu$ and using relations in (46), we can bring them to the following form:

$$
\begin{aligned}
&g(q)_{\mu\nu}(x^\mu\xi^\nu - q^{-2}\xi^\mu x^\nu) = 0 \ , \\
&\bar{u}^{ms}(q)_{\mu\nu}(x^\mu\xi^\nu + \xi^\mu x^\nu) = 0 \ , \\
&\bar{w}^{mn}(q)_{\mu\nu}(x^\mu\xi^\nu - q^2\xi^\mu x^\nu) = 0 \ ;
\end{aligned}
$$

$$
\begin{aligned}
&g(q)_{\mu\nu}(\partial^\mu\xi^\nu - q^{-4}\xi^\mu\partial^\nu) = 0 \ , \\
&\bar{u}^{ms}(q)_{\mu\nu}(\partial^\mu\xi^\nu + q^{-2}\xi^\mu\partial^\nu) = 0 \ , \\
&\bar{w}^{mn}(q)_{\mu\nu}(\partial^\mu\xi^\nu - \xi^\mu\partial^\nu) = 0 \ ;
\end{aligned}
$$

$$
\begin{aligned}
&g(q)_{\mu\nu}(\partial^\mu x^\nu - q^4 x^\mu\partial^\nu) = [2] \ , \\
&\bar{u}^{ms}(q)_{\mu\nu}(\partial^\mu x^\nu + q^2 x^\mu\partial^\nu) = 0 \ , \\
&\bar{w}^{mn}(q)_{\mu\nu}(\partial^\mu x^\nu - x^\mu\partial^\nu) = 0 \ .
\end{aligned} \tag{54}
$$

Here $g(q)_{\mu\nu}$, $\bar{u}^{ms}(q)_{\mu\nu}$ and $\bar{w}^{mn}(q)_{\mu\nu}$ are indeed the quantum C-G coefficients of $SL_q(2,C)$, some of which have been presented in Eqs(31) and (40). To give an insight what these relations are, we list here the cross commutation relations between the coordinates and the derivatives as follows.

$$
\begin{aligned}
&[2]\partial_+x^+ = [2](1+q^2x^+\partial_+) + \{-x^3\partial_3 + x^0\partial_3 - x^3\partial_0 + x^0\partial_0\}dq^2 \ , \\
&[2]\partial_+x^3 = 2qx^3\partial_+ - dx^0\partial_+ + dqx^-\partial_3 + dqx^-\partial_0 \ , \\
&[2]\partial_+x^0 = -dq^2x^3\partial_+ + q(q^2+q^{-2})x^0\partial_+ + dqx^-\partial_3 + dqx^-\partial_0 \ , \\
&\partial_+x^- = x^-\partial_+ \ ; \\
&[2]\partial_3x^+ = 2qx^+\partial_3 + dx^+\partial_0 + dqx^3\partial_- - dqx^0\partial_- \ , \\
&[2]\partial_3x^3 = [2] + d(q^2+2)x^+\partial_+ - (q^3-4q+q^{-1})x^3\partial_3 + dq^2x^0\partial_3 - dq^2x^3\partial_0 \\
&\qquad +dq^2x^0\partial_0 - dq^2x^-\partial_- \ , \\
&[2]\partial_3x^0 = dx^+\partial_+ - dq^2x^3\partial_3 + q(q^2+q^{-2})x^0\partial_3 + dx^3\partial_0 - dq^2x^-\partial_- \ , \\
&[2]\partial_3x^- = -d(2q+q^{-1})x^3\partial_+ + dqx^0\partial_+ + 2qx^-\partial_3 - dq^2x^-\partial_0 \ ; \\
&[2]\partial_0x^+ = dq^2x^+\partial_3 + q(q^2+q^{-2})x^+\partial_0 - dqx^3\partial_- + dqx^0\partial_- \ , \\
&[2]\partial_0x^3 = -dx^+\partial_+ + dq^2x^3\partial_3 + dx^0\partial_3 + q(q^2+q^{-2})x^3\partial_0 + dq^2x^-\partial_- \ , \\
&[2]\partial_0x^0 = [2] + dq^2x^+\partial_+ + dq^2x^3\partial_3 + q(q^2+q^{-2})x^0\partial_0 + dq^2x^-\partial_- \ , \\
&[2]\partial_0x^- = dqx^3\partial_+ + dq^{-1}x^0\partial_+ - dx^-\partial_3 + q(q^2+q^{-2})x^-\partial_0 \ ; \\
&\partial_-x^+ = x^+\partial_- \ , \\
&[2]\partial_-x^3 = -d(2q+q^{-1})x^+\partial_3 - dqx^+\partial_0 + 2qx^3\partial_- + dq^2x^0\partial_- \ , \\
&[2]\partial_-x^0 = -dqx^+\partial_3 + dq^{-1}x^+\partial_0 + dx^3\partial_- + q(q^2+q^{-2})x^0\partial_- \ , \\
&[2]\partial_-x^- = [2](1+x^+\partial_+ + q^2x^-\partial_-) + d(q^2+2)x^3\partial_3 - dx^0\partial_3 + dx^3\partial_0 + dq^2x^0\partial_0 \ .
\end{aligned} \tag{55}
$$

The relations(51) among the coordinates coincide with those given in Ref[11], and all others are new.

4. Quantum Dirac Equation

Now we are ready to set up the q-deformation of the Dirac equation on the non-commutative quantum Minkowski space M_q. As has been shown in Eqs(18) and (36), the bispinor $X^\alpha{}_{\dot\alpha}$ transforms under the Lorentz transformation as $(\frac{1}{2},\frac{1}{2})$ representation

$$X^\alpha{}_{\dot\alpha} \longrightarrow M^\alpha{}_\beta X^\beta{}_{\dot\beta} M^{+\dot\beta}{}_{\dot\alpha} \ . \tag{56}$$

As in the classical case we may use the ϵ-symbol to raise and lower the indices on $X^\alpha{}_{\dot\alpha}$(or equivalently on $\tau_\mu{}^\alpha{}_{\dot\alpha}$):

$$-\tau_\mu(q)^\alpha{}_{\dot\alpha}\epsilon(q)_{\alpha\beta}\epsilon(q)^{\dot\alpha\dot\beta} = \bar\tau_\mu(q^{-1})^{\dot\beta}{}_\beta \ .$$

Corresponding to the transformation (56), it will transform as

$$\bar\tau_\mu(q^{-1})^{\dot\beta}{}_\beta \longrightarrow M^{-1\gamma}{}_\beta \bar\tau_\mu(q^{-1})^{\dot\gamma}{}_\gamma \bar M^{\dot\beta}{}_{\dot\gamma}$$

which can not be written as $\bar M\bar\tau_\mu(q^{-1})M^{-1}$ as in the classical case, since now the entries of M and $\bar M$ are not commuting. Instead we can check that

$$\hat\tau_\mu(q)^{\dot\alpha}{}_\alpha \equiv \check R(q^{-1})^{\dot\alpha\beta}{}_{\alpha\dot\beta}\bar\tau_\mu(q^{-1})^{\dot\beta}{}_\beta \longrightarrow \bar M^{\dot\alpha}{}_{\dot\beta}\hat\tau(q)^{\dot\beta}{}_\beta M^{-1\beta}{}_\alpha \tag{57}$$

as desired. Then we see immediately that

$$[\tau_\mu(q)\hat\tau_\nu(q)]^\alpha{}_\beta \longrightarrow M^\alpha{}_{\alpha'}[\tau_\mu(q)\hat\tau_\nu(q)]^{\alpha'}{}_{\beta'}(M^{-1})^{\beta'}{}_\beta \ ,$$

$$[\hat\tau_\mu(q)\tau_\nu(q)]^{\dot\alpha}{}_{\dot\beta} \longrightarrow \bar M^{\dot\alpha}{}_{\dot\alpha'}[\hat\tau_\mu(q)\tau_\nu(q)]^{\dot\alpha'}{}_{\dot\beta'}(M^+)^{\dot\beta'}{}_{\dot\beta} \ . \tag{58}$$

So they transform as $(1\oplus 0,0)$ and $(0,1\oplus 0)$ respectively. We list here the concrete matrix form of the matrices $\tau^\mu(q)^\alpha{}_{\dot\alpha}$ and $\hat\tau^\mu(q)^{\dot\beta}{}_\beta$

$$\begin{aligned}
\tau^+(q) &= \begin{pmatrix} 0 & q^{-\frac{3}{2}} \\ 0 & 0 \end{pmatrix}, & \hat\tau^+(q) &= \begin{pmatrix} 0 & -q^{-\frac{3}{2}} \\ 0 & 0 \end{pmatrix}, \\
\tau^3(q) &= \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}[2]^{-\frac{1}{2}}, & \hat\tau^3(q) &= \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix}[2]^{-\frac{1}{2}}, \\
\tau^-(q) &= \begin{pmatrix} 0 & 0 \\ q^{\frac{3}{2}} & 0 \end{pmatrix}, & \hat\tau^-(q) &= \begin{pmatrix} 0 & 0 \\ -q^{\frac{3}{2}} & 0 \end{pmatrix}, \\
\tau^0(q) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q[2]^{-\frac{1}{2}}, & \hat\tau^0(q) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q^{-1}[2]^{-\frac{1}{2}}.
\end{aligned} \tag{59}$$

Then it follows from a tedious but straightforward calculation that

$$\begin{aligned}
A^{\mu\nu}(q)^\alpha{}_\beta &\equiv \tau^\mu(q)^\alpha{}_{\dot\alpha}\hat\tau^\nu(q)^{\dot\alpha}{}_\beta \\
&= [2]^{1/2}\tau^m(q)^\alpha{}_\beta\bar u_{m+}(q^{-1})^{\mu\nu} + \delta^\alpha{}_\beta\bar v(q^{-1})^{\mu\nu} \ , \\
B^{\mu\nu}(q)^{\dot\alpha}{}_{\dot\beta} &\equiv \hat\tau^\mu(q)^{\dot\alpha}{}_\alpha\tau^\nu(q)^\alpha{}_{\dot\beta} \\
&= [2]^{1/2}\tau^m(q)^{\dot\alpha}{}_{\dot\beta}\bar u_{m-}(q^{-1})^{\mu\nu} + \delta^{\dot\alpha}{}_{\dot\beta}\bar v(q^{-1})^{\mu\nu} \ .
\end{aligned} \tag{60}$$

It implies that

$$\begin{aligned}
Q^{(0)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}A^{\kappa\lambda}(q)^\alpha{}_\beta &= \delta^\alpha{}_\beta\bar v(q^{-1})^{\mu\nu} \ , \\
Q^{(2)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}A^{\kappa\lambda}(q)^\alpha{}_\beta &= 0 \ ; \\
Q^{(0)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}B^{\kappa\lambda}(q)^{\dot\alpha}{}_{\dot\beta} &= \delta^{\dot\alpha}{}_{\dot\beta}\bar v(q^{-1})^{\mu\nu} \ , \\
Q^{(2)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}B^{\kappa\lambda}(q)^{\dot\alpha}{}_{\dot\beta} &= 0 \ .
\end{aligned} \tag{61}$$

Define

$$\begin{aligned}
\sigma^{\mu\nu}(q)^\alpha{}_\beta &\equiv Q^{(1)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}A^{\kappa\lambda}(q)^\alpha{}_\beta = [2]^{1/2}\tau^m(q)^\alpha{}_\beta\bar u_{m+}(q^{-1})^{\mu\nu} \ , \\
\bar\sigma^{\mu\nu}(q)^{\dot\alpha}{}_{\dot\beta} &\equiv Q^{(1)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}B^{\kappa\lambda}(q)^{\dot\alpha}{}_{\dot\beta} = [2]^{1/2}\tau^m(q)^{\dot\alpha}{}_{\dot\beta}\bar u_{m-}(q^{-1})^{\mu\nu} \ ,
\end{aligned} \tag{62}$$

and we see that they are nothing but the q-analogues of Lorentz generators[12] in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations respectively. The relations in Eq(61) can be rewritten in a more compact form:

$$[\tau^\mu(q)\hat\tau^\nu(q) + q^{-1}\check{\mathcal{R}}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\tau^\kappa(q)\hat\tau^\lambda(q)]^\alpha{}_\beta = q\ g(q^{-1})^{\mu\nu}\delta^\alpha{}_\beta \ ,$$

$$[\hat\tau^\mu(q)\tau^\nu(q) + q^{-1}\check{\mathcal{R}}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\hat\tau^\kappa(q)\tau^\lambda(q)]^{\dot\alpha}{}_{\dot\beta} = q\ g(q^{-1})^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta} \ . \tag{63}$$

