Fermilab Library
0 1160 0010869 0
BUFB-92-4
BUFB-92-04

BUFB-92-04

9

QUANTUM GRASSMANNIAN MANIFOLDS

M. ARIK[1], F. AYDIN[2], S. ÇELİK[3], E. HIZEL[4]

[1]Department of Physics, Boğaziçi University ,Bebek,İstanbul, Turkey

[2]Department of Mathematics, Yıldız University ,Şişli, İstanbul, Turkey

[3]Department of Mathematics, Mimar Sinan University ,Beşiktaş, İstanbul, Turkey

[4]Department of Mathematics, İstanbul Technical University ,Maslak,İstanbul, Turkey

1992

Abstract

We discuss the q-deformation of complex Grassmannian manifolds $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$. We show that the coordinates of this quantum manifold can be taken to be an mxn dimensional q-oscillator. We present (n+m)x(n+m) quantum matrix representations and show that each quantum Grassmannian manifold corresponds to an element of a singular Hecke algebra.

August 1992

The discovery of q-deformations of Lie groups and Lie algebras, the so called quantum groups and algebras [1] has led to renewed interest in the theory of braids and knots [2]. The emphasis of the connection between the two theories is usually stated through the R-matrix and the Yang-Baxter equation [3] and quantum Lie algebras. In this paper we will show that an explicit construction of the quantum Grassmannian manifolds reveals that each such manifold corresponds to an element of a Hecke algebra. The Hecke algebra representations of the braid group suggest a direct link between quantum group cosets and braids.

Recently it has been shown that the unitary quantum group $U_q(n)$ can be constructed in terms of $n(n-1)/2$ q-oscillators and n-commuting phases [4]. This approach shows that the quantum group cosets $SU_q(n+1)/U_q(n) \equiv CP_q(n)$ can be identified with n q-oscillators. The normalization of these q-oscillators are chosen such that in the $q \longrightarrow 1$ limit the oscillator creation and annihilation operators commute and give c-numbers.

In this paper we construct the quantum Grassmannian manifolds which are identical to the quantum group cosets $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$. The additional ingredient to build these cosets in terms of q-oscillators, turns out to be the braid group relation $A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$, where each A_i uniquely corresponds to a q-oscillator. Finally , we discuss the braids and the knots and links defined by our representations.

First let us consider the quantum group $SU_q(2)$, the elements of which

can be expressed as

$$M = A_1 X_1,$$

$$A_1 = \begin{bmatrix} a & (1-a^*a)^{1/2} \\ -(1-aa^*)^{1/2} & a^* \end{bmatrix}, X_1 = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix}, \tag{1}$$

where

$$1 - aa^* = q^2(1 - a^*a) \tag{2}$$

and α can be taken to be a real phase which commutes with the q-oscillator operators a and a^*. We will refer to A_1 as a q-oscillator matrix. The quantum matrix M satisfies

$$M^* M^T = I \tag{3}$$

where $*$ denotes Hermitian conjugation on all elements of M (without transpose) and T denotes ordinary transposition. Using (2), individual elements of the matrix M can be shown to satisfy the usual quantum group commutation relations.

From this representation it follows that, since the second matrix in (1) is an element of $U(1) = U_q(1)$, the first matrix can be taken to be the canonical form of a coset representative of $CP_q(1) = SU_q(2)/U(1)$.

In [4] it was shown that an element of $SU_q(3)$ can be expressed as

$$M = A_1 A_2 A_1 X_1 X_2, \tag{4}$$

where each factor A_i denotes a matrix which contains a q-oscillator matrix in the 2x2 block starting at the i-th diagonal element, 1's on the remaining

diagonals and 0's everywhere else. Thus

$$A_1 = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \tag{5}$$

and similarly

$$X_1 = \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 & 0 \\ 0 & X \end{bmatrix}.$$

In equation (4), the arguments of all the q-oscillators matrices A_i are independent (commuting) q-oscillators. Thus equation (4) shows that the quantum group $SU_q(3)$ can be parametrized in terms of three independent q-oscillators and two phase angles. The factors of X_i in equation(4) can be commuted to any desired position. This just amounts to a redefinition of the q-oscillators and the phases. Using the methods of [4] it can be shown that an element of $SU_q(3)$ can also be expressed as

$$M = A_2 A_1 A_2 X_1 X_2. \tag{6}$$

Comparing (6) and (4), and omitting the phase matrices X_i we find

$$A_1 A_2 A_1 = A_2 A_1 A_2 \tag{7}$$

which is the braid group relation. The equality of (6) and (4) actually implies a relation of the form

$$A_1(u) A_2(v) A_1(w) X_1(\lambda) X_2(\mu) = A_2(a) A_1(b) A_2(c), \tag{8}$$

where for given q-oscillators a,b,c, the q-oscillators u,v,w and the phases λ and μ are determined. The reason for omitting the phases in equation (7) is

that for the canonical form of a coset representative of $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$ the phases are actually absent. Using the matrices A_i and the quantum subgroup $SU_q(3)$ generated by $A_i, A_{i+1}, X_i, X_{i+1}$ we obtain the braid group relation

$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}. \tag{9}$$

From the representations found in [1] an element of $SU_q(n)$ can be expressed as

$$M = \prod_{k=1}^{n-1} \prod_{i=1}^{n-k} A_i \prod_{i=1}^{n-1} X_i, \tag{10}$$

which for $SU_q(3)$ reduces to (4) and for $SU_q(4)$ is given by

$$M = A_1 A_2 A_3 A_1 A_2 A_1 X_1 X_2 X_3. \tag{11}$$

Construction of the quantum coset $SU_q(4)/SU_q(2)\times SU_q(2)\times U(1)$ amounts to obtaining a factor belonging to $SU_q(2)\times SU_q(2)\times U(1)$ on the right end of formula (11) by using the braid group relation (9) and the fact that $A_i A_j = A_j A_i$ for $|i-j| \neq 1$. Thus

$$A_1 A_2 A_3 A_1 A_2 A_1 = A_1 A_2 A_1 A_3 A_2 A_1 = A_2 A_1 A_2 A_3 A_2 A_1 =$$

$$A_2 A_1 A_3 A_2 A_3 A_1 = A_2 A_3 A_1 A_2 A_1 A_3. \tag{12}$$

The factor $A_1 A_3$ on the right together with the omitted phases $X_1 X_2 X_3$ is just the required factor. Hence we can take the canonical representative of $SU_q(4)/SU_q(2)\times SU_q(2)\times U(1)$ to be

$$M = A_2 A_3 A_1 A_2. \tag{13}$$

It can be shown that the necessary arrangement to obtain a factor $SU_q(k)\times SU_q(n-k)\times U(1)$ on the right end of the formula (10) can always be performed. The canonical coset representative of $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$ is then given by

$$M = \prod_{l=0}^{m-1}\prod_{k=0}^{n-1} A_{m+k-l}. \tag{14}$$

As an example, for $SU_q(5)/SU_q(3)\times SU_q(2)\times U(1)$ (14) becomes $M = A_2A_3A_4A_1A_2A_3$ and for $SU_q(6)/SU_q(3)\times SU_q(3)\times U(1)$ it becomes $M = A_3A_4A_5A_2A_3A_4A_1A_2A_3$. Formula (14) shows that the "coordinates" of the quantum Grassmannian manifolds $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$ consist of $m\times n$ independent q-oscillator creation and annihilation operators satisfying the commutation relation (2).

In all our formulas such as (10) or (14) the q-oscillators on which the matrices A_i depend are taken to be independent. This means that in addition to the braid group relation

$$A_iA_j = A_jA_i \quad , \qquad |i-j| \neq 1$$

$$A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1} \tag{15}$$

with the additional property

$$A_i^2 = A_i \tag{16}$$

which explicitly reads

$$A_i(a)A_i(b) = A_i(u)X_i(\lambda)$$

which determine the q-oscillator u and the phase λ, given the q-oscillator annihilation operators a and b and their hermitean conjugates. This follows from the fact that the product of two quantum matrices is also a quantum matrix provided that the elements of the two matrices are taken to be independent. If relation (16) is replaced by the existence of A_i^{-1} then relations (15) and (16) would define the braid group. As a quantum matrix the inverse A_i^{-1} of the matrix A_i exists, but has the parameter q^{-1}. Moreover, the oscillator which corresponds to A_i^{-1} is the same as the one corresponding to A_i. However, in our representations of quantum Grassmannian manifolds the q-oscillators corresponding to the matrices A_i have to be independent (commuting). Hence A_i^{-1} does not belong to the class of matrices we consider in equations (14) or (10). Relations (15) and (16) actually define the singular Hecke algebra H(0,n) [5], i.e. generators are not invertible. This of course has to do with the fact that although the inverse oscillator matrices

$$A^{-1}(a) = \begin{bmatrix} a^* & -q^{-1}(1-a^*a)^{1/2} \\ q(1-aa^*)^{1/2} & a \end{bmatrix}$$

exist, they never occur in our expressions.

Expressions such as (10) and (14) give well-defined elements of the braid group, since they never contain the inverse of the q-oscillator matrix A_i and we never have to use relation (16) in our manipulations of formulas (10) and (14). Thus using (10) and (14) , to each quantum group $SU_q(n)$ and to each quantum group coset $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$ there corresponds a well-defined element of the braid group. The closure of the

braid associated with $SU_q(n)$ gives a link of l components, where l is equal to the integer part of $(n+1)/2$, while the closure of the braid corresponding to $SU_q(n+m)/SU_q(n)\times SU_q(m)\times U(1)$ gives a link of l components, l being the greatest common divisor of m and n.

We have presented an explicit matrix representation of the quantum Grassmannian manifolds. Our construction has led us to a close relationship between quantum matrix groups, the braid group and the Hecke algebra. The essential ingredient to establish this relationship turns out to be the oscillator representation of the quantum matrices.

Acknowledgement: This work was supported in part by T.B.T.A.K. the Turkish Scientific and Technical Research Council.

References

[1] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhajan, preprint LOMI, 1987; V.G. Drinfeld, Quantum Groups, Proc. Int. Congr. Math., Berkeley I, 798-820 (1986); M. Jimbo, Lett. Math. Phys. 11, 247 (1986); S.L. Woronowicz, Commun. Math. Phys. 111, 613 (1987).

[2] "New Developments in the Theory of Knots" ed. T. Kohno, World Scientific, Singapure, 1989.

[3] V. Turaev "The Yang-Baxter Equation and Invariants of Links", Invent. Math.92 (1988), 527-553. A.N. Kirillov, N.Yu. Reshetikhin, "Representations of the Algebra $U_q(Sl(2))$, q-Orthogonal Polynomials and Invariants of Links, in Infinite Dimensional Lie Algebras and Groups". ed. V.G. Kac, World Scientific, 1989. T. Kohno "Monodromy Representations of Braid Groups and Yang-Baxter Equations" Ann. Inst. Fourier 37 (1987), 139-160. reprinted in [2]. M. Jimbo in [1].

[4] M. Arık and S. Çelik, "Unitary quantum groups, quantum projective spaces, and q-oscillators", Boğaziçi University preprint BUFB-92-3.

[5] V.Jones "Hecke algebra representations of braid groups and link polynomials" Annals of Mathematics, 126 (1987), 335-388. reprinted in [2].