CCAST – 91 – 46
August 1991

Poisson Loop Group, Classical and Quantum Generalized Periodic Toda Lattices

GUO Mao-Zheng QIAN Min WANG Bai-Ling

CCAST (World Laboratory)
P.O.Box 8730, Beijing 100080, China

and

Department of Mathematics, Peking University
Beijing 100871, China[1]

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Abstract

We use r-matrix with a spectral parameter to discuss the solution of infinite dimensional integrable systems. The intrinsic geometric pictures are given in the sense that the solutions of the integrable system can be described as geodesics on an infinite dimensional Riemannian manifold. As an application to generalized periodic Toda lattices,we obtain the reduction to a finite dimensional symplectic manifold, the quantum pictures of the theories can be given by geometric quantization of this symplectic manifold. Double loop group is easily derived,which is a loop group with Poisson structure, it may serve as the origin of quantum groups.

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[1]Supported by NNSFC Grant. to the first two authors.

CCAST-91-46

§1. Introduction and main results

It has been well known that the classical and quantum r-matrix play an important role in classical and quantum integrable systems [2,3,5,15,20,21]. Many solutions of r-matrix have been given and well understood [1,2,3,10,15,16,20,21]. For a given classical r-matrix, the integrable systems can be constructed in principle and the solutions of Hamiltonian equations can also be obtained by representation theory [13]. Ten years ago, Perelomov and Olshanetsky [17] posed the following question as a conjecture: for all integrable systems, do the solutions correspond to geodesics on some homogeneous space? They also illustrated this property in the case of some finite dimensional integrable systems. Olive [6] has proved this conjecture for non-periodic Toda lattices. But for infinite dimensional integrable systems, this conjecture is open since the theory of infinite dimensional geometry is imcomplete. But recently the infinite dimensional theory becomes more interesting from many aspects. Such systems were studied by methods of algebra and algebraic geometry in some case, the solution has been give in terms of Riemann's theta functions [4,19]. So the question whether Perelomov's conjecture also holds true arises naturally. In this paper,we use the theory of loop group and loop algebra to construct an infinite dimensional homogeneous space and give the proof about the geodesic property of the solution. All the method we used can also applies to finite dimensional case, what we get is then the theory just given in [6].

To be specific, we first recall the equations of the Toda system:

Let $\mathcal{G}$ be a complex semisimple Lie algebra . $\mathcal{LG} = \mathcal{G} \otimes \mathcal{C}[\lambda, \lambda^{-1}]$, we denote the generators of $\mathcal{LG}$ in standard form by $\{h_i, i = 1, \cdots, n;\ e_{\pm\alpha_j}, j = 0, \cdots, n\}$, their commutative relations will be given in next section. Let

$$\begin{aligned} A &= \textstyle\sum_{i=1}^{n} p_i h_i + \sum_{i=0}^{n} c_i exp(\sum_{j=1}^{n} \alpha_i(h_j) q_j)(e_{\alpha_i} + e_{-\alpha_i}) \\ B &= \textstyle\sum_{i=0}^{n} c_i exp(\sum_{j=1}^{n} \alpha_i(h_j) q_j)(e_{\alpha_i} - e_{-\alpha_i}) \end{aligned} \tag{1}$$

where$\{q_1, \cdots, q_n\}$ are free variables and $p_i = \frac{dq_i}{dt}$. We consider the equation in Lax form:

$$\frac{dA}{dt} = [B, A] \tag{2}$$

This equation describs the motion of Toda particles in the language of Lie

algebra. For the purpose of r-matrix approach, we will introduce in §3 (25) a linear operator on the loop algebra $\mathcal{LG}$ such that

$$\begin{aligned} R(e_{\pm\alpha_i}) &= \pm e_{\pm\alpha_i} \quad (i=0,\cdots,n) \\ R(h_i) &= 0 \qquad (i=1,\cdots,n) \end{aligned} \tag{3}$$

By the definition, we easily rewrite Lax equation (2) in form:

$$\frac{dA}{dt} = [R(A), A] \tag{4}$$

This Lax equation will be studied deeply in this paper, we will see that the operator R is just the generalized Koszul operator of homogeneous spaces, see [15], it can also define a classical r-matrix satisfying the modified Yang-Baxter equation.

The detailed theories about classical r-matrix, classical doubles, dressing transformations on loop algebra and loop group will be given elsewhere. These are the basis for infinite dimensional integrable systems. In finite dimensional case, these were studied by [15,16].

We now outline the main results in the form of theorems. Since the proof are scattered through the next sections according to logical order, they are renumbered into propositions and theorems in the main context.

Let G be a semisimple Lie group over a complex field, $\mathcal{G}$ its Lie algebra, $\mathcal{G}^*$ the dual space of $\mathcal{G}$, $\mathcal{LG}$=$C^\infty(S^1,\mathcal{G})$ is the loop algebra of $\mathcal{G}$. $\mathcal{LG}^*$ is the smooth part of the dual space of $\mathcal{LG}$. LG=$C^\infty(S^1,G)$, the smooth maps from S^1 to G or the loop group of G. Let K is a maximal compact subgroup of G with trivial centre, LK is the loop group of K. We have

[Theorem A]

(1) LG can be decomposed into two factors:

$$LG = LK \cdot S \tag{5}$$

where the subgroup S of LG can be equipped with a Riemannian structure in infinite dimension sense.

(2) On the homogenous space LG/LK, there exists a standard Riemannian

metric such that LG/LK is a symmetric Riemannian space (similar to the finite dimension case).

(3) There exists a geodesic immersion :

$$\Psi : LG/LK \longrightarrow LG \tag{6}$$

we identify LG/LK with the image of Ψ.

(4) Under the identification (6),there is an isometric diffeomorphism :

$$\Phi : S \longrightarrow LG/LK \tag{7}$$

This theorem is the generalization of Cartan's immersion theorem in finite dimensional Lie group theory [8]. This generalization is the mathematical basis for our theory. We note that the group property of S leads to considering dynamic systems on the adjoint or coadjoint orbits of the Lie algebra of S. Using theorem A ,we can identify S with LG/LK. Under this identification, we can give different pictures of the solution of the infinite dimensional integrable systems whose equation is (4), and also prove Perelomov's conjecture about the geodesic property.

Let $\mathcal{S}$ be Lie algebra of S, $\mathcal{S}^*$ the dual space of $\mathcal{S}$. Since there is a nongenerate scaler product on $\mathcal{G}$, the Killing form, we can obtain a nongenerate bilinear form $<,>$ on $\mathcal{LG}$ by integrating over the unit circle, which is invariant under the adjoint action of LG. We then identify $\mathcal{LG}^*$ with $\mathcal{LG}$ by setting

$$y^*(\xi) =< y, \xi >= \frac{1}{2\pi}\int_0^{2\pi} Kill(y(\theta), \xi(\theta))d\theta \tag{8}$$

where $\xi \in \mathcal{LG}, y^* \in \mathcal{LG}^*$ and y^* is identified with $y \in \mathcal{LG}$. Under this identification, one embeds $\mathcal{S}^*$ into $\mathcal{LG}$, the image of this embedding is not $\mathcal{S}$, this is the subtle point of particular importance. In fact, if one uses the Riemannian metric on S, $\mathcal{S}^*$ can, in a different fashion, be identified with $\mathcal{S}$, and what we get is a different geometric picture. We denote the first identification with $\mathcal{S}^* \subset \mathcal{LG}$, and the second identification with $\mathcal{S}^* = \mathcal{S}$, one has to distinguish these carefully.

Motivated by the equation (4), we define a general Lax equation related to the loop algebra $\mathcal{LG}$:

$$\frac{dA}{dt} = [R(A), A] \tag{9}$$

where R is the operator on $\mathcal{LG}$ defined by our classical r-matrix explicitly.

Because we have identified $\mathcal{LG}^*$ with $\mathcal{LG}$, so equation (9), in fact, defines a standard Hamilton system on the coadjoint-orbits in $\mathcal{LG}^*$ with Hamiltonian $H = \frac{1}{2} < A, A >$, from which we obtain another geometric descriptions of the solutions of equation (9).

[Theorem B1] (Coadjoint)Adjoint-Orbits Picture of the Solutions

(1) For a fixed initial condition $A(0) = A_0 \in \mathcal{S}^*$, the Lax equation(9) has a unique solution, expressed in (coadjoint)adjoint-orbit form:

$$\begin{aligned} A(t) &= Ad(s(expt A_0)).A_0 \\ &= Ad(k^{-1}(expt A_0)).A_0 \end{aligned} \tag{10}$$

where the element $s(t) = s(expt A_0)$ and $k(t) = k(expt A_0)$ are defined from $exp(tA_0)$ by the decomposition of (5), *i.e.* $exp(tA_0) = k(t)s(t)$.

(2) If one pulls back the tangent vector at $s(t)$ by right action of LG, *i.e.* take the function

$$A(t) = \frac{ds(t)}{dt} \cdot s^{-1}(t)$$

which belongs to $\mathcal{S}^*$ under the identification $\mathcal{S}^* = \mathcal{S}$, then $A(t)$ satisfies Lax equation (9).

[Theorem B2] Geodesic Picture of the Solutions

(1) If we denote

$$x(t) = \Phi(s(expt A)) = \Phi(s(t)) \tag{11}$$

where Φ is defined by (7), and $s(expt A) = s(t)$ is defined by Theorem B, then

$$\frac{d}{dt}(\dot{x}(t) \cdot x^{-1}(t)) = 0 \tag{12}$$

With the knowledge of infinite dimensional geometry given in next sections, the equation (12) is that of the geodesics flow in LG/LK.

(2) Given a geodesic $x(t)$ in LG/LK, we consider

$$\Phi_*^{-1}(\dot{x}(t)x^{-1}(t))$$

in $\mathcal{S}$, where Φ_* is the tangent map of Φ. If we identify this in $\mathcal{S}^*$, and call it $A(t)$. Then $A(t)$ is the solution of equation (9).

For a reduction to Toda lattices, we first define a family of manifolds depending on $(c_0, \cdots, c_n) \in (\mathcal{C}\backslash\{0\})^{n+1}$ modulo some relations, denoted by $Q(c_0, \cdots, c_n)$,

$$Q(c_0, \cdots, c_n) = \{A \in \mathcal{LG} \mid A = \sum_{i=1}^{n} p_i h_i + \sum_{i=0}^{n} c_i x_{\alpha_i}(e_{\alpha_i} + e_{-\alpha_i})\} \tag{13}$$

where $\alpha_1, \cdots, \alpha_n$ are simple roots of $\mathcal{G}$, and

$$x_{\alpha_i} = \begin{cases} exp(\sum_{k=1}^{n} q_k \alpha_i(h_k)) & i = 1, \cdots, n \\ \prod_{j=1}^{n} x_{\alpha_j}^{-a_j} & i = 0 \end{cases}$$

and $(a_1, \cdots, a_n)$ is defined by the highest root Θ of $\mathcal{G}$, *i.e.* $\Theta = \sum_{i=1}^{n} a_i \alpha_i$.

[Theorem C] (1) The phase space of generalized periodic Toda lattices can be decomposed into the union of $Q(c_0, \cdots, c_n)$, each leaf $Q(c_0, \cdots, c_n)$ has a natural symplectic structure derived from the classical r-matrix.

(2) For a given initial element $A(0) \in Q(c_0, \cdots, c_n)$, the solutions of Toda equation (2) are just a geodesic projection from some geodesics of LG/LK.

It is obvious that the quantum theory of Toda systems can be obtained by quantizing the phase space such as $Q(c_0, \cdots, c_n)$, the procedure of geometric quantization is applied in a straightfroward way (See section 5) to get the same partial results as in Goodman and Wallach's papers [7].

From the decomposition (5) or the classical r-matrix, we obtain the Lie bialgebra structure on the infinite dimensional Lie algebra $\mathcal{LG}$. By the construction of Semenov-Tian-Shansky [20], there exists a classical double in our case. This will be given explicitly by the decomposition (5), we will also see some hidden relations between the quantum theory of Toda systems and quantum group. The later leads to many interesting problems in recent years.

§2.Preliminaries

In this section, we review some fundmental results about Lie algebra, Lie group, loop group and loop algebra which are involved in our paper. We give them in the form of lemmas without proof. The concepts and the proofs can be found in the references. In our paper, all notations can be found in this section.

[Lemma 1] Construction of Complex Semisimple Lie Algebra[9]

Let $\mathcal{G}$ be a semisimple Lie algebra over a complex field $\mathcal{C}$, $\mathcal{H}$ the Cartan subalgebra of $\mathcal{G}$, $\mathcal{H}^*$ the dual space of $\mathcal{H}$. Then there exists a finite set Δ in $\mathcal{H}^*$, called the root system of $(\mathcal{G}, \mathcal{H})$, it has the following properties:

(a) For any $\alpha \in \Delta$, we take $\mathcal{G}_\alpha = \{x \in \mathcal{G} \mid [h, x] = \alpha(h)x \text{ for any } h \in \mathcal{H}\}$, then

$$dim_{\mathcal{C}}\mathcal{G}_\alpha = 1$$

and $\mathcal{G}$ has the root spaces decomposition:

$$\mathcal{G} = \mathcal{H} \bigoplus \sum_{\alpha \in \Delta} \mathcal{G}_\alpha \tag{14}$$

(b) Let $Kill(x, y) = Tr(adxady)$, where $x, y \in \mathcal{G}$, then

(i) $Kill(\mathcal{G}_\alpha, \mathcal{G}_\beta) = 0$, for any $\alpha, \beta \in \Delta$ and $\alpha + \beta \neq 0$.

(ii) The Killing form is nondegenerate on $\mathcal{H} \times \mathcal{H}$ and $\mathcal{G}_\alpha \times \mathcal{G}_{-\alpha}$ for any $\alpha \in \Delta$. In particular, the Killing form is nongenerate on $\mathcal{G} \times \mathcal{G}$.

(c) There exist non-zero elements $E_\alpha \in \mathcal{G}_\alpha$ for each $\alpha \in \Delta$ such that

(i) $Kill(E_\alpha, E_{-\alpha}) = 1$

(ii) If $\alpha, \beta \in \Delta$, $\alpha + \beta \neq 0$, then

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & (\alpha + \beta \in \Delta) \\ 0 & (\alpha + \beta \notin \Delta) \end{cases}$$

where $N_{\alpha,\beta}$ is a non-zero real number.

(iii) $[E_\alpha, E_{-\alpha}] = H_\alpha$ for any $\alpha \in \Delta$, and $Kill(H_\alpha, H_{-\alpha}) = \alpha(H)$ for any $H \in \mathcal{H}$. One can construct the real vector space $\mathcal{H}_\mathcal{R}$ spanned by H_α, *i.e.*

$$\mathcal{H}_\mathcal{R} = \sum_{\alpha \in \Delta} \mathcal{R} \cdot H_\alpha$$

where $\mathcal{R} \cdot H_\alpha$ is spaned by H_α over $\mathcal{R}$. Then

$$\mathcal{H} = \mathcal{H}_\mathcal{R} + i\mathcal{H}_\mathcal{R}$$

$$\mathcal{H}_\mathcal{R} \bigcap i\mathcal{H}_\mathcal{R} = \{0\}$$

and $Kill|_{\mathcal{H}_\mathcal{R} \times \mathcal{H}_\mathcal{R}}$ is the positive definite inner product.

(d) There is a subset $\Pi \in \Delta$, whose element is called a simple roots, such that any root in Δ can be expressed in the linear combination of simple roots whose coefficients are all positive or all negative integers, we call them positive or negative root respectively. To simplify, we denote $\Delta = \Delta_+ \bigcup \Delta_-$. Then $\{H_\alpha, \alpha \in \Pi;\ E_\alpha, \alpha \in \Delta\}$ is the Chevalley basis of $\mathcal{G}$, and the collection $\{H_\alpha, E_\alpha | \alpha \in \Pi\}$ is a set of generators of $\mathcal{G}$.

(e) Let

$$\mathcal{K} = i\mathcal{H}_\mathcal{R} + \sum_{\alpha \in \Delta} \mathcal{R}(E_\alpha - E_{-\alpha}) + \sum_{\alpha} \mathcal{R} \cdot i(E_\alpha + E_{-\alpha})$$

$$\mathcal{P} = \mathcal{H}_\mathcal{R} + \sum_{\alpha \in \Delta} \mathcal{R}(E_\alpha + E_{-\alpha}) + \sum_{\alpha \in \Delta} \mathcal{R} \cdot i(E_\alpha - E_{-\alpha})$$

Then $Kill|_{\mathcal{K} \times \mathcal{K}}$ is negative definite, $Kill|_{\mathcal{P} \times \mathcal{P}}$ is positive definite, $\mathcal{K}$ is orthogonal to $\mathcal{P}$ with respect to Killing form. In particular, we have the commutative relations

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K} \quad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P} \quad [\mathcal{P}, \mathcal{P}] \subset \mathcal{K}$$

Moreover there exists an involution σ on $\mathcal{G}$ such that $\sigma(E_\alpha) = -E_{-\alpha}$, which has eigenvalues +1 and -1 with eigenspaces $\mathcal{K}$ and $\mathcal{P}$ respectively. We call σ the Cartan involution.

(f) (Iwasawa decomposition of $\mathcal{G}$)

Let $\mathcal{A}$=$\mathcal{H}_\mathcal{R}$, $\mathcal{N}$=$\sum_{\alpha \subset \Delta_+} \mathcal{C} \cdot E_\alpha$, then

$$\mathcal{G} = \mathcal{K} + \mathcal{A} + \mathcal{N}$$

is the direct sum of Lie algebras .

[Lemma 2] (Construction of Loop Algebra) [11]

We denote $\mathcal{LG}$=$\mathcal{C}^\infty(S^1, \mathcal{G})$, *i.e.* the smooth maps from S^1 to $\mathcal{G}$. $\mathcal{LG}$ is called the loop algebra of $\mathcal{G}$. Then

(a) By Fourier's expansion, one has

$$\mathcal{LG} = \bigoplus_{k\in\mathcal{Z}} \lambda^k \mathcal{H} \bigoplus \sum_{k\in\mathcal{Z}\ \alpha\in\Delta} \lambda^k \mathcal{G}_\alpha$$

(b) The Cartan involusion σ can be extended to $\mathcal{LG}$ by

$$\sigma(\lambda^k x) = \lambda^{-k}\sigma(x) \quad (k \in \mathcal{Z}, x \in \mathcal{G})$$

Then

$$\begin{aligned}
\mathcal{LK} &= \mathcal{C}^\infty(S^1, \mathcal{K}) \\
&= \{x \in \mathcal{LG} | \sigma(x) = x\} \\
&= \oplus_{k\in\mathcal{Z}}(\lambda^k + \lambda^{-k}) i\mathcal{H}_\mathcal{R} \oplus_{k\in\mathcal{Z}}(\lambda^k - \lambda^{-k})\mathcal{H}_\mathcal{R} \\
&\quad \oplus(\textstyle\sum_{\alpha\in\Delta}\mathcal{R}\cdot(\lambda^k E_\alpha - \lambda^{-k}E_{-\alpha}) + \sum_{\alpha\in\Delta}\mathcal{R}\cdot i(\lambda^k E_\alpha + \lambda^{-k}E_{-\alpha})) \\
\mathcal{LP} &= \mathcal{C}^\infty(S^1, \mathcal{P}) \\
&= \{x \in \mathcal{LG} | \sigma(x) = -x\} \\
&= \oplus_{k\in\mathcal{Z}}(\lambda^k + \lambda^{-k})\mathcal{H}_\mathcal{R} \oplus_{k\in\mathcal{Z}} i(\lambda^k - \lambda^{-k})\mathcal{H}_\mathcal{R} \\
&\quad \oplus(\textstyle\sum_{\alpha\in\Delta}\mathcal{R}\cdot(\lambda^k E_\alpha + \lambda^{-k}E_{-\alpha}) + \sum_{\alpha\in\Delta}\mathcal{R}\cdot i(\lambda^k E_\alpha - \lambda^{-k}E_{-\alpha}))
\end{aligned}$$

(c) The set $\{\lambda^k H_\alpha, \alpha \in \Pi, \quad \lambda^k E_\alpha, \alpha \in \Delta k \in \mathcal{Z}\}$ is the Chevalley basis of $\mathcal{LG}$ with the following commutative relations:

$$[\lambda^k H_\alpha, \lambda^l H_\beta] = 0 \qquad (\alpha, \beta \in \Pi)$$

$$[\lambda^k E_\alpha, \lambda^l E_\beta] = \begin{cases} \lambda^{k+l} N_{\alpha,\beta} E_{\alpha+\beta} & (\ \alpha, \beta, \alpha+\beta \in \Delta) \\ \lambda^{k+l} H_\alpha & (\ \alpha = -\beta \in \Delta) \\ 0 & (\ otherwise) \end{cases} \tag{15}$$

$$[\lambda^k H_\alpha, \lambda^l E_\beta] = \beta(H_\alpha)\lambda^{k+l} E_\beta \qquad (\alpha, \beta \in \Delta)$$

where $k, l \in \mathcal{Z}$ and $N_{\alpha,\beta}$ given by Lemma 1(c).

(d) Three is a nondegenerate bilinear form on $\mathcal{LG}$ defined by

$$< \xi, \eta >= \frac{1}{2\pi}\int_0^{2\pi} Kill(\xi(\theta), \eta(\theta))d\theta \tag{16}$$

Then $<,>_{\mathcal{LK}\times\mathcal{LK}}$ is negative definite, $<.>_{\mathcal{LP}\times\mathcal{LP}}$ is positive definite and $\mathcal{LK}$ is orthogonal to $\mathcal{LP}$.

(e) If one set $\Theta = \sum_{i=1}^n a_i\alpha_i$, the highest root of Δ, then collection $\{E_{\pm\alpha}, \alpha \in \Pi, \lambda^{-1}E_\Theta, \lambda E_{-\Theta}\}$ is a set of generators of $\mathcal{LG}$.

[Lemma 3] About Lie Group and Loop Group [8,18]

Let G be the complex semisimple Lie group with Lie algebra $\mathcal{G}$, K is the maximal compact subgroup with trivial centre, LG=$\mathcal{C}^\infty(S^1, G)$, LK=$\mathcal{C}^\infty(S^1, K)$. Then

(a) The Lie algebra of LG and LK are respective to $\mathcal{LG}$ and $\mathcal{LK}$.

(b) There exists an involution on LG, its differential is given by Lemma 2(b), its fixed points is LK.

(c) LG/LK is an infinite dimentional Riemannian symmetric space. This statement will be given in detail later.

(d) Iwasawa decomposition of G

G can be decomposed into:

$$G = KAN$$

its infinitesimal form is just the Iwasawa decomposition of $\mathcal{G}$.

(e) Birkhoff decomposition of LG

Let ΩK be the subgroup of LK whose elements are through the unit of K, L^+G the subgroup of LG, whose elements are the boundary values of holomorphic maps in the unit disk. Then we have

$$LG = \Omega K \cdot L^+G \tag{17}$$

To construct the decomposition that we need exactly, we change the form of (17) in the following way :

[Corollary 4] Let G be a complex semisimple Lie group, K its maximal compact subgroup with trivial centre, LG and LK are loop groups respective to G and K. We denote S the subgroup of LG which satisfies the following property:

For $\gamma \in S$, γ can be extended holomorphically to the unit disk with $r(0) \in AN$.

where A, N are given by Lemma 3(d). Then

$$LG = LK \cdot S \tag{18}$$

Proof: We first express $\gamma \in L^+G$ of Lemma 3(c) in the form of series: $\gamma = \gamma_0 + \gamma_1 z + \cdots + \gamma_n z^n + \cdots$. Note that $\gamma_0 = \gamma(0)$ can be decomposed by Iwasawa decomposition of G, *i.e.* $\gamma_0 = kan$. So $\gamma = k(an + k^{-1}\gamma_1 z + \cdots + k^{-1}\gamma_n z^n + \cdots) \in K \cdot S$. Then combining with the Birkhoff decomposition of LG and $LK = \Omega K \cdot K$, one obtains $LG = LK \cdot S$. This complete the corollary. Q.E.D.

We take $\mathcal{S}$ for the Lie algebra of S, which is important for our paper, then

$$\begin{aligned} \mathcal{S} &= \{\xi \in \mathcal{LG} | \xi \text{ is holomorphic} \\ &\quad \text{inside of } S^1 \text{ with } \gamma(0) \in \mathcal{A} \oplus \mathcal{N}\} \\ &= \mathcal{H}_{\mathcal{R}} \oplus \textstyle\sum_{\alpha \in \Delta_+} \mathcal{G}_\alpha \oplus \sum_{k>0} \lambda^k \mathcal{G} \end{aligned} \tag{19}$$

§3. Construction of Generalized Periodic Toda Lattices

In many interesting examples , the Hamilton systems can be constructed on the dual space of a Lie algebra which admit sufficiently many integrals of motion in involution . They can also be restricted to a coadjoint orbit. It should be pointed out that these constructions may be viewed as the more general r-matrix method. Seminov-Tian-Shansky has developed into a elegant Lie algebraic scheme in [20,21]. In this section we use the construction of Seminov-Tian-Shansky to give the Hamilton systems of generalized periodic Toda lattices, we choose a r-matrix with a spectral parameter. We first give two equivalent definitions of classical r-matrix, the details may be found in [3.20].