Now since the derivative operator ∂_μ behaves as a four-vector, we can put it into the spinorial form as in the classical case

$$\partial^\alpha{}_{\dot\alpha} = \sigma^\mu(q)^\alpha{}_{\dot\alpha}\partial_\mu \ , \quad \hat\partial^{\dot\alpha}{}_\alpha = \hat\sigma^\mu(q)^{\dot\alpha}{}_\alpha\partial_\mu \tag{64}$$

where we have scaled the τ^μ and $\hat\tau^\mu$ to the conventional normalized ones $\sigma^\mu = [2]^{1/2}\tau^\mu$ and $\hat\sigma^\mu = [2]^{1/2}\hat\tau^\mu$. Then

$$\partial \longrightarrow M\partial M^+ \ , \quad \hat\partial \longrightarrow \bar M\hat\partial M^{-1} \ . \tag{65}$$

To set up the field equation for particles with spin $\frac{1}{2}$ and nonvanishing rest mass we must introduce a pair of elementary spinors φ^α and $\chi^{\dot\alpha}$, which are independent to each other. By making use of the spinorial derivative operators we may set

$$i\sigma^\mu(q)^\alpha{}_{\dot\alpha}\partial_\mu\chi^{\dot\alpha} = -m\varphi^\alpha \ , \quad i\hat\sigma^\mu(q)^{\dot\alpha}{}_\alpha\partial_\mu\varphi^\alpha = -m\chi^{\dot\alpha} \tag{66}$$

with m the analogue of the rest mass. This set of equations is manifestly covariant. It is the q-Dirac equation in spinorial form. To bring it to the customary form we group φ^α and $\chi^{\dot\alpha}$ together to form a four-component spinor, or Dirac spinor

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \chi^{\dot\alpha} \\ \varphi^\alpha \end{pmatrix} \tag{67}$$

and define

$$\alpha^\mu = \begin{pmatrix} \sigma^\mu & 0 \\ 0 & \hat\sigma^\mu \end{pmatrix} \ , \quad \beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{68}$$

Then Eqs(66) can be rewritten as

$$i\alpha^\mu\partial_\mu\psi = m\beta\psi \ . \tag{69}$$

Or by introducing $\gamma^\mu = \beta\alpha^\mu$, it becomes

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{70}$$

where

$$\gamma^0 = \beta\alpha^0 = \begin{pmatrix} 0 & -q^{-1} \\ -q & 0 \end{pmatrix} \ , \quad \vec\gamma = \beta\vec\alpha = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix} \tag{71}$$

are the Dirac matrices in the chiral representation. It follows from Eq(63) that

$$\gamma^\mu(q)\gamma^\nu(q) + q^{-1}\check{\mathcal{R}}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa(q)\gamma^\lambda(q) = (1+q^2)g(q^{-1})^{\mu\nu} \ . \tag{72}$$

A similar relation has also been obtained by Zumino in a completely different consideration[13]. This can be considered as the definition relation for the quantum Clifford algebra. Eq(72) can also be written in the projection form:

$$Q^{(0)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa(q)\gamma^\lambda(q) = g(q^{-1})^{\mu\nu} \ , \quad Q^{(2)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa(q)\gamma^\lambda(q) = 0 \tag{73}$$

while

$$[2]^{-1}Q^{(1)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa(q)\gamma^\lambda(q) = \begin{pmatrix} \bar\sigma^{\mu\nu}(q) & 0 \\ 0 & \sigma^{\mu\nu}(q) \end{pmatrix} \equiv \Sigma^{\mu\nu}(q) \ . \tag{74}$$

Now by applying a operator $(-i\gamma^\mu\partial_\mu - m)$ on Eq(70) we can easily see that each component of ψ satisfies a q-deformed Klein-Gorden equation

$$(\partial^\nu\partial_\nu + m^2)\psi = 0 \ . \tag{75}$$

which is invariant under the quantum Lorentz transformation (37). Here we have inserted a complete set of projection operators to obtain

$$\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu = \partial_\mu\partial_\nu\left(Q^{(0)}(q^{-1}) + Q^{(1)}(q^{-1}) + Q^{(2)}(q^{-1})\right)^{\mu\nu}{}_{\kappa\lambda}\gamma^\kappa\gamma^\lambda \tag{76}$$

and noticed the relation (73) and the fact that

$$\partial_\mu\partial_\nu Q^{(1)}(q^{-1})^{\mu\nu}{}_{\kappa\lambda} = 0 \ , \tag{77}$$

from (46) and (50). Equation (70) or (75), together with the cross commutation relations given in Section 3, constructs the basic equation in the quantum Minkowski space.

The details of these results and related topics will be published elsewhere.

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