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New Solutions to Braid Group Yang-Baxter Equation and Integrable Fermion Systems

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The generators of braid group B_N are Δ, Δ^{-1} and b_1, b_1^{-1},

$$\Delta = b_1 b_2 \cdots b_{N-1} \ , \tag{8}$$

such that all elements are expressed by these four generators,

$$b_\alpha = \Delta^{\alpha-1} b_1 \Delta^{-\alpha+1} \ . \tag{9}$$

1.4 Yang-Baxter Equation

In an N-body quantum system with particle labelled by $i = 1, 2, \cdots, N$, $R(\lambda)$ is an operator defined on every pair of particles, say, R_{ij} is defined on the i-th and j-th particles, and $\lambda \in \mathbf{C}$ is called spectral parameter. The following equation about $R_{ij}(\lambda)$ is called Yang-Baxter equation,

$$R_{12}(\lambda) R_{13}(\lambda+\mu) R_{23}(\mu) = R_{23}(\mu) R_{12}(\lambda+\mu) R_{23}(\lambda) \ . \tag{10}$$

Such equations exist for every two particles. The simplest solution of Yang-Baxter equation was found by C. N. Yang[2]. There is another form of Yang-Baxter equation, different from (10),

$$\hat{R}_{12}(\lambda) \hat{R}_{23}(\lambda+\mu) \hat{R}_{12}(\mu) = \hat{R}_{12}(\mu) \hat{R}_{23}(\lambda+\mu) \hat{R}_{12}(\lambda) \ , \tag{11}$$

where

$$\hat{R}_{12} = \sigma_{12} \circ \Re_{12} \ , \quad \hat{R}_{13} = \sigma_{13} \circ \Re_{13} \ , \quad \hat{R}_{23} = \sigma_{23} \circ \Re_{23} \ . \tag{12}$$

Here σ_{ij} is an element from permutation group. The action of σ_{ij} is interchