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Matrix Models with Random Sources

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ABSTRACT

We discuss some aspects of matrix models under the influences of random sources. Our formalism does not rely on stochastic equations of the Langevin's type, and thus provides a possible way of performing calculations beyond the tree levers in the double expansions of the couplings and the sources. We apply this formalism especially to matrix models with potentials $V(\phi) = \phi^2/2 + g\phi^4/4$ *(pure gravity) and* $V(\phi) = log(1-\phi) + \phi$ *(the Penner's model), to see how their critical behavior changes after introducing the random sources. In the first case our result agrees with that obtained by F. David by using a different functional method, showing that the Parisi-Sourlas dimensional reduction occurs at least for matrix models with polynomial potentials. In the second case, however, the result indicates that instead of appearance of a naive reduction of the target dimensions of strings, some new phases, such as the crumpled worldsheets, may emerge.*

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Matrix models may provide a nonperturbative definition of strings coupled to 2D quantum gravity. However, if one takes this point seriously, the question arises as to how to adjust the suitable target dimensions of strings in the theory. Here the main difficulty is that too few matrix models, mostly in dimensions less than or equal to 1, are known to be solvable. It will be fruitful, therefore, to explore various ways of relating matrix models in some dimensions to those in the other dimensions.

The most popular way of linking different matrix models is known as the employment of the KdV flows to interpolate between the adjacent multi-critical points. Another useful way, which was initiated by F.David several years ago[1], is to introduce a random background matrix field as a mathematical tool to alter the critical behavior of the underlying model. For d=0 matrix models with the ϕ^3-interaction, David argued that adding such a random source will cause the well-known Parisi-Sourlas dimensional reduction[2].

David's argument is based on the use of a stochastic equation of the Langevin's type. This amounts to take the tree-level approximation in the perturbation theory --- i.e., one first performs a double expansion in powers of the coupling and of the source, and then drops all terms corresponding to the Feymann diagrams with index loops, hoping that these terms contribute to a less singular part which is invisible in the double scaling limit. For the ϕ^3 matrix model, detailed calculations show that such an approximation is indeed reliable[1].It becomes clear that as soon as the stochastic equation is established, one can map the

underlying model into a supersymmetric theory, which leads to an effective dimensional reduction in turn.

When the interaction in a matrix model becomes more complicated, however, one may cast doubts on the appearance of a naive dimensional reduction, since there are no general principles to guarantee the supersymmetry in the theory. Take the Penner's model[3] as an example. This is a zero-dimensional matrix model with a transcendental potential, but it looks like a theory of random surfaces embedded in one-dimensional target space[4]. If we couple such a model to a random background field and then simply employ the above tree-level argument qualitatively, we will face the ambiguity of dertermining its effective target dimensions. In general, it is not clear at all how a random source influences the critical behavior of a matrix model with interaction other than the polynomial form.

The main purpose of this paper is thus to provide a functional method to study matrix models with random sources. Without reference to the stochastic equations, our method has the advantage of obtaining free energies beyond the tree-level approximation. As we shall see, by applying this method, it is possible to derive an exact (and thus nonperturbative) expression of the free energy of the Penner's matrix model coupled to a random source. This method also allows us to give an independent check of the result in ref.[1].

To begin with, let us consider the following integral over NxN hermitian matrices Φ :

$$Z(\beta,h) = \int d\Phi \; exp(-\beta\, trV(\Phi)+tr(h\Phi)), \tag{1}$$

where β is a parameter of order N and h is an NxN hermitian random matrix with gaussian probability distribution

$$P(h)\cdot dh = exp\left(-\frac{1}{2\sigma}\cdot trh^2\right)\cdot dh \tag{2}$$

One may also choose the parameter σ in (2) to be of order N when N becomes large.

The standard way of dealing with the integral (1) is to reduce the number of integration variables from N^2 matrix elements of Φ to its N eigenvalues, $y_1, y_2, \cdots, y_N$, by writing the matrix as $\Phi = U^{+}\mathrm{diag}(y_1, y_2, \cdots, y_N)U$, where U is an element in SU(N). In the case of $h=0$, the integrand of (1) does not depend on U, which can then be integrated out. This is not the case when the external matrix field h is presented, since the term $tr(h\Phi)$ violates the SU(N)-invariance. However, even though U does not decouple to the background source, the integral over it can be performed, following the classic work of Itzykson and Zuber[5] and of Mehta[6]. The result of such an integral may be summarized as follows:

$$Z(\beta,h) = \frac{(2\pi)^{N(N-1)/2}}{\Delta(x_1,\cdots,x_N)}\int dy_1\cdots dy_N\; \Delta(y_1,\cdots,y_N)\, exp\left(-\beta\sum_{i=1}^{N} V(y_i)+\sum_{i=1}^{N} x_i y_i\right) \tag{3}$$

where $x_1,\cdots,x_N$ are the eigenvalues of the matrix source h and $\Delta(x_1,\cdots,x_N) = \prod_{i<j}(x_i-x_j)$ is the Vandermonde determinant. It will be convenient to carry out a further reduction of the N-dimensional integral in (3) with the aid of the "moments"

$$\mu_j(x) = \int dy \cdot y^j \exp(-\beta V(y) + xy) \tag{4}$$

and rewrite the partition function $Z(\beta, h)$ as

$$Z(\beta, h) = \frac{(2\pi)^{N(N-1)/2}}{\Delta(x_1, \cdots, x_N)} \cdot \det[\mu_{j-1}(x_i)]_{i,j=1,\cdots,N}. \tag{5}$$

Next, given the gaussian distribution (2), we wish to consider the following average of the connected generating function over it:

$$\overline{\log Z(\beta, h)} = A \cdot \int dh \, \log Z(\beta, h) \, \exp\left(-\frac{1}{2\sigma} \cdot trh^2\right) \tag{6}$$

where $A^{-1} = \int dh \, \exp(-1/2\sigma \cdot trh^2)$ is a normalization constant. In other words, we are interested in the average of the free energy over the random source.

From the point of view of string theory, one may regard Eq(6) as an ensemble average of the string vacuum amplitude. To evaluate such an average we notice that the integrand in (6) depends only on the eigenvalues of the random source and therefore the "angular part" of the integral can be performed. After performing this part of integral one finds

$$\overline{\log Z} = B \int dx_1 \cdots dx_N \Delta^2(x_1, \cdots, x_N) \exp\left(-\frac{1}{2\sigma} \sum_{i=1}^{N} x_i^2\right) \log\left(\frac{\det \mu_{j-1}(x_i)}{\Delta(x_1, \cdots, x_N)}\right) \tag{7}$$

where the constant B is now given by:

$$B^{-1} = \int dx_1 \cdots dx_N \Delta^2(x_1, \cdots, x_N) \exp(-1/2\sigma \cdot \sum_{i=1}^{N} x_i^2)$$

$$= (2\pi)^{N/2} \cdot \sigma^{N^2/2} N!(N-1)! \cdots 3!2!1!. \tag{8}$$

Alternatively, one may derive (7) by using the following replica trick[7]:

$$\log Z = \lim_{n \to 0} (Z^n - 1)/n.$$

It means that instead of $\log Z$ itself one consider the n replications of the original matrix system. To carry out the average of Z^n over the random source, one meets the multi-matrix model on a lattice which forms a star diagram with n edges. Since in this case there are no closed loops in the target space, one expect[8] that the path integrals over the matrices in the model can be reduced to integrals over their eigenvalues, i.e., all the angular degrees of freedom decouple. Actually, one can perform calculations along the lines of [6] to obtain an explicit formula for $\overline{Z^n}$, which leads to Eq(7) via the replica trick.

To proceed further, let us consider two concrete examples. The first example we shall discuss is a matrix model with the potential

$$V(\phi) = \frac{1}{2}\phi^2 + \frac{1}{4} g \phi^4 \quad . \tag{9}$$

This potential describes a model of random surfaces (triangulated by square tiles) embedding in zero-dimension target space, which corresponds to a system of pure gravity. Here the coupling g plays the role of the cosmological constant. Now we couple this this system to the background random field h. In order to analysis the

critical behavior of the free energy, we have to evaluate the integral

$$\mu_j(\beta x) = \int_{-\infty}^{\infty} dy \cdot y^j \exp\left(-\beta\left(\frac{1}{2}y^2+\frac{1}{4}gy^4-xy\right)\right) . \qquad (4')$$

For β large, the evaluation may be carried out by using the method of steepest descents: if y_0 is a root of the equation

$$y+gy^3=x \qquad (10)$$

such that $V''(y_0)=1+3gy_0^2>0$, then for β large,

$$\mu_j(\beta x) \simeq y_0^j \cdot \left(\frac{2\pi}{\beta(1+3gy_0^2)}\right)^{1/2} \exp\left(-\frac{\beta}{2}y_0^2-\frac{\beta}{4}gy_0^4+\beta x y_0\right). \qquad (11)$$

It is sufficient to choose y_0 as the unique real root of (10), which is determined by

$$y_0 = \left[\frac{x}{2g}+\left(\frac{1}{27g^3}+\frac{x^2}{4g^2}\right)^{1/2}\right]^{1/3}+\left[\frac{x}{2g}-\left(\frac{1}{27g^3}+\frac{x^2}{4g^2}\right)^{1/2}\right]^{1/3}.$$

One may expand y_0 in powers of x. More generally, if $f(y_0)$ is a smooth function of y_0, then the following Lagrange expansion (see e.g. the Appendix of [9]) holds:

$$f(y_0) = f(x)+\sum_{n=1}^{\infty}(-g)^n/n! \cdot \frac{d^{n-1}}{dx^{n-1}}\left(x^{3n}f'(x)\right) .$$

This formula together with (11) gives rise to

$$\log\left[\frac{\det\mu_{j-1}(\beta x_k)}{\Delta(x_1,\cdots,x_N)}\right] \simeq F_1(x_1,\cdots,x_N)+F_2(x_1,\cdots,x_N), \qquad (12)$$

where

$$F_1(x_1,\cdots,x_N)=\beta\sum_{i=1}^{N}\sum_{n=0}^{\infty}(-g)^n\frac{(3n)!}{n!(2n+2)!}x_i^{2n+2} \qquad (13)$$

and

$$F_2(x_1\cdots,x_N)=\sum_{i<j}\log\left\{1+\sum_{n=1}^{\infty}(-g)^n\frac{(3n)!}{n!(2n+1)!}\left(\frac{x_i^{2n+1}-x_j^{2n+1}}{x_i-x_j}\right)\right\}. \qquad (14)$$

Now we are in a position to derive an asymptotic expansion for the free energy (7). If we introduce the symbol $\tilde{F}$ as an integral of the function $F(x_1,\cdots,x_N)$ with weight $B\beta^{N^2}\Delta^2(x_1,\cdots,x_N)\cdot\exp(-\beta^2/2\sigma\cdot\sum_{i=1}^{N}x_i^2)$, we may rewrite (7) as

$$\overline{\log Z} = \tilde{F}_1+\tilde{F}_2. \qquad (15)$$

First we look at the integral $\tilde{F}_1$. In order to evaluate this integral, we consider

$$I_n=\int dx_1\cdots dx_N\Delta^2(x_1,\cdots,x_N)\exp(-\beta^2/2\sigma\cdot\sum_{i=1}^{N}x_i^2)\cdot\left(\sum_{k=1}^{N}x_k^{2n}\right)$$

$$=\left(\frac{2\sigma}{\beta^2}\right)^{N^2/2+n} N!\prod_{i=0}^{N-1}(2^{-i}i!\pi^{1/2})\sum_{j=0}^{N-1}\frac{\int dx x^{2n}H_j^2(x)\exp(-x^2)}{\int dx H_j^2(x)\exp(-x^2)} . \qquad (16)$$

where $H_j(x)$ are the Hermitian polynomials. By applying an integral relation between the Hermitian and the Laguerre polynomials[10]

$$\int dx H_j^2(x) exp(-x^2) cos(xt) = 2^j j! \pi^{1/2} exp(-t^2/4) L_j(t^2/2),$$

one easily derives

$$I_n = (-1)^n (2n-1)!! N! \prod_{i=0}^{N-1} (2^{-i} i! \pi^{1/2}) \left(\frac{2\sigma}{\beta^2}\right)^{N^2/2+n} \frac{d^n}{dt^n} \left(e^{-t/2} L_{N-1}^{(1)}(t)\right)_{t=0} \tag{17}$$

where

$$L_{N-1}^{(1)}(t) = \sum_{k=0}^{N-1} \binom{N}{N-k-1} \frac{(-t)^k}{k!}$$

is the generalized Laguerre polynomial. Using (13) and (16) one can expand the integral $\tilde{F}_1$ in terms of I_n to obtain the following asymptotic series:

$$\tilde{F}_1 = \frac{\sigma}{2\beta} \sum_{n=0}^{\infty} \frac{(3n)!}{n!(n+1)!} c_{n+1}(N) \cdot (-g\sigma/2\beta^2)^n \tag{18}$$

where

$$c_n(N) = \frac{d^n}{dt^n} \left[e^t L_{N-1}^{(1)}(-2t)\right]_{t=0} = \sum_{k\geq 1} 2^{k-1} \binom{N}{k} \binom{n}{k-1} \tag{19}$$

There is a combinatoric meaning of the coefficient (19). Suppose that we have a fixed 2n-gon P_n with its sides consecutively around the boundary. One may perform some operations on P_n, such as

an N-coloring Φ of the vertices or an edge-pairing τ. We say that the edge identification τ is compatible with Φ, if two edges of P_n to be identified under τ have the property that the left end of each edge has the same color as the right end of the other. Now a combinatoric problem arises as to count the number of these compatible operator pairs (Φ,τ). In ref.[11], Harer and Zagier solved this problem by applying the method of matrix integration together with a combinatoric argument. They showed that the number of pairs (Φ,τ) of a fixes 2n-gon P_n is nothing but $(2n-1)!!c_n(N)$.

From the work of Harer and Zagier, one may regard the asymptotic series (18) as a diagramatical expansion. A typical diagram in such an expansion is an N-colored 2(n+1)-gon with edges compatibly identified in pairs, which will be denoted by $P_{n+1}/(\Phi,\tau)_{n+1}$. To each diagram $P_{n+1}/(\Phi,\tau)_{n+1}$ we attach a weight factor proportional to g^n. Then summing over all these diagrams with weights reproduces $\tilde{F}_1$.

Similarly, one may derive a diagramatical expansion for the integral $\tilde{F}_2$. In this case we attach each weight factor $-g_n$ to all "disconnected" diagrams $(P_k/(\Phi,\tau)_k) \cup (P_l/(\Phi,\tau)_l)$ with $k+l=n$. It is not difficult to see that for n large the number of these disconnected diagrams (with total edges equal to 2n) is much less than the number of diagrams $P_{n+1}/(\Phi,\tau)_{n+1}$ (which have 2n+2 edges), so we expect that the integral $\tilde{F}_2$ contributes to a less divergent term in the free energy (15), and therefore

$$\overline{\log Z} \simeq \tilde{F}_1 = \frac{\sigma}{2\beta} \sum_{n=0}^{\infty} \frac{(3n)! c_{n+1}(N)}{n!(n+1)!} (-g\sigma/2\beta^2)^n \tag{20}$$

To make the above discussion be more precise, we have to estimate some related integrals. Let us decompose the integrals $\tilde{F}_1$ and $\tilde{F}_2$ into

$$\tilde{F}_{1,2} = \tilde{F}_{1,2}(|x|<a) + \tilde{F}_{1,2}(|x|\geq a) \ , \tag{21}$$

where $a \ll 1$ is a cut-off and

$$\tilde{F}_{1,2}(x\in S) = B\beta^{N^2}\cdot\int_{x\in S} dx_1\cdots dx_N \Delta^2(x_1,\cdots,x_N)\exp\left(-\frac{\beta^2}{2\sigma}\sum_{i=1}^{N} x_i^2\right)F_{1,2}(x_1\cdots x_N)$$

Then for $a\sim 0^+$, we find

$$\tilde{F}_1(|x|\geq a) \simeq \text{const.}\cdot\left(\frac{1}{a}\right)^{4/3}, \tag{22}$$

while

$$\tilde{F}_2(|x|\geq a) \simeq \text{const.}\cdot \log\left(\frac{1}{a}\right). \tag{23}$$

From (22) and (23) we see that as $a\to 0^+$ the integral $\tilde{F}_2(|x|\geq a)$ is much less singular than $\tilde{F}_1(|x|\geq a)$, and we can neglect the former integral when it is compared to the latter. On the other hand, it is not difficult to see that as the cut-off a approaches to zero, both the integrals $\tilde{F}_1(|x|<a)$ and $\tilde{F}_2(|x|<a)$ scale as const.$\cdot a^2$, so we can not establish a simple inequality like $\tilde{F}_2(|x|<a) \ll \tilde{F}_1(|x|<a)$. Nevertheless, one may easily show that in the limit $a\to 0$, the leading term of $\tilde{F}_2(|x|<a)$ consists of a polynomial in the coupling g, which is irrelevant and can be discarded from the free

energy by taking a few derivatives. Thus Eq(20) indeed holds.

In order to extract the critical behavior from the divergent series (20), one has to perform resummation over it. Let us rewrite (20) as

$$\overline{\log Z} = \frac{\sqrt{3}}{4\pi}\cdot\frac{\beta}{\sigma}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/3)\Gamma(n+2/3)}{(n+1)!}\, c_{n+1}(N)(-z)^n, \tag{24}$$

where $z = 27g\sigma/(2\beta^2)$. One may then use the integral representation of the Γ-function and Eq(19) to derive

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1/3)\Gamma(n+2/3)}{(n+1)!}\, c_{n+1}(N)(-z)^n = \frac{N}{z}\Gamma(-1/3)\Gamma(-2/3)$$

$$-\frac{1}{z}\int_0^{\infty}dt\, e^{-t}t^{-2/3-1}(2zt)^{1/3}\int_0^{\infty}ds\exp\left(-\frac{1+zt}{2zt}s\right)s^{-4/3}L_{N-1}^{(1)}(s)$$

$$= -\frac{N\Gamma(-1/3)}{z}\int_0^{\infty}dt\, e^{-t}t^{-2/3-1}\left\{(1+zt)^{1/3}\,{}_2F_1\left(-N+1,-1/3,2,\frac{2zt}{1+zt}\right)-1\right\} \tag{25}$$

where ${}_2F_1(\alpha,\beta,\gamma,z)$ is the hypergeometric function. The free energy (24) is thus represented by

$$\overline{\log Z} = -\frac{\sqrt{3}\sigma N\Gamma(-1/3)}{4\pi\beta z}\int_0^{\infty}dt\, e^{-t}t^{-2/3-1}\left[(1-zt)^{1/3}\,{}_2F_1\left(-1/3,N+1,2,\frac{2zt}{1-zt}\right)-1\right] \tag{26}$$

Now for $\sigma\sim\beta\sim N$ large, we have $z\sim 1/N$, and

$$(1-zt)^{1/3}\,{}_2F_1\left(-1/3,N+1,2,\frac{2zt}{1-zt}\right) \sim {}_1F_1(-1/3,2,\lambda t). \tag{27}$$

where $\lambda=2(N+1)z=27(N+1)\sigma g/\beta^2$ is the scaled cosmological constant,

which keeps fixed in the large N limit, and ${}_1F_1(-1/3,2,\lambda t)$ is the confluent hypergeometric function. Substituting (27) into (26) one finds, up to the leading order of the large-N expansion,

$$\overline{\log Z} = -\frac{27\sqrt{3}}{2\pi}\left(\frac{\sigma}{\beta}\right)\Gamma(-2/3)\Gamma(-1/3)\cdot N^2\cdot\frac{{}_2F_1(-1/3,-2/3,2,\lambda)-1}{\lambda}. \quad (28)$$

From this expression, it is clear to see that $\lambda_c=1$ is a critical point of the free energy. In fact, as $\lambda\sim\lambda_c=1$, one has the following critical behavior:

$$\overline{\log Z} \simeq \text{const.}\cdot N^2\cdot(\lambda-\lambda_c)^3\log(\lambda-\lambda_c) + \textit{less singular terms} +$$

$$+ \textit{polynomials in } (\lambda-\lambda_c) \quad (29)$$

which gives rise to the sting susceptibility $\gamma_{str}=-1$ and the effective target dimension $d_{eff}=-2$.

Eq(29) agrees with the result obtained in ref.[1], there a Φ^3-matrix model coupled to the random source was investigated. More generally, one may extend the above calculations to matrix models with generic polynomial potentials $V(\Phi)=\Sigma g_i\Phi^i$, to determine how the random source influences their critical behavior. If we choose the couplings $\{g_i\}$ to be away from some "catastrophic points" and set them in general positions, we will find that the critical behavior of $\overline{\log Z}$ takes the same form as Eq(29). Therefore, for matrix models with polynomial interactions, the Parisi-Sourlas dimensional reduction is a universal phenomenon.

In the rest of this paper we will turn to our second

example, the Penner's matrix model. This model was originally used by mathematicians to calculate Euler characteristic of moduli space of Riemann surfaces [3] [11]. Recently, it has been recognized[4] that the Penner's model at the critical point behaves as a d=1 string theory coupled to 2D quantum gravity. The partition function of this model is given by

$$Z = \int d\Phi\ \exp\left(Nt\cdot tr(\log(1-\Phi))+\Phi\right)$$

$$= \int d\Phi\ \det(1-\Phi)^{Nt}\cdot\exp(Nt\cdot tr\Phi), \quad (30)$$

where Φ is an NxN hermitian matrix and t is the coupling constant. Let $M=1-\Phi$. In order that the matrix integral (30) to be well-defined, one has to restrict M to positive definite matrices. The perturbative expansion for this model may be written as

$$\log Z = \sum_{g,n} N^{2-2g}t^{2-2g-n}\cdot\chi_{g,n} \quad (31)$$

where $\chi_{g,n}$ is the Euler character of moduli space of Riemann surfaces of genus g with n punctures. It can be shown that there exists a double scaling limit $N\to\infty$, $t\to-1$ and $\mu=-N(t+1)$ fixed, in which Eq(31) becomes[4]

$$\log Z = \frac{1}{2}\mu^2\log\mu-\frac{1}{4}\mu^2-\frac{1}{2}\mu+\frac{B_2}{2}\log\mu+\sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\mu^{2-2g} \quad (32)$$

where B_{2g} are the Bernoulli numbers.The scaling behavior of (32) coincides with that of d=1 matrix models[12].

Now we introduce a random matrix source h into the Penner's model. The partition function has, as before, the following form:

$$Z(h) = \int d\Phi \, \det(1-\Phi)^{Nt} \cdot \exp(N^2 t + \mathrm{tr}\Phi + \mathrm{tr}(h\Phi)),$$

$$= \exp(N^2 t + \mathrm{tr}h) \int dM \cdot \det M^{Nt} \exp(-N\,\mathrm{tr}M - \mathrm{tr}(hM)). \qquad (33)$$

Notice that for the last integral in (33) we should integrate over positive definite matrices M. To carry out such an integral, we diagonalize both the matrix source and the matrix field

$$h = \mathrm{diag}(x_1, \cdots, x_N), \qquad M = \mathrm{diag}(y_1, \cdots, y_N)$$

to derive

$$Z(h) = \frac{(2\pi)^{N(N-1)/2}}{\Delta(x_1, \cdots, x_N)} \exp(N^2 t + \sum_{i=1}^{N} x_i) \int_0^\infty dy_1 \cdots dy_N \Delta(y_1, \cdots, y_N)$$

$$\cdot \prod_{i=1}^{N} \left(y_i^{Nt} \exp(-Nt + x_i) y_i \right)$$

$$= (2\pi)^{N(N-1)/2} \exp(N^2 t + \sum_{i=1}^{N} x_i) \cdot \frac{\det[\mu_{j-1}(x_k)]}{\Delta(x_1, \cdots, x_N)}, \qquad (34)$$

where the moments $\mu_{j-1}(x)$ are determined by

$$\mu_{j-1}(x) = \int_0^\infty dy\, y^{Nt+j-1} e^{-(Nt+x)y} = \frac{\Gamma(Nt+j)}{(Nt+x)^{Nt+j}}, \quad j=1,2,\cdots,N. \qquad (35)$$

The determinant of these moments may be evaluated as follows:

$$\det[\mu_{j-1}(x_k)] = \begin{vmatrix} \frac{\Gamma(Nt+1)}{(Nt+x_1)^{Nt+1}} & \frac{\Gamma(Nt+1)}{(Nt+x_2)^{Nt+1}} & \cdots & \frac{\Gamma(Nt+1)}{(Nt+x_N)^{Nt+1}} \\ \frac{\Gamma(Nt+2)}{(Nt+x_1)^{Nt+2}} & \frac{\Gamma(Nt+2)}{(Nt+x_2)^{Nt+2}} & \cdots & \frac{\Gamma(Nt+2)}{(Nt+x_N)^{Nt+2}} \\ \vdots & \vdots & & \vdots \\ \frac{\Gamma(Nt+N)}{(Nt+x_1)^{Nt+N}} & \frac{\Gamma(Nt+N)}{(Nt+x_2)^{Nt+N}} & \cdots & \frac{\Gamma(Nt+N)}{(Nt+x_N)^{Nt+N}} \end{vmatrix}$$

$$= \prod_{k=1}^{N} \frac{\Gamma(Nt+k)}{(Nt+x_k)^{Nt+1}} \cdot \det\left[\left(\frac{1}{Nt+x_i}\right)^{j-1}\right]_{1\le i,j\le N}$$

$$= \prod_{k=1}^{N} \frac{\Gamma(Nt+k)}{(Nt+x_k)^{Nt+1}} \cdot \prod_{i<j} \frac{x_j - x_i}{(Nt+x_i)(Nt+x_j)}. \qquad (36)$$

