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Physikalisches Institut

VACUUM FLUCTUATIONS IN RADIATION- AND MATTER-FIELDS +)

by

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Abstract: We discuss general notions of the regularisation and renormalisation of ground-state observables in quantum field theory. We evoke a theorem by Ramanujan to introduce the concept of the cut-off independent parts of vacuum observables and discuss their measurement and methods for their calculation. The role of Atiyah-Patodi-Singer spectral invariants in determining the vacuum structure of relativistic matter fields is illustrated.

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I. INTRODUCTION

A discussion of physical properties of the vacuum should begin with a few general remarks attempting to elucidate general notions of the problem. Consider a cavity made of "matter" enclosing a section of vacuum. Quantum theory has a unique prediction for the structure of the ground state of the latter: each oscillator degree of freedom (for a precise definition see below) contributes $\varepsilon_\alpha = \frac{1}{2}\hbar\omega_\alpha$, ω_α is the frequency of this degree of freedom, to the energy of the ground state; degrees of freedom of the matter walls are not accounted for and we have

$$E_{vac} = \sum_\alpha \frac{1}{2}\hbar\omega_\alpha \quad . \tag{1.1}$$

Now, the frequency spectrum of the modes in our cavity is unbounded from above and we conclude

$$E_{vac} = \infty \quad . \tag{1.2}$$

However, this infinity depends on the quantum-field theoretical model and its parameters, on the geometry and the physical properties of the cavity and, therefore, is of physical interest. In order to extract information on the structure of the vacuum from this quantity we have to classify divergences, clarify dependences on the cavity parameters and model measuring procedures.

Let us begin with a 'Gedankenexperiment' and consider an electro-magnetic radiation field enclosed in cavities of identical shape but made of different materials. We decouple thermal black-body radiation modes by going to zero temperature: only 'vacuum fluctuations' or 'zero point oscillations' contribute to the field energy. Different materials used for building the cavities provide natural cut-offs for the spectrum of the zero point modes, in particular for its high-frequency components. Hence, in actual fact the vacuum energy is given by a cut-off sum

$$E_{vac} = \frac{\hbar}{2}\sum \omega_\alpha C_\alpha(\Lambda) \tag{1.3}$$

where the $C_\alpha(\Lambda)$ are material-dependent functions which are expected, with reasonable confidence, to provide a cut-off and to lead to a convergent series for the vacuum energy. Its value, however, then is predominantly determined by the experimental arrangement and the question of physical interest arises. Can we identify and experimentally disentangle components in E_{vac} which unambigously reflect the structure of the theory for the fields making up the vacuum state and which are independent of the apparatus, in particular of the cavity materials?

The parameter Λ stands for a length characterising the cut-off scale such that

$$C_\alpha(\Lambda)\Big|_{\Lambda=0} = 1 \, . \tag{1.4}$$

The other parameter in this game carrying a dimension is a length, L, characterising the spatial extension of the cavity.

We expect E_{vac} to contain terms proportional to the volume, to the surface etc. of the cavity. Therefore, we write E_{vac} as a series

$$E_{vac} = a_0 L^3 \Lambda^{-4} + a_1 L^2 \Lambda^{-3} + a_2 L \Lambda^{-2} + a_3 \Lambda^{-1} + a_4 L^{-1} + a_5 L^{-2}\Lambda + \dots \tag{1.5}$$

which we read as an asymptotic series for small Λ. The coefficient a_i can be determined by measuring E_{vac} as a function of Λ (remember that we are discussing a Gedankenexperiment!), varying the cavity materials. It is the very point of this enterprise to realise that a_4 has a universal meaning.

This is guaranteed by a theorem[1)] due to Ramanujan which we state in a loose form appropriate for our purposes.

Let

$$\sum_{n=1}^{\infty} a_n = \infty \tag{1.6}$$

be a divergent series such that

$$a_n \underset{n\to\infty}{\sim} n^k \quad , \quad k > -1 \tag{1.7}$$

and $\{C_n(\Lambda)\}$ a set of functions with a sufficiently rapid decrease for $n \to \infty$ such that the series

$$\sum_{n=1}^{\infty} a_n C_n(\Lambda) \quad , \quad \Lambda \neq 0 \tag{1.8}$$

is convergent. Choose $\{C_n(\Lambda)\}$ such that

$$C_n(0) = 1 \; ; \tag{1.9}$$

the Λ-independent part of (1.8) is then independent of the choice of the set $\{C_n(\Lambda)\}$ as long as (1.9) holds.