[Definition 1] Let $\mathcal{L}$ be finite or infinite dimensional Lie algebra, we call $r \in \mathcal{L} \otimes \mathcal{L}$ a classical r-matrix if r satisfies the following modified Yang-Baxter equation:

$$\begin{aligned} &[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \quad \text{is } \mathcal{L}-\text{invariant}. \\ &r^{12} + r^{21} = 0 \end{aligned} \tag{20}$$

where $\mathcal{L}$-action is the adjoint action, $r^{12}, r^{13}, r^{23}, r^{21} \in \otimes \mathcal{U}(\mathcal{L})^3$, *i.e.* if one sets $\{I_\mu\}$ for a basis of $\mathcal{L}$, $r = \sum_{\mu,\nu} r^{\mu\nu} I_\mu \otimes I_\nu$, then

$$\begin{aligned} r^{12} &= \sum_{\mu,\nu} r^{\mu\nu} I_\mu \otimes I_\nu \otimes 1 \\ r^{13} &= \sum_{\mu,\nu} r^{\mu\nu} I_\mu \otimes 1 \otimes I_\nu \\ r^{23} &= \sum_{\mu,\nu} r^{\mu\nu} 1 \otimes I_\mu \otimes I_\nu \\ r^{21} &= \sum_{\mu,\nu} r^{\mu\nu} I_\nu \otimes I_\mu \otimes 1 \end{aligned}$$

In this case, the Lie algebra can be equipped with a Lie bialgebra structure and $(\mathcal{L} \oplus \mathcal{L}^*, \mathcal{L}, \mathcal{L}^*)$ is called a Manin triple. [3.15.20].

If there exists a non-degenerate invariant bilinear form on $\mathcal{L}$, then Definition 1 of the classical r-matrix is equivalent to the following definition:

[Definition 2] Let $\mathcal{L}$ be a finite or infinite dimensional Lie algebra, a linear operator $R \in End(\mathcal{L})$ is called a classical r-matrix if the bracket given by

$$[X, Y]_R = [R(X), Y] + [X, R(Y)] \qquad for \ \ X, \ Y \in \mathcal{L} \tag{21}$$

is still a Lie algebra bracket on $\mathcal{L}$. The Lie algebra with this classical r-matrix is called the double Lie algebra.

The equivalence between these two definitions means: for a fixed basis $\{I_\mu\}$ of $\mathcal{L}$, we can found an another basis $\{I^\mu\}$ such that $< I_\mu, I^\nu >= \delta_{\mu\nu}$, this is guaranteed by the non-degenerate bilinear form $<.>$ on $\mathcal{L}$. Then if a classical r-matrix given by Definition 1, *i.e.*

$$r = \sum_{\mu,\nu} r^{\mu\nu} I_\mu \otimes I_\nu$$

it can be expressed in the form of Definition 2 :

$$R(X) = \sum_{\mu,\nu} r^{\mu\nu} I_\mu < I_\nu, X > \qquad for \ \ X \in \mathcal{L} \tag{22}$$

On the contrary, if a classical r-matrix given by Definition 2 , we can construct it in the tensor-form, *i.e.*

$$r = \sum_\mu R(I_\mu) \otimes I^\mu$$

We keep this equivalence in mind and constructe Hamilton systems of generalized periodic Toda lattices.

Let $\{h_i\}_{i=1}^n$ be an orthogonal basis for $\mathcal{H}_R$. In $\mathcal{U}(\mathcal{LG})$, the universal enveloping algebra of $\mathcal{LG}$, we regard $\mathcal{LG}\otimes 1$ and $1\otimes\mathcal{LG}$ as realized in $(\mathcal{G}\otimes C[\lambda,\lambda^{-1}])\otimes 1$ and $1\otimes(\mathcal{G}\otimes C[\mu,\mu^{-1}])$ respectively. We intruduce a classical r-matrix simply in tensor-form [1,10]:

$$\begin{aligned} r &= r(\lambda,\mu) \\ &= \textstyle\sum_{\alpha\in\Delta} sign(\alpha)E_\alpha\otimes E_{-\alpha} + \frac{\mu-\lambda}{\mu+\lambda}(\sum_{i=1}^n h_i\bigotimes h_i + \sum_{\alpha\in\Delta} E_\alpha\bigotimes E_{-\alpha}) \end{aligned} \tag{23}$$

where $sign(\alpha) = \begin{cases} +1 & for \quad \alpha\in\Delta_+ \\ -1 & for \quad \alpha\in\Delta_- \end{cases}$

we can express (23) as follows :

$$\begin{aligned} r &= \textstyle\sum_{\alpha\in\Delta} sign(\alpha)E_\alpha\otimes E_{-\alpha} + \sum_{i=1}^n\sum_{k\in Z\backslash 0} sign(k)\lambda^k h_i\otimes\mu^{-k}h_i \\ &\quad + \textstyle\sum_{\alpha\in\Delta}\sum_{k\in Z\backslash 0} sign(k)\lambda^k E_\alpha\otimes\mu^{-k}E_{-\alpha} \end{aligned} \tag{24}$$

where $sign(k) = \begin{cases} 1 & for \quad k>0 \\ -1 & for \quad k<0 \end{cases}$

From (22) (24), we know the operator R on $\mathcal{LG}$ could be realized as follows :

$$\begin{aligned} &R|_{\mathcal{H}} = 0 \\ &R|_{\mathcal{G}_\alpha} = sign(\alpha)Id \quad for \quad \alpha\in\Delta \\ &R|_{\lambda^k\mathcal{G}} = sign(k)Id \quad for \quad k\neq 0 \end{aligned} \tag{25}$$

So we obtain :

[Proposition 1] Let $\mathcal{G}$ be a complex semisimple Lie algebra, $\mathcal{LG}$ its loop algebra, then the classical r-matrix (23-25) gives a classical double structure on $\mathcal{LG}$.

Proof: The r-matrix (23) (24) is given by Belavin and Drinfel'd in [4], which is satisfied the classical Yang-Baxter equation (20). We also know that the new bracket (21) is still a Lie bracket. And the classical double structure is easily obtained by the Seminov-Tian-Shansky construction [20,21]. Q.E.D.

Using the classical r-matrix (25), we have a new Lie bracket on $\mathcal{LG}$ by definition 2, then the new commutative relations of Lemma 2(c) in section 2 become:

$$\begin{aligned} &[\lambda^k H_\alpha, \lambda^l H_\beta]_R = 0 && for\ k,l\in Z\ ,\ \alpha,\ \beta\in\Pi \\ &[H_{\alpha_i}, E_{\pm\alpha}]_R = \alpha(H_{\alpha_i})E_{\pm\alpha} && for\ \alpha\in\Delta_+,\ \ \alpha_i\in\Pi \\ &[E_\alpha, E_\beta]_R = 0 && for\ \ \alpha,\in\Delta_+\ \ \beta\in\Delta_- \\ &[E_\alpha, E_\beta]_R = 2N_{\alpha,\beta}E_{\alpha+\beta} && for\ \ \alpha,\beta,\alpha+\beta\in\Delta_+ \\ &[\lambda^k H_{\alpha_i}, \lambda^l E_{\pm\alpha}]_R = (sign(k)+sign(l))(\pm\alpha(H_{\alpha_i}))\lambda^{k+l}E_{\pm\alpha} \\ &\qquad for\ k,l\in Z\backslash 0\ \ \alpha\in\Delta_+, \alpha_i\in\Pi \end{aligned} \tag{26}$$

If we choose coordinates in $\mathcal{LG}^*$ to be $\{h_{i,k}, x_{\alpha,k} | k\in Z, i=1,\cdots,n, \alpha\in\Delta\}$ such that

$$\begin{aligned} h_{i,k}(L) &= <\lambda^k h_i, L> \\ x_{\alpha,k}(L) &= <\lambda^k E_\alpha, L> \end{aligned}$$

for $L\in\mathcal{LG}$. Then we can define the Poisson bracket on $C^\infty(\mathcal{LG}^*)$:

$$\{f_1,f_2\}_R(L) = L([\nabla f_1(L), \nabla f_2(L)]_R) \tag{27}$$

where

$$\nabla f_i(L) = \sum_{i,k}\frac{\partial f_i}{\partial h_{i,k}}(L)\lambda^k H_\alpha + \sum_{k,\alpha}\frac{\partial f_i}{\partial x_{\alpha,k}}(L)\lambda^k E_\alpha \in \mathcal{LG}$$

and $f_1, f_2\in C^\infty(\mathcal{LG}^*), L\in\mathcal{LG}^*$.

Combining (26-27), one knows that $\mathcal{LG}^*$ is a infinite dimensional Poisson manifold. In particular ,the Poisson brackets between coordinate functions are :

$$\begin{aligned} &\{h_{i,k}, h_{j,l}\}_R = 0 \\ &\{h_i, x_{\pm\alpha}\}_R = \alpha(h_i)x_{\pm\alpha} \quad for \quad \alpha\in\Delta_+ \\ &\{x_\alpha, x_{-\beta}\}_R = 0 \quad for \quad \alpha,\beta\in\Delta_+ \\ &\{x_\alpha, x_\beta\}_R = 2N_{\alpha,\beta}x_{\alpha+\beta} \quad for \quad \alpha,\beta,\alpha+\beta\in\Delta_+ \end{aligned} \tag{28}$$

where h_i, x_α simply denote $h_{i,0}$, $x_{\alpha,0}$ respectively.

Now let

$$A = \sum_{i=1}^{n} p_i h_i + \sum_{i=0}^{n} c_i exp(\sum_{k=1}^{n} q_k \alpha_i(h_k))(e_{\alpha_i} + e_{-\alpha_i}) \tag{29}$$

where $e_{\alpha_i} = \begin{cases} E_{\alpha_i} & for\ i = 1, \cdots, n \\ \lambda E_{-\Theta} & for\ i = 0 \end{cases}$

$$e_{-\alpha_i} = \begin{cases} E_{-\alpha_i} & for\ i = 1, \cdots, n \\ \lambda^{-1} E_{\Theta} & for\ i = 0 \end{cases}$$

then

$$R(A) = \sum_{i=0}^{n} c_i exp(\sum_{k=1}^{n} q_k \alpha_i(h_k))(e_{\alpha_i} - e_{-\alpha_i}) \tag{30}$$

So we obtained the equation of generalized periodic Toda lattices [10]:

$$\frac{dA}{dt} = [R(A), A] \tag{31}$$

Because we can identify $\mathcal{LG}^*$ with $\mathcal{LG}$ by the nondegenerate bilinear form on $\mathcal{LG}$ (See Lemma 2.2(d)). Restricted to coadjoint-orbit of $\mathcal{LG}^*$ with the symplectic structure given by (27) (28), then Lax equation (31) is just the equation of Hamilton systems whose Hamiltonnian is $\frac{1}{2} < A, A >$. We will give the proof that the phase space of (31) is the union of the following family symplectic manifolds, each leaf is $Q(c_0, \cdots, c_n)$, depending on $(c_0, \cdots, c_n) \in (\mathcal{C}\backslash\{0\})^{n+1}$, is defined by

$$Q(c_0, \cdots, c_n) = \{A \in \mathcal{LG} \mid A = \sum_{i=1}^{n} p_i h_i + \sum_{i=0}^{n} c_i x_{\alpha_i}(e_{\alpha_i} + e_{-\alpha_i})\}$$

where

$$x_{\alpha_i} = \begin{cases} exp(\sum_{k=1}^{n} q_k \alpha_i(h_k)) & i = 1, \cdots, n \\ \prod_{j=1}^{n} x_{\alpha_j}^{-a_j} & i = 0 \end{cases}$$

and $(a_1, \cdots, a_n)$ is defined by the highest root $\Theta = \sum_{i=1}^{n} a_i \alpha_i$ in $\mathcal{G}$.

It is easy to see that

$$Q(c_0, \cdots, c_n) = Q(c_0', \cdots, c_n') \ if\ and\ only\ if\ there\ exists$$

$$(q_1, \cdots, q_n) \in \mathcal{C}^n\ such\ that\ \ c_i' = c_i exp(\sum_{k=1}^{n} \alpha_i(h_k) q_k) \tag{*}$$

The existence of $(q_1, \cdots, q_n)$ is obtained by the nondegenerate matrix ($\alpha_i(h_j)$). So we choose $(c_0, \cdots, c_n) \in (\mathcal{C}\backslash\{0\})^{n+1}$ with modulo the relation (*).

We have finished the construction of generalized periodic Toda lattices by the standard r-matrix method. The classical and quantum theories will be given in detail later.

§4. Property of the Solutions

In this section , we shall discuss the solution of Lax equation by using the theories of the infinite dimensional geometry. In general, because the coadjoint orbits have symplectic structures, the Hamiltonian systems defined on the coadjoint orbits are very important and have been well developed. Especially, if the Lie algebra has a nondegenerate bilinear form, then the Hamiltonian equation can be expressed in Lax form. In section 3, we have constructed Generalized Toda Lattices in this sense. From section 3, we have the r-matrix (24) and the operator (25) on $\mathcal{LG}$, then we only consider the following Lax equation:

$$\frac{dA}{dt} = [R(A), A] \tag{32}$$

where A valued in $\mathcal{LG}$. Of course, this equation may be trivial for some A, but if we choose $A \in \mathcal{S}^* \subset \mathcal{LG}$, the dual space of $\mathcal{S}$ (19), then many interesting results can be obtained.

[Proposition 2] Under the identification of $\mathcal{S}^* \subset \mathcal{LG}$, the Lax equation (32) is the equation of Hamilton flow on coadjoint orbits of $\mathcal{LG}^*$, whose Poisson structure given by (27), with the Hamiltonian $H = \frac{1}{2} < A, A >$.

Proof: First note that, under the identification of $\mathcal{LG}^* = \mathcal{LG}$, $Ad^*\gamma = Ad\gamma$ for $\gamma \in LG$ and $ad^* = -ad$. Since H is $Ad-invariant$ function, so the Hamiltonian on the coadjoint orbits is well defined. The Hamilton's equation is the form:

$$\frac{dA}{dt} = \{H, A^*\}_R \tag{33}$$

where $A \in \mathcal{LG}, A^* =< A, \cdot >$ is a function on $\mathcal{LG}$. By the definition of Poisson bracket $\{,\}_R$, the equation is just (32). Q.E.D.

This Hamilton system is integrable. Because the Poisson structure on $\mathcal{LG}^*$ is induced by the standard Poisson-tensor π on LG, *i.e.*

$$\pi(\gamma) = l_{\gamma_*}\mathrm{r} - r_{\gamma_*}\mathrm{r}$$

where $\gamma \in LG$, $l_{\gamma_*}, r_{\gamma_*}$ are the tangent maps of left multiplication γ and right multiplication γ respectively, and r is the r-matrix (23-24). So the integrablity is followed by the V.G.Drinfel'd in involution theorem [2] and [5].

One easily gives a differential structure on LG/LK, for there exists a local isometric diffeomorphism between $\mathcal{LG}$ and LG, *i.e.* exponential map. The quotient space is diffeomorphic to $\mathcal{LP}$ locally (see Lemma 2(b) in section 2). By the left multiplication of LG, we obtain a smooth cover on LG/LK which gives the manifold structure. An obvious result is that the tangent space at any point on LG/LK is isomorphic $\mathcal{LP}$. Since there exists a positive definite inner product, we can obtain a LG-invariant metric on left coset space LG/LK. From Lemma 3(b) in section 2, LG/LK is an infinite dimensional Riemannian symmetric space.

[Theorem 1] There exists a geodesic immersion:

$$\Psi: \quad LG/LK \longrightarrow LG$$

where LG/LK is equipped with LG-invariant metric above and LG is equipped with the standard Riemannian structure.