Using this result together with Eq(34) one finds:

$$\log Z(h) = N^2 t + \sum_{k=1}^{N} x_k + \sum_{k=1}^{N} \log\Gamma(Nt+k) - N^2(t+1)\log(Nt)$$

$$- \sum_{k=1}^{N} N(t+1)\log(1+x_k/Nt). \qquad (37)$$

Now we are in a position to take the average of the free energy (37) over the random source h with gaussian distribution (2). Denoting

$$\overline{(\cdots)} = \frac{\int dx_1 \cdots dx_N \Delta^2(x_1, \cdots, x_N) \exp(-\frac{1}{2\sigma}\sum_{i=1}^{N} x_i^2) \cdot (\cdots)}{\int dx_1 \cdots dx_N \Delta^2(x_1, \cdots, x_N) \exp(-\frac{1}{2\sigma}\sum_{i=1}^{N} x_i^2)}, \qquad (38)$$

we have

$$\overline{\log Z(t)} = N^2 t + \sum_{k=1}^{N} x_k + \sum_{k=1}^{N} \log\Gamma(Nt+k) - N^2(t+1)\log(Nt)$$

$$-N(t+1)\overline{\sum_{k=1}^{N} \log(1+x_k/Nt)} \ . \qquad (39)$$

Expanding $\log(1+x_k/Nt)$ in powers of x_k and then calculating $\overline{\sum_{k=1}^{N} x_k^n}$ by using the method we have discussed before, we obtain the following exact formula of the free energy:

$$\overline{\log Z} = N^2 t + \sum_{k=1}^{N} x_k + \sum_{k=1}^{N} \log\Gamma(Nt+k) - N^2(t+1)\log(Nt)$$

$$+ \frac{N(t+1)}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)\, c_n(N)}{n} z^n \ , \qquad (40)$$

where the coefficients $c_n(N)$ are given by (19), and z is a parameter related to the coupling constant t:

$$z = \frac{2\sigma}{N^{3/2}} \cdot 1/N \ . \qquad (41)$$

It is not difficult to derive the following summation formula:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\, c_n(N)}{n!} z^n = \frac{N\sqrt{\pi}}{(1+z)^{1/2}} \cdot {}_2F_1(1/2, N+1, 2, \frac{2z}{1+z}) \ . \qquad (42)$$

Using this formula we may rewrite (40) as

$$\overline{\log Z} = N^2 t + \sum_{k=1}^{N} x_k + \sum_{k=1}^{N} \log\Gamma(Nt+k) - N^2(t+1)\log(Nt)$$

$$+\frac{1}{2}N^2(t+1)\cdot\int_0^{\infty}\frac{ds}{s}e^{-s}\left[\frac{{}_2F_1(1/2,N+1,2,\frac{2zs}{1+zs})}{(1+zs)^{1/2}} -1\right] \ . \qquad (43)$$

This result may be regarded as a nonperturbative solution of the Penner's model coupled to the random source. Here we are interested in extracting the critical behavior from the leading term of (43) in the large-N limit. First we notice that as $N\to\infty$,

$$\frac{1}{(1+zs)^{1/2}}\cdot{}_2F_1(1/2,N+1,2,\frac{2zs}{1+zs}) \to {}_1F_1(1/2,2,\eta^2 s) \ , \qquad (44)$$

where $\eta^2=4\sigma/N$ is the scaled intensity of the random source. So if we substitute (44) into the last term in Eq(43) and then perform the integral over s, we find the following result:

$$\frac{1}{2}N^2(t+1)\cdot\int_0^{\infty}\frac{ds}{s}e^{-s}\left[\frac{{}_2F_1(1/2,N+1,2,\frac{2zs}{1+zs})}{(1+zs)^{1/2}} -1\right]$$

$$= \frac{1}{2}N^2(t+1)\left[(2\log 2+1)-2\log(1+\{1-(\eta/t)^2\}^{1/2})\right.$$

$$\left.-2(t/\eta)^2(1-\{1-(\eta/t)^2\}^{1/2})\right] + \textit{terms with lower of N.} \qquad (45)$$

Next we consider $\sum_{k=1}^{N}\log\Gamma(Nt+k)$. This sum can be rewritten as follows:

$$\sum_{k=1}^{N}\log\Gamma(Nt+k) = N\log\Gamma(Nt)+\sum_{k=1}^{N}\sum_{j=0}^{k-1}\log(Nt+j) \ . \qquad (46)$$