The set $\{C_n(\Lambda)\}$ is realised by choosing a C^∞-function $\lambda(x)$ with

$\lambda(0) = 1$ for $x = 0$

$\lambda(x) \to 0$ sufficiently rapidly
$x \to \infty$

and putting

$$C_n(\Lambda) = \lambda\left(\frac{n\Lambda}{d}\right)$$

where d is an irrelevant length scale.

Since the use of cavities made of different materials translates into different sets of functions $\{C_n(\Lambda)\}$ we conclude that

$$E_{vac}^{(u)} = \Lambda\text{-independent part of } \frac{\hbar}{2} \sum \omega_n C_n(\Lambda) \tag{1.10}$$

is also independent[2)] of the physical properties of the experimental set up and thus represents the universal component of the vacuum energy which only follows from the quantum theory for the 'radiation'-field (actually containing all degrees of freedom making up the ground-state of the world) and the global geometrical properties of the 'piece' of vacuum enclosed by the cavity.

How do we measure and compute in practice this universal component? Let us consider an old story in the new light of (1.10): the original Casimir effect[3)]. The experimental arrangement originally consisted of two infinitely extended parallel metal plates at a distance d. The frequency spectrum of the electro-magnetic field modes is given by

$$\omega_n = \pi \sqrt{\left(\frac{n}{d}\right)^2 + k_x^2 + k_y^2} \tag{1.11}$$

$$n \in \mathbb{Z} \; , \; k_{x,y} \text{ continuous.}$$

We choose an exponential cut-off

$$C_n(\Lambda) = e^{-\frac{\Lambda \omega_n}{\pi}} \tag{1.12}$$

and obtain for the vacuum energy (1.3) per unit area

$$\begin{aligned} E_{vac}(d,\Lambda) &= \frac{\hbar}{2} \sum_{n=-\infty}^{+\infty} 2\pi \int_0^\infty k\,dk \; \pi \sqrt{\left(\frac{n}{d}\right)^2 + k^2} \; e^{-\frac{\Lambda \omega_n}{\pi}} \\ &= \frac{\hbar \pi^2}{2d} \frac{d^2}{d\Lambda^2} \sum_{n=0}^{\infty} \frac{B_n}{n!} \left(\frac{\Lambda}{d}\right)^{n-2} + \frac{2\hbar\pi^2}{\Lambda^3} \\ &= \frac{\hbar\pi^2}{2}\left(\frac{6d}{\Lambda^4} + \frac{3}{\Lambda^3} - \frac{1}{360} d^{-3} + \dots\right); \end{aligned} \tag{1.13}$$

we observe consecutively the volume term (proportional to d since we are considering an effectively one-dimensional problem), a surface term (proportional to d^0 accounting for the point-like surface) and the Λ-independent part

$$E_{vac}^{(u)}(d) = \frac{\hbar\pi^2}{720} d^{-3} \quad , \tag{1.14}$$

the Casimir potential.

To measure the latter in its pure form we consider an experimental set-up of three parallel, conducting, infinitely extended plates: two plates fixed at a distance D, the third plate between them at a distance d from one of the plates such that

$$d \ll D .$$

The vacuum energy per unit area is then

$$\begin{aligned} E_{vac}(d,D,\Lambda) = \frac{\hbar\pi^2}{2}\Big(&\frac{6}{\Lambda^4}(d + D - d) + \frac{6}{\Lambda^3} \\ &- \frac{1}{360}\left(d^{-3} + (D-d)^{-3}\right) + \dots \Big) . \end{aligned} \tag{1.15}$$

The Casimir potential is measured by varying d, we neglect the $(D-d)^{-3}$ term and note that the cut-off dependent volume and surface terms are irrelevant constants not

contributing to the force between the two close plates. We see that in the light of our definition of a universal component in the vacuum energy (which can be literally extended to other observables), no artificial subtraction procedures have to be introduced, procedures which are not modelled in experiment.

A final remark seems in place. If logarithmic terms appeared in the expansion (1.5), the Λ-independent part would be obviously ill-defined and the vacuum energy would not contain a universal component in the sense discussed above.

We now turn to the question of how to compute this universal component in quantum theory and distinguish two cases. We first treat theories in which the single particle dispersion is determined by the Laplace or, rather, the Laplace-Beltrami operator, i.e. theories of the electro-magnetic radiation field and of non-relativistic matter fields. The case of relativistic matter-fields has to be discussed separately since the single-particle dispersion operators are not bounded from below and the phenomenon of anti-particle states has to be accounted for.

II. THE VACUUM ENERGY FOR LAPLACIAN MODELS

The Laplace-Beltrami operator

$$A = -\Delta + V(\vec{x})$$
$$\Delta = |g|^{-1/2}\partial_\mu |g|^{1/2} g^{\mu\nu}\partial_\nu \tag{2.1}$$

where $g^{\mu\nu}(\vec{x})$ is the metric tensor (and $|g|$ its determinant) on a position space manifold parametrised by coordinates $\vec{x}$ on which our system is found to move, is known to describe the single-particle excitation spectrum of the radiation-field (V=0) and of non-relativistic particles moving in the potential $V(\vec{x})$. Our goal is to compute the universal cut-off independent part of the vacuum energy described in the preceding chapter. To this end we regularise E_{vac}

$$E_{vac} = \frac{\hbar}{2}\sum \varepsilon_\alpha = \frac{\hbar}{2} tr\, A \tag{2.2}$$

by introducing the same cut-off function $C_\alpha(\Lambda)$ as in our calculation of the Casimir effect

$$C_\alpha(\Lambda) = e^{-\Lambda\varepsilon_\alpha} \tag{2.3}$$

such that

$$E_{vac}(\Lambda) = \sum_\alpha \varepsilon_\alpha e^{-\Lambda\varepsilon_\alpha} = tr\left(A e^{-\Lambda A}\right)$$
$$= -\frac{d}{d\Lambda} tr\left(e^{-\Lambda A}\right) =: -\frac{d}{d\Lambda} Z(\Lambda) \ . \tag{2.4}$$

The partition function $Z(\Lambda)$, i.e. Λ^{-1} can be visualised as temperature, allows for an interesting 'high-temperature' expansion, asymptotic for $\Lambda \to 0$, which is precisely the expansion we have been anticipating in (1.5). We now give a short description of the method.

The central object is the 'heat-kernel' $K_\Lambda(\vec{x},\vec{x}')$ determined from the equation

$$\left(-\frac{\partial}{\partial\Lambda} - A\right) K_\Lambda(\vec{x},\vec{x}') = 0 \tag{2.5}$$

with

$$K_\Lambda(\vec{x},\vec{x}')\Big|_{\Lambda\to 0} = |g|^{-1/2}\delta(\vec{x}-\vec{x}') \tag{2.6}$$

We brutally abbreviate a long and fascinating story[4-6] on local differentiability and global topological properties of geometrical entities if we simply state that this heat-kernel has an asymptotic expansion for $\Lambda \to 0$. By taking its trace and thus computing $Z(\Lambda)$ we obtain the asymptotic expansion used in (1.3), (1.4) and (1.10) and its Λ-independent part.

The point is that there exist algorithms for a systematic computation of the coefficients $c_k(\vec{x},\vec{x}')$ in the said asymptotic expansion

$$K_\Lambda(\vec{x},\vec{x}') \underset{\Lambda\to 0}{\sim} (4\pi\Lambda)^{-d/2} e^{-\frac{r^2}{4\Lambda}} \sum_{k=0}^{\infty} \Lambda^{k/2} c_k(\vec{x},\vec{x}') \tag{2.7}$$

where r denotes the geodesic distance in d-dimensions between $\vec{x}$ and $\vec{x}'$. For the regularised vacuum energy we then find

$$E_{vac}(\Lambda) \underset{\Lambda \to 0}{\sim} (4\pi)^{-d/2} \sum_{k=0}^{\infty} \frac{k-d}{2} \Lambda^{\frac{k-d}{2}-1} \int d^d x \; c_k(\vec{x},\vec{x}) \tag{2.8}$$

and, hence,

$$E_{vac}^{(u)} = (4\pi)^{-d/2} \int d^d x \; c_k(\vec{x},\vec{x}) \Big|_{k=d+2} \quad . \tag{2.9}$$

The coefficients $c_k(\vec{x},\vec{x}')$ can be found in refs. /5-11/ for the case of compact manifolds without boundary and in ref. /12/ and ref. /13/ for compact manifolds with boundary in d=2 and d=3 respectively. It is clear that the last case corresponds to the physical situation, vacuum enclosed in a cavity, discussed in the previous chapter.

For the sake of completeness we cite the first few coefficients; more precisely, we give only the diagonal parts since those are the ones needed in the computation of $E_{vac}(\Lambda)$ and $E_{vac}^{(u)}$:

<u>d=3:</u>

$$c_0(\vec{x},\vec{x}) = 1$$

$$c_1(\vec{x},\vec{x}) = \frac{\sqrt{\pi}}{2}\sqrt{g^{33}}\,\delta_B$$

$$c_2(\vec{x},\vec{x}) = \frac{R}{6} - V + \left\{\frac{2}{3}H\sqrt{g^{33}} + \frac{1}{2}\sqrt{g^{33}}\,\partial_3\right\}\delta_B$$

$$c_3(\vec{x},\vec{x}) = \frac{\sqrt{\pi}}{2}\left\{V + \frac{1}{8}H^2 - \frac{5}{24}K - \frac{R}{8}\right\}\sqrt{g^{33}}\,\delta_B - \frac{5}{16}\sqrt{\pi}\,H\sqrt{g^{33}}\,\partial_3\delta_B - \frac{1}{8}(g^{33})^{3/2}\partial_3 g_{33}\partial_3\delta_B$$

etc.; (2.10)

δ_B is the δ-function with support on the boundary of our 3-dimensional manifold which we parametrised with normal coordinates, the 3-direction being the normal on the sufficiently smooth boundary; R is the Riemannian curvature scalar, H the average- and

K the Gaussian curvature of the surface $x_3=0$. Dirichlet boundary conditions have been imposed. Similar formulae hold for Neumann boundary conditions.[13]

Summarising we may state that heat-kernel expansions provide a general method to extract $E_{vac}^{(u)}$, the cut-off independent part of $E_{vac}(\Lambda)$, from a given model for single-particle excitations. Generally speaking, for odd dimensions, the latter is a pure boundary effect and depends only on the global geometry of the cavity since odd coefficients $c_{2k+1}(\vec{x},\vec{x})$ can be shown to be proportional to δ_B and its normal derivatives.

We should stress again that only the coefficient $c_{d+2}(\vec{x},\vec{x})$ leads to a physically relevant contribution in $E_{vac}(\Lambda)$: the cut-off functions $C_\alpha(\Lambda)$ provided by a specific material used to construct the cavity are, in general, not parametrised by $e^{-\Lambda t_\alpha}$ and the coefficients a_i, $i \neq 4$, in (1.5), hence, are not reproduced in a heat-kernel expansion since they do depend on the set $\{C_\alpha(\Lambda)\}$.

III. THE VACUUM ENERGY OF RELATIVISTIC, MASSIVE FIELDS

We now turn to the more involved case of relativistic systems. Here the main complications come from the fact that the single-particle dispersion operator A is unbounded from below and the question of stability poses itself The generally accepted solution of this problem is to imbed a given single-particle theory into a many-body theory formulated in terms of local quantum fields, i.e. into local quantum field theory: a spectrum of single-antiparticle states and the corresponding pair-creation processes appear. We should like to give a cursory sketch of the main issues pertinent for a discussion of vacuum properties.

We treat simultaneously the cases of spin 0 bosons and spin 1/2 fermions, the single-particle spectrum of which we take to be given by

Spin 0 Bosons: $$A = \left(-(\vec{\partial} + ie\vec{A})^2 + m^2\right)^{1/2} + eA_0$$

Spin 1/2 Fermions: $$A = \frac{1}{i}\vec{\alpha}\cdot(\vec{\partial} + ie\vec{A}) + \beta m + eA_0 \tag{3.1}$$

where $(\vec{A}, A_0)$ is the electro-magnetic 4-potential, m the particle mass, $\vec{\alpha}$, β the usual Dirac matrices; both signs of the square-root are understood.

We shall work in the Schrödinger picture of quantum field theory[14], more precisely speaking in its coordinate representation. Here, the coordinate is a complex field

acting as a multiplicative operator and the adjoint momentum acts as the functional derivative versus this field :

coordinate: $\varphi(\vec{x})$, complex field acting as a multiplicative operator

momentum: $\pi(\vec{x}) = -i\dfrac{\delta}{\delta\varphi(\vec{x})}$.

The dynamics of a physical system is determined by a Hamiltonian

$$\mathcal{H} = \mathcal{H}[\varphi;\pi] \quad ; \tag{3.2}$$

quantum states are represented by functionals

$$\Psi = \Psi[\varphi;t]$$

the time-dependence of which is governed by the Schrödinger equation

$$(i\partial_t - \mathcal{H})\Psi = 0. \tag{3.3}$$

In this formulation quantum field theory is directly interpreted as the wave-mechanics of a continuum of degrees of freedom, as quantum continuum mechanics.

Generalising the Hamiltonian of an harmonic oscillator accordingly we have

$$\mathcal{H} = \int d^3x \left\{ -\pi^*\pi(\vec{x}) + \varphi^* A^2 \varphi(\vec{x}) \right\} . \tag{3.4}$$

Its diagonalisation proceeds as follows. We first use the spectrum of A

$$A\psi_\alpha(\vec{x}) = \varepsilon_\alpha \psi_\alpha(\vec{x}) \tag{3.5}$$

to define the operator $\tilde{A}$

$$\tilde{A} := A \,\mathrm{sign}\, A \tag{3.6}$$

by its eigenvalues

$$\tilde{\varepsilon}_\alpha := |\varepsilon_\alpha| . \tag{3.7}$$

The square-root $\tilde{A}^{1/2}$ being well-defined we then introduce two sets of creation and annihilation operators

$$\begin{aligned}
a^+(\vec{x}) &:= -\frac{\tilde{A}^{-1/2}}{\sqrt{2}}\frac{\delta}{\delta\varphi(\vec{x})} + \frac{\tilde{A}^{1/2}}{\sqrt{2}}\varphi^*(\vec{x}) \\
a(\vec{x}) &:= \frac{\tilde{A}^{-1/2}}{\sqrt{2}}\frac{\delta}{\delta\varphi^*(\vec{x})} + \frac{\tilde{A}^{1/2}}{\sqrt{2}}\varphi(\vec{x})
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
b^+(\vec{x}) &:= -\frac{\tilde{A}^{-1/2}}{\sqrt{2}}\frac{\delta}{\delta\varphi^*(\vec{x})} + \frac{\tilde{A}^{1/2}}{\sqrt{2}}\varphi(\vec{x}) \\
b(\vec{x}) &:= \frac{\tilde{A}^{-1/2}}{\sqrt{2}}\frac{\delta}{\delta\varphi(\vec{x})} + \frac{\tilde{A}^{1/2}}{\sqrt{2}}\varphi^*(\vec{x})
\end{aligned} \tag{3.9}$$

with the usual commutation (bosons) or anticommutation (fermions) rules

$$\begin{aligned}
&\left[a(\vec{x}), a^+(\vec{x}')\right]_\pm = \left[b(\vec{x}), b^+(\vec{x}')\right]_\pm = \delta(\vec{x}-\vec{x}') \\
&\left[a(\vec{x}), b(\vec{x})\right]_- = 0 \qquad \text{etc.}
\end{aligned} \tag{3.10}$$

where the fermion anticommutators are calculated by considering the fermion field as an anticommuting Grassmann variable.

The Hamiltonian is then written in diagonal form[15)]

$$\mathcal{H} = \int d^3x \left\{ a^+(\vec{x})\tilde{A}a(\vec{x}) + b^+(\vec{x})\tilde{A}b(\vec{x}) \right\} \pm \mathrm{tr}\,\tilde{A} \tag{3.11}$$

where the (+)-sign holds for bosons, the (-)-sign for fermions.

It is obviously diagonal in a Fock space built up by creation and annihilation operators

$$\left.\begin{aligned}
a_\alpha^+ &:= \int d^3x\, a^+(\vec{x})\,\psi_\alpha(\vec{x}) \\
a_\alpha &= \int d^3x\, \psi_\alpha^*(\vec{x})\, a(\vec{x})
\end{aligned}\right\} \longrightarrow \Psi^{(a)}$$

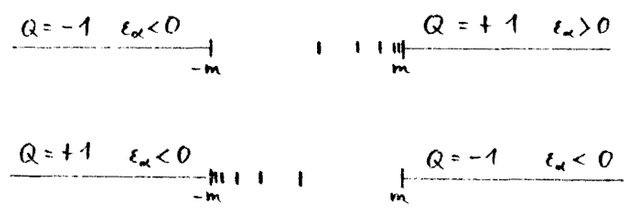
and

$$\left.\begin{aligned} b_\alpha^\dagger &:= \int d^3x\, \psi_\alpha^*(\vec{x})\, b^\dagger(\vec{x}) \\ b_\alpha &= \int d^3x\, b(\vec{x})\, \psi_\alpha(\vec{x}) \end{aligned}\right\} \longrightarrow \Psi^{(b)} \;;$$

the Fock states are direct products

$$\Psi = \Psi^{(a)} \otimes \Psi^{(b)} \;; \tag{3.12}$$

the two factors are distinguished by their charge structure which we depict schematically by plotting the (a)- and (b)-spectra



(a)

(b)

Because of the external charges distinguishing the charges $Q = \pm 1$, the spectral densities on the left- and right-hand branches are different.

Charge conjugation now serves to cross-identify the (a) and (b) spectra; the single-particle and -antiparticle states are then doubly counted in the states (3.12).

Defining occupation numbers

$$n_\alpha^{(\pm)} = \begin{cases} \text{occupation number for particles } Q = +1 \\ \text{occupation number for antiparticles } Q = -1 \end{cases}$$

we have for the energy of (3.12), defined as

$$\mathcal{H}\Psi = 2E[n_\alpha^{(+)}, n_\alpha^{(-)}]\Psi \tag{3.13}$$

the following expression

$$\begin{aligned} E[n_\alpha^{(+)}, n_\alpha^{(-)}] &= \frac{1}{2}\langle \Psi | \mathcal{H} | \Psi \rangle \\ &= \sum_{\varepsilon_\alpha > 0} n_\alpha^{(+)} \tilde{\varepsilon}_\alpha + \sum_{\varepsilon_\alpha < 0} n_\alpha^{(-)} \tilde{\varepsilon}_\alpha \pm \frac{1}{2}\, tr\, \tilde{A} . \end{aligned} \tag{3.14}$$

The vacuum energy obtains if we set

$$n_\alpha^{(+)} = n_\alpha^{(-)} = 0 \quad \text{for all } \alpha \tag{3.15}$$

and we find the relativistic analogon for (1.1) and (2.2) $(\hbar = 1)$

$$\begin{aligned} E_{vac} &= \pm \frac{1}{2}\, tr\, \tilde{A} \\ &= \pm \frac{1}{2}\Big(\sum_{\varepsilon_\alpha > 0} \varepsilon_\alpha - \sum_{\varepsilon_\alpha < 0} \varepsilon_\alpha \Big) . \end{aligned} \tag{3.16}$$

We emphasise that only positive energies enter in our diagonalisation of $\mathcal{H}$, states Ψ correspond to physical states.

The vacuum energy is infinite and needs a regularisation. The procedures introduced in the previous chapter can be applied literally if we start from

$$E_{vac}(\Lambda) = \pm \frac{1}{2}\, tr\left(\tilde{A}\, e^{-\Lambda \tilde{A}^2} \right) \tag{3.17}$$

and extract the universal component $E_{vac}^{(u)}$.

We shall not pursue this line any longer but rather discuss another intriguing aspect of the vacuum: the charge density induced by vacuum deformation, effected e.g. by cavities.

The charge density is given as the first order response of the total energy versus the time component of the external 4-potential; the vacuum energy, in particular, leads to the vacuum charge density ρ_{vac}.

We find

$$\rho_{vac}(\vec{x}) = \frac{\delta E_c}{\delta e A_0(\vec{x})}$$
$$= \pm \frac{1}{2}\left(\sum_{\varepsilon_\alpha > 0} |\psi_\alpha(\vec{x})|^2 - \sum_{\varepsilon_\alpha < 0} |\psi_\alpha(\vec{x})|^2 \right) . \tag{3.18}$$

The physical meaning of this formula is clear: the vacuum charge density is induced by an external field distinguishing positive and negative charges.

The total induced charge, i.e. the integral

$$Q_{vac} = e \int d^3x \, \rho_{vac}(\vec{x}) \tag{3.19}$$

is of particular interest since it is nothing but the famous Atiyah-Patodi-Singer spectral invariant[16] which is non-trivial for topologically non-trivial external fields produced e.g. by monopoles, dyons etc. There is a number of very interesting physical phenomena[17], subsumed under the name of fermion fractionisation[18] related to this quantity.

Up to now we have covered the cases of charged bosons or fermions interacting with an external electromagnetic field. If we promote the latter to a dynamical degree of freedom by adding its kinetic term to the Hamiltonian (3.4) using e.g. Coulomb gauge we end up with the full Hamiltonian of quantum electrodynamics (QED). The construction of Fock states proceeds by introducing single-particle orbitals. The latter are produced in a mean-field ansatz, the simplest of which is, perhaps, the Hartree-field: in (3.1) we write instead of eA_0

$$eA_0\big|_H = \frac{e^2}{4\pi} \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}') \tag{3.20}$$

where

$$\rho(\vec{x}) = \sum_{\varepsilon_\alpha > 0} n_\alpha^{(+)} |\psi_\alpha(\vec{x})|^2 - \sum_{\varepsilon_\alpha < 0} n_\alpha^{(-)} |\psi_\alpha(\vec{x})|^2$$
$$+ \rho_{vac}(\vec{x}) \tag{3.21}$$

and determine the ψ_α self-consistently.

We have recently shown[19] that the use of generalised Hartree mean-fields, the g-Hartree fields, allows for a unique determination of the energy shift produced by vacuum deformations, which one might call the Casimir effect in atoms.

The g-Hartree potential contains an exchange term $eA_0\big|_{exch}$ and reads

$$eA_0\big|_{g\text{-}H} = g\, eA_0\big|_H - (1-g)\, eA_0\big|_{exch} \tag{3.22}$$

where $g = g_0$ can be chosen such that

$$E_{QED} = E[n_\alpha^{(+)}, n_\alpha^{(-)}]\Big|_{g=g_0} , \tag{3.23}$$

i.e. such that the exact total energy predicted by QED is given by g-Hartree single particle excitations. Since the vacuum contribution to the latter can be identified unambiguously, it can be extracted from the experimentally measured E_{QED}. Its measurement in highly ionised atoms is of great interest since it gives insight into the structure of the vacuum component due to all charged quanta; not only electrons and positrons but also other charged fundamental fields contribute to it.

IV. CONCLUSIONS

We have discussed the regularisation of the vacuum energy which in actual experiments is provided by the physical properties of the cavity walls enclosing the region of vacuum under consideration. A very powerful theorem by Ramanujan[1] led us to the notion of the cut-off independent component in the vacuum energy which is solely determined by the structure of the quantum theory describing the degrees of freedom building up the vacuum state and by the geometry of the cavity. To compute the cut-off independent part we are free to choose a convenient regularisation: the heat-kernel provides a cut-off and allows for an algorithm for the computation of the cut-off independent part; global geometric quantities like integrated curvature and Riemann scalar for the cavity walls appear in it.

The case of relativistic matter fields requires further analysis in order to incorporate the antiparticle spectrum in a physically adequate manner: the diagonalisation of (3.4) under the requirement that only positive energies appear in its spectrum leads to (3.11) from which we read off that the eigenvalues are determined by the positive $\tilde{\varepsilon}_\alpha$ and that a constant contribution $\pm \frac{1}{2} \operatorname{tr} \tilde{A}$ has to be taken into

account which is finally due to this positivity. This is the vacuum energy which is a non-trivial infinity since it depends on physical degrees of freedom and geometrical properties of boundaries in an essential way. This becomes apparent if we compute the vacuum charge density $\rho_{vac}(\vec{x})$ the integral of which is the famous Atiyah-Patodi-Singer spectral invariant[16] which led to the interesting phenomenon of fermion fractionisation[18].

We pointed out, in an extension of the theory to full QED, that energy shifts self-consistently produced by $\rho_{vac}(\vec{x})$ are physically interesting quantities, measureable in highly ionised atoms.

The case of self-interacting fields like the one described by the Landau-Ginzburg Hamiltonian and its relativistic generalisations, the scalar φ^4-models, remains to be treated. We refrain from doing so here and merely refer to the literature[20].

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