Proof : For LK is the fixed points of the involution automorphism σ, so $\sigma(\gamma)^{-1}\gamma$ is independent of the presentation elements in the coset $[\gamma] = LK.\gamma$, *i.e.* $\sigma(\gamma)^{-1}\gamma$ gives the parameterized coordinates of LG/LK. We define $\Psi([\gamma]) = \sigma(\gamma)^{-1}\gamma$. This is well defined. Up to a positive scaler. Ψ is isometric. Then we can normalize it such that Ψ is a geodesic immersion. Q.E.D

In the following discussion, we will identify LG/LK with the image of Ψ and think the element of LG/LK in form of $\sigma(\gamma)^{-1}\gamma$. From corollary 4 in section 2, LG have decomposed into two factors: $LG = LK \cdot S$, then for a coset $[\gamma]$, let $\gamma = k(\gamma)s(\gamma)$, we have $[\gamma] = [s(\gamma)]$. So we reparameterize LG/LK with the elements of S, *i.e.* $\Phi(s) = [s]$ is one-to-one from S to LG/LK, under the identification of Ψ, $\Phi(s) = \sigma(s)^{-1}s$. Because Killing form restricted to S, the tangent space of S, is

degenerate, so S has not a Riemannian structure from Ad-invariant bilinear form. But we knows that Φ is injective and surjective smooth map, then Riemannian metric on LG/LK can be pulled back to S through Φ such that Φ is isometric of course. One can easily know that this metric on S is induced by the inner product of $\mathcal{LP} = \mathcal{S}^*$ on cotangent space of S (up to a scaler) at any point. To conclude this paragraph, we claim that the covariant Riemannian tensor of this metric on LG/LK can be expressed through the identification of Ψ in following form:

$$ds^2 = \frac{1}{2\pi}\int_0^{2\pi} Kill(x^{-1}dx, x^{-1}dx)d\theta \tag{34}$$

Let $x(t)$ be a smooth curve in LG/LK , recall that the first variation formula gives the equation of geodesic flow. Directly calculation gives:

$$\frac{d}{dt}(\dot{x}(t)\cdot x^{-1}(t)) = 0 \tag{35}$$

where $\dot{x}(t)$ denote $\frac{dx}{dt}$. So the geodesic equation in this infinite dimensional case is the same as that in the finite dimensional case.

The reader will obtain the following Lemma without any difficulty.

[Lemma] Let $A \in \mathcal{S}$, if we identify it with a element $\tilde{A}$ in $\mathcal{S}^* = \mathcal{LP}$ by the induced inner product on $\mathcal{S}$, then $\tilde{A} = \frac{1}{2}(A - \sigma(A))$. Moreover $R(\tilde{A}) = \frac{1}{2}(A + \sigma(A))$. If $A \in \mathcal{S}^*$, then $A + R(A) \in \mathcal{S}$.

Now we give the following main theorem of classical theories.

[Theorem 2] Properties of the Solutions

(1) For a fixed initial condition $A(0) = A_0 \in \mathcal{S}^*$, the Lax equation (32) has a unique solution , expressed in (coadjoint)adjoint-orbit form:

$$\begin{aligned} A(t) &= Ad(s(expt A_0)).A_0 \\ &= Ad(k^{-1}(expt A_0)).A_0 \end{aligned} \tag{36}$$

where the element $s(t) = s(expt A_0)$ and $k(t) = k(expt A_0)$ are defined by $exp(tA_0)$ in the decomposition of (18), *i.e.* $exp(tA_0) = k(t)s(t)$.

(2) If one pulls back the tangent vector at $s(t)$ by right action of LG, *i.e.* take the function

$$A(t) = \frac{ds(t)}{dt}\cdot s^{-1}(t)$$

which belongs to $\mathcal{S}^*$ under the identification $\mathcal{S}^* = \mathcal{S}$, then $A(t)$ satisfies Lax equation (32).

(3) If we denote

$$x(t) = \Phi(s(exp tA)) = \Phi(s(t)) \tag{37}$$

where Φ is defined by (7),and $s(exp tA) = s(t)$ is defined by Theorem B, then

$$\frac{d}{dt}(\dot{x}(t) \cdot x^{-1}(t)) = 0 \tag{38}$$

With the knowledge of infinite dimensional geometry given above, the equation (38) is that of the geodesic in LG/LK.

(4) Given a geodesic $x(t)$ in LG/LK, we consider

$$\Phi_*^{-1}(\dot{x}(t)x^{-1}(t))$$

in $\mathcal{S}$. If we identify this in $\mathcal{S}^*$, and call it $A(t)$. Then $A(t)$ is the solution of equation (32).

Proof: Note that $s(exp(tA_0)) = k(exp tA_0)^{-1}exp(tA_0)$ and $Ad(exp(tA_0)).A_0 = A_0$, so the second identity of (36)is obvious. First we calculate :

$$\begin{aligned}
&\dot{s}(exp(tA_0))s^{-1}(exp(tA_0)) \\
&= \frac{d}{dt}(k^{-1}(exp(tA_0))exp(tA_0))exp(tA_0)^{-1}k(exp(tA_0)) \\
&= Ad(k^{-1}(exp(tA_0)).A_0 - k^{-1}(exp(tA_0))\,\dot{k}\,(exp tA_0) \qquad (*) \\
&= A(t) + R(A(t)) \qquad (\ by\ Lemma\ 1\ )
\end{aligned}$$

where the third identity is obtained by the first part and the second part of $(*)$ belong to $\mathcal{LP}$ and $\mathcal{LK}$ respectively, and $\mathcal{LK}$ is orthogonal to $\mathcal{LP}$. So we have

$$\begin{aligned}
\frac{dA}{dt} &= \frac{d}{dt}(Ad(s(exp(tA_0))).A_0) \\
&= [\dot{s}(t)s^{-1}(t), A(t)] \\
&= [A(t) + R(A(t)), A(t)] \\
&= [R(A(t)), A(t)]
\end{aligned}$$

so (1) and (2) are obtained. Secondly, we obtain easily that

$$\dot{x}(t)x^{-1} = 2\sigma(s)^{-1}A(t)\sigma(s)$$

where $s = s(exp\ tA_0)$. Using the equation (32), (3) and (4) of Theorem 2 are obtained by directly calculation. Q.E.D.

So far, we have described the classical pictures of the infinite dimensional integrable system. As an application , we reduct to the generalized periodic Toda systems constructed in section 3. We will give that the phase of Toda lattices can be decomposed into the union of leaves $Q(c_0, \cdots, c_n)$ and the solutions are the geodesic projection of the geodesics in LG/LK.

We introduce a quotient algebra $\mathcal{B}$ of $\mathcal{S}$, *i.e.*

$$\mathcal{B} = \bigoplus_{i=1}^{n} q_i h_i + \bigoplus_{i=0}^{n} x_{\alpha_i} e_{\alpha_i}$$

where h_i, e_{α_i} are given in section 3. The commutative relations between them are :

$$[h_i, h_j] = [e_{\alpha_i}, e_{\alpha_j}] = 0 \qquad [h_i, e_{\alpha_j}] = \alpha_j(h_i)e_{\alpha_j}$$

The dual space of $\mathcal{B}$ can be identified with $\mathcal{B}^* = \sum_{i=1}^{n} p_i h_i + \sum_{i=0}^{n} x_{\alpha_i}(e_{\alpha_i} + e_{-\alpha_i})$ by the nondegenerate bilinear form on $\mathcal{LG}$. In $\mathcal{B}^*$, we construct a family of symplectic leaves, each leaf denoted by $Q(c_0, \cdots, c_n)$, depending on $(c_0, \cdots, c_n) \in (\mathcal{C}\backslash 0)^{n+1}$ modulo relations $(*)$ in section 3, is defined by

$$Q(c_0, \cdots, c_n) = \{A \in \mathcal{LG} \mid A = \sum_{i=1}^{n} p_i h_i + \sum_{i=0}^{n} c_i x_{\alpha_i}(e_{\alpha_i} + e_{-\alpha_i})\}$$

where

$$x_{\alpha_i} = \begin{cases} exp(\sum_{k=1}^{n} q_k \alpha_i(h_k)) & i = 1, \cdots, n \\ \prod_{j=1}^{n} x_{\alpha_j}^{-a_j} & i = 0 \end{cases}$$

and $(a_1, \cdots, a_n)$ is defined by the highest root Θ of $\mathcal{G}$, *i.e.* $\Theta = \sum_{i=1}^{n} a_i \alpha_i$, $\alpha_1, \cdots, \alpha_n$ are simple roots of $\mathcal{G}$.

[Theorem 3] $Q(c_0, \cdots, c_n)$ inherits the Poisson structure from $\mathcal{B}^*$ and is a symplectic manifold with the symplectic form:

$$\omega = \sum_{i,j=1}^{n} a^{ij} x_{\alpha_j}^{-1} dp_i \bigwedge dx_{\alpha_j} \tag{39}$$

where (a^{ij}) is the inverse matrixs of $(a_{ij}) = (\alpha_i(h_j))$.

Proof: The Poisson structure on $\mathcal{B}^*$ are induced by (27) (28), the Poisson brackets between the coordinate functions should to be the following form:

$$\begin{aligned} \{p_i, x_{\alpha_j}\}_{PB} &= \alpha_j(h_i)x_{\alpha_j} \quad for \quad j = 1, \cdots, n \\ \{p_i, x_{\alpha_0}\}_{PB} &= -\Theta(h_i)x_{\alpha_0} \\ \{p_i, p_j\}_{PB} &= \{x_{\alpha_i}, x_{\alpha_j}\}_{PB} = 0 \end{aligned} \tag{40}$$

This identities are obtained by the formulae (27), we only illustrate the second identity as an example. Note that

$$\nabla p_i = h_i, \quad \nabla x_{\alpha_0} = \lambda^{-1}E_\Theta$$

Then

$$\begin{aligned} & \{p_i, x_{\alpha_0}\}_{PB}(L) \\ = & < [\nabla p_i, \nabla x_{\alpha_0}]_R, L > \\ = & < [h_i, \lambda^{-1}E_\Theta]_R, L > \\ = & < [R(h_i), \lambda^{-1}E_\Theta] + [h_i, R(\lambda^{-1}E_\Theta)], L > \\ = & < -[h_i, \lambda^{-1}E_\Theta], L > \\ = & -\Theta(h_i) < \lambda^{-1}E_\Theta, L > \\ = & -\Theta(h_i)x_{\alpha_0} \end{aligned}$$

One knows that the Poisson-tensor on $\mathcal{B}^*$ is given by

$$\pi_0 = a_{ji}x_{\alpha_j}\frac{\partial}{\partial p_i} \bigwedge \frac{\partial}{\partial x_{\alpha_j}} - \Theta(h_i)x_{\alpha_0}\frac{\partial}{\partial p_i} \bigwedge \frac{\partial}{\partial x_{\alpha_0}}$$

To calculate the Poisson structure on each leaf, we define a projection :

$$\rho: \quad B^* \longrightarrow \quad Q(c_0, \cdots, c_n)$$

by

$$\begin{aligned} & \rho(\textstyle\sum_{i=1}^n p_i h_i + \sum_{i=0}^n x_{\alpha_i}(e_{\alpha_i} + e_{-\alpha_i})) \\ & = \textstyle\sum_{i=1}^n p_i h_i + \sum_{i=1}^n c_i x_{\alpha_i}(e_{\alpha_i} + e_{-\alpha_i}) + \prod_{j=1}^n x_{\alpha_j}^{-\eta_j}(e_{\alpha_0} + e_{-\alpha_0}) \end{aligned}$$

Then the inherited Poisson-tensor on $Q(c_0, \cdots, c_n)$ is given by:

$$\pi = \rho_*(\pi_0) = a_{ji}x_{\alpha_j}\frac{\partial}{\partial p_i} \bigwedge \frac{\partial}{\partial x_{\alpha_j}}$$

since $\rho_*(\frac{\partial}{\partial x_{\alpha_0}}) = 0$. This Poisson structure π is nondegenerate, and the Poisson bracket on $C^\infty(Q(c_0, \cdots, c_n))$ is given by

$$\{f, g\} = \pi(f \bigotimes g) = \pi(df \bigotimes dg)$$

We define a 2-form by

$$\omega = \sum_{i,j=1}^n a^{ij}x_{\alpha_j}^{-1}dp_i \bigwedge dx_{\alpha_j}$$

and let $X_f = df \rfloor \pi$. Then

$$X_f \rfloor \omega = -df$$

$$\{f, g\} = \omega(X_f, X_g) = X_f(g)$$

So ω is a nondenegerate closed 2-form from that the Poisson tensor π is nongenerate and the Poisson bracket $\{,\}$ is satisfied the Jacobi condition. This completes the proof of Theorem 3. Q.E.D.

Note that $\mathcal{B}$ is a direct sum of two commutative subalgebras, and the second subalgebra is an ideal of $\mathcal{B}$. So we can integrate them and obtain a decomposition for the Lie group B:

$$B = exp(\sum_{i=1}^n q_i h_i)exp(\sum_{i=0}^n x_{\alpha_i}e_{\alpha_i})$$

And there exists a projection which is isometric :

$$\pi: \quad S \longrightarrow B$$

For an initial condition $A_0 \in \mathcal{B}^* \subset \mathcal{LP}$, we know $A(t) = Ad(s(exp(tA_0)))A_0$ is a solution of Lax equation (32). We also denote the differential of π with the same symbol. Then $A(t) = Ad(\pi(s(expt A_0)))A_0$ is the solution of the generalized periodic Toda equation, since if we let

$$\pi(s(expt A_0)) = exp(\sum_{i=1}^n q_i(t)h_i)exp(\sum_{i=0}^n x_{\alpha_i}e_{\alpha_i})$$

Then

$$\pi(\dot{s}(expt A_0)s^{-1}(expt A_0))$$

$$= \sum_{i=1}^n q_i h_i + \sum_{j=0}^n \dot{x}_{\alpha_j} exp(\sum_{i=1}^n \alpha_j(h_i)q_i(t))e_{\alpha_j}$$

$$= A(t) + R(A(t))$$

so $A(t) = \sum_{i=1}^n h_i + \sum_{j=0}^n \frac{1}{2}\dot{x}_{\alpha_j} exp(\sum_{i=1}^n \alpha_j(h_i)q_i(t))(e_{\alpha_j} + e_{-\alpha_j})$. By the proof of Theorem 2, we know $A(t)$ is the solution of Toda equation (31). One can verify that x_{α_j} is independent of the time t by the following lemma.

[Proposition 3] Let $A_0 \in \mathcal{B}^*$, $c_j =< A_0, e_{\alpha_j} >$ $(i = 0, \cdots, n)$. Then $Q(c_0, \cdots, c_n)$ is the coadjoint orbit of B through A_0.

Proof: Note that $B = exp(\sum_{i=1}^n q_i h_i) exp(\sum_{i=0}^n x_{\alpha_i} e_{\alpha_i})$ is a product of two commutative Lie groups and $exp(\sum_{i=0}^n x_{\alpha_i} e_{\alpha_i})$ is a normal subgroup of B. Then this proposition is obtained by some tedious direct calculations. Q.E.D.

Using this proposition, we know that $< A(t), e_{\alpha_j} >=< A_0, e_{\alpha_j} >$, so

$$\lim_{t\to 0} < A(t), e_{\alpha_j} >$$

$$= \lim_{t\to 0} \frac{1}{2}\dot{x}_{\alpha_j} exp(\sum_{i=1}^n \alpha_j(h_i)q_i(t))$$

$$= \frac{1}{2}\dot{x}_{\alpha_j} = constant$$

So $A(t) \in Q(c_0, \cdots, c_n)$ for $c_j = \frac{1}{2}\dot{x}_{\alpha_j}$. Up to now , we finish the decomposition of the phase space of Toda systems. The geodesic property of the solution follows from Theorem 2 and the geodesic projection π .

§5 Geometric Quantization of Toda Systems

In this section, we will give the quantum picture of Toda systems by the standard procedure of geometric quantization [22]. We only consider the reduced Lie group B and its Lie algebra $\mathcal{B}$. By Theorem 3, one knows that $Q(c_0, \cdots, c_n)$ is a

symplectic manifold with the symplectic form:

$$\omega = \sum_{i,j=1}^n a^{ij} x_{\alpha_j}^{-1} dp_i \bigwedge dx_{\alpha_j}$$

If we change the coordinates by $x_{\alpha_i} = exp(\sum_{k=1}^n q_k \alpha_i(h_k))$ $i = 1, \cdots, n$, then

$$\begin{aligned}
\omega &= \sum_{i,j=1}^n a^{ij} x_{\alpha_j}^{-1} dp_i \wedge dx_{\alpha_j} \\
&= \sum_{i,j=1}^n a^{ij} (exp(\sum_{k=1}^n q_k \alpha_i(h_k)))^{-1} dp_i \wedge d(exp(\sum_{k=1}^n q_k \alpha_j(h_k))) \\
&= \sum_{i,j=1}^n a^{ij} (exp(\sum_{k=1}^n q_k \alpha_i(h_k)))^{-1} exp(\sum_{k=1}^n q_k \alpha_j(h_k)) a_{jk} dp_i \wedge dq_k \\
&= \sum_{i=1}^n dp_i \wedge dq_i
\end{aligned}$$

One knows that the symplectic potential is $\theta = \sum_{i=1}^n p_i dq_i$. Let the prequantum linebundle is trivial bundle ,*i.e.* $L = M \times \mathbb{C}$, where M denotes $Q(c_0, \cdots, c_n)$. Let the connection potential $\beta = -\frac{i}{\hbar}\theta$. Under the trivialization, we have a nonvanish global section s_0 such that any other section s is the form $s = \phi s_0$ with $\phi \in \mathcal{C}^\infty(M)$. Then we define a connection ∇ on L by

$$\nabla s_0 = \beta \bigotimes s_0 \tag{41}$$

then

$$\nabla_X s_0 = (X \rfloor \beta) s_0$$

so

$$\begin{aligned}
\nabla_X(\phi s_0) &= X(\phi)s_0 + \phi \nabla_X s_0 \\
&= (X(\phi) + \phi(X \rfloor \beta))s_0
\end{aligned}$$

for X is a vector field.