In order to obtain the leading term of (46) in the large N-limit, one uses the asymptotic expansion for $N\log\Gamma(Nt)$ and then applies the Maclauring summation formula twice to the double sum in (46). The result reads:

$$\sum_{k=1}^{N} \log\Gamma(Nt+k) = \frac{1}{2}N^2t^2(1+1/t)^2\log(1+1/t)+N^2t\log(Nt)-\frac{3N^2}{4}(2t+1)$$

$$+ \textit{ terms with lower order of } N. \qquad (47)$$

Now from (43), (45) and (47) we obtain, up to the leading order of N, the free energy of the Penner's matrix model coupled to the random source:

$$\overline{\log Z} = E_0(t)+E_1(t,\eta) \quad , \qquad (48)$$

where

$$E_0(t) = \frac{1}{2}N^2t^2(1+1/t)^2\log(1+1/t)-N^2\log(Nt)-\frac{N^2}{4}(2t+3) \qquad (49)$$

and

$$E_1(t,\eta) = \frac{1}{2}N^2(t+1)\left[(2\log 2+1)-2\log(1+\{1-(\eta/t)^2\}^{1/2})\right.$$

$$\left.-2(t/\eta)^2(1-\{1-(\eta/t)^2\}^{1/2})\right] \qquad (50)$$

It is interesting to note that $E_0(t)$ does not depend on the intensity of the random source η and $E_1(t,\eta)$ approaches to zero as $\eta\to 0$. It means that if we "switch off" the random source, then the free energy (48) becomes

$$\overline{\log Z}|_{\eta=0} = \frac{1}{2}N^2t^2(1+1/t)^2\log(1+1/t)-N^2\log(Nt)-\frac{N^2}{4}(2t+3) \quad ,$$

which admits a double scaling limit $N\to\infty$, $t\to -1$, and $N(t+1)=\mu$ fixed. In this double scaling limit, one finds

$$\overline{\log Z}|_{\eta=0} = \frac{1}{2}\mu^2\log\mu + \textit{polynomials in } \mu. \qquad (51)$$

This equation agrees with the genus zero part of the free energy (32) of the Penner's model without random sources, as we can expect. On the other hand, if we "turn on" the random source (i.e. put $\eta\neq 0$) and renormalize the free energy $\overline{\log Z}$ by subtracting $\overline{\log Z}|_{\eta=0}$ from it, we find

$$\overline{\log Z}|_{ren} = \frac{1}{2}N^2(t+1)\left[(2\log 2+1)-2\log(1+\{1-(\eta/t)^2\}^{1/2})\right.$$

$$\left.-2(t/\eta)^2(1-\{1-(\eta/t)^2\}^{1/2})\right] \qquad (52)$$

Eq(52) shows that $t_c=-\eta$ is a critical point of the free energy. When the coupling t is close to this critical point, the free energy behaves as

$$\overline{\log Z}|_{ren} = \text{const.}\cdot N^2(t+1)(t+\eta)^{3/2} \quad . \qquad (53)$$

From (53) we see that a different choice of the values of η will lead to a different phase of the model. In particular, if we take $\eta=1$, then the free energy looks like the partition function of pure quantum gravity, and the corresponding string susceptibility exponent $\gamma_{str.}$ is equal to -1/2. On the other hand, if we put η in general positions (i.e. $\eta\neq1,0$), then the free energy behaves as the generating function of the branched polymers. In this phase we have $\gamma_{str.}=1/2$, which seems to indicate that in the continuum limit the worldsheets of strings are crumpled.It would be very interesting to find out a combinatoric explanation of this phenomenon. In all cases, however, we can never find a naive Parisi-Sourlas dimensional reduction $d\to d_{eff}=d-2$, since the effective dimension of the matrix model equals neither -2 nor -1. Consequently, a supersymmetric formalism based on the stochastic equation for the Penner's model can no longer be trusted.

In conclusion, we have developed a direct functional integration method to deal with matrix models coupled to random sources. This approach allows us to compute the free energies beyond the tree-level approximation, and it may be useful to obtain some nonperturbative results. Although we have restricted ourselves in this paper to the coupling $tr(h\cdot\Phi)$ of the matrix field Φ and the random source h, it is possible to extend our calculations to another type of interaction between Φ and h, for example, $tr(h\Phi^m)$. In this case it will be interesting to investigate how such a random source influences the critical behavior of the underlying matrix model.

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