It is obvious that $curvature \nabla = d\beta - \beta \wedge \beta = -\frac{i}{\hbar} d\theta = -\frac{i}{\hbar}\omega$, in other words we have

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -\frac{i}{\hbar}\omega(X, Y)$$

for any tangent vector fields X, Y. We also introduce some useful identities, for each $f \in \mathcal{C}^\infty(M)$, X_f is the associated Hamiltonian vector field, *i.e.* $X_f \rfloor \omega = -df$,

then

$$(1)\ \{f,g\} = \omega(X_f, X_g) = X_f g = -X_g f$$

$$(2)\ [X_f, X_g] = X_{\{f,g\}}$$

$$(3)\ [\nabla_{X_f}, g] = X_f g$$

where $\{,\}$ denotes the Poisson bracket on $\mathcal{C}^\infty(M)$.

Now we define the prequantization operators acting on the sections of the linebundle L. For each $f \in \mathcal{C}^\infty(M)$ the associated prequantization operator is defined by

$$\hat{f} = -i\hbar\, \nabla_{X_f} + f \tag{42}$$

Then

$$\begin{aligned}
&[\hat{f}, \hat{g}] \\
= \ & [-i\hbar\, \nabla_{X_f} + f, -i\hbar\, \nabla_{X_g} + g] \\
= \ & -\hbar^2 [\nabla_{X_f}, \nabla_{X_g}] - i\hbar[\nabla_{X_f}, g] + i\hbar[f, \nabla_{X_g}] \\
= \ & -\hbar^2\, \nabla_{[X_f, X_g]} - i\hbar X_f g + i\hbar X_g f + i\hbar\omega(X_f, X_g) \\
= \ & -i\hbar^2\, \nabla_{X_{\{f,g\}}} - i\hbar\{f, g\} \\
= \ & -i\hbar\widehat{\{f, g\}}
\end{aligned}$$

This is Dirac's quantum condition in quantum systems. The Hilbert space is the completion of square integrable sections with the scaler product:

$$< s_1 | s_2 > = \int_M < s_1, s_2 > \omega^{dim M}$$

where $<,>$ is the unique Hermitian form compatible to ∇

$$X < s_1, s_2 > = < \nabla_X s_1, s_2 > + < s_1, \nabla_X s_2 >$$

and $\omega^{dim M}$ is the *Liouville form* on M. One easily knows the operator $\hat{f}$ is a self-adjoint operator with this inner product. But this space is too large to represent the phase space of a physically reasonable system. One must choose some sections that are parallel along a polarization of the classical phase space. In our case,

we choose the polarization by the distribution $P \subset T^{\mathcal{C}}(M)$, where $P = \{X \in T^{\mathcal{C}}(M) \mid X \rfloor \theta = 0\} = span_{\mathcal{C}}\{\frac{\partial}{\partial p_i}\}_{i=1}^n = span_{\mathcal{C}}\{X_{q_i}\}_{i=1}^n$.

The polarized section is $s = \phi s_0$ which satisfies

$$\nabla_{X_{q_i}} s = 0$$

i.e.

$$\frac{\partial \phi}{\partial p_i} s_0 + \phi\, \nabla_{X_{q_i}} s_0 = \frac{\partial \phi}{\partial p_i} s_0 = 0$$

So the polarized condition for ϕ is $\frac{\partial \phi}{\partial p_i} = 0$. The another problem arises since the integral of $< s|s >$ for polarized sections does not converge, so this leads to **Half-density quantization**. By the procedure of half-density quantization, the physical Hilbert space should be identified with the square integrable functions of $(q_1, \cdots, q_n)$ with the volume element $dq_1 \wedge \cdots \wedge dq_n$. So the operators correspond to classical observables, must act on the polarized sections space invariantly, we call them the quantizable classical observables. Let f be a quantizable function on M, then we have

$$\nabla_{X_{q_i}} (\hat{f} s) = 0$$

for any polarized section s. By directly calculation, we obtain

$$\begin{aligned}
&\nabla_{X_{q_i}} (\hat{f} s) \\
&= -i\hbar\, \nabla_{X_{q_i}} \nabla_{X_f} s + \nabla_{X_{q_i}} (f s) \\
&= -i\hbar\, \nabla_{X_f} \nabla_{X_{q_i}} (s) - i\hbar\, \nabla_{[X_{q_i}, X_f]} s - \omega(X_{q_i}, X_f) s + f\, \nabla_{X_{q_i}} s \\
&= -i\hbar\, \nabla_{[X_{q_i}, X_f]} s \\
&= 0
\end{aligned}$$

This implies that

$$[X_{q_i}, X_f] \in P$$

so we know that the condition for the quantizable functions is

$$\frac{\partial^2 f}{\partial p_i^2} = 0 \qquad (*)$$

In particular, if we take h_i, e_{α_j} for $i = 1 \cdots n, j = 0 \cdots n$, as a function on M by the bilinear form on $\mathcal{B}$. Then $h_i = p_i$ and $e_{\alpha_j} = x_{\alpha_j}$, which satisfy the condition $(*)$ obviously . So we obtain a representation of $\mathcal{B}$ on the polarized sections space, which will be described as follows. Note that $\hat{f}$ acts on ϕs_0 as $-i\hbar X_f - X_f](\sum_{k=1}^n p_k dq_k) + f$ on ϕ for f satisfies the condition $(*)$. Then

$$\hat{p}_i = -i\hbar\frac{\partial}{\partial q_i} \qquad \hat{q}_i = i\hbar\frac{\partial}{\partial p_i} + q_i$$

By the polarized condition or the identified Hilbert space of square integrable functions of q_i's, we have

$$\hat{p}_i = -i\hbar\frac{\partial}{\partial q_i} \qquad \hat{q}_i = q_i$$

which is the familiar Schrodinger repesentation.

Then the representation of $\mathcal{B}$ acting on the physical Hilbert space are given by

$$h_i = -i\hbar\frac{\partial}{\partial q_i}$$

$$e_{\alpha_j} = \begin{cases} c_j exp(\sum_{k=1}^n \alpha_j(h_k)q_k) & for \quad j = 1, \cdots, n \\ c_0 exp(-\sum_{k=1}^n \Theta(h_k)q_k) & for \quad j = 0 \end{cases} \tag{43}$$

We can write the quantum Hamiltonian by the Casimir element of $\mathcal{B}$:

$$H = \sum_{i=1}^n h_i^2 + \sum_{j=0}^n e_{\alpha_j}^2 \tag{44}$$

by (43), we obtain the quantum Hamiltonian of generalized periodic Toda lattices:

$$H = -\sum_1^n \hbar\frac{\partial^2}{\partial q_i^2} + \sum_{j=0}^n c_j^2 exp(2\sum_{k=1}^n \alpha_j(h_k)q_k) \tag{45}$$

where α_0 denotes $-\Theta$. One easily knows that our construction of quantum Hamiltonian is obtain by the important theorem of Kostant [14], which implies that, in geometric quantization, the quantum operators can give the representation of the associated Lie algebra by the means of moment map. So the quantum picture of generalized periodic Toda lattices is obtained by the procedure of geometric quantization.

Acknowledgements

The third auther wishes to thank the staff of CCAST and ITP for their great hospitality, especially H. Y. Guo, Z. M. Qiu and K. Wu.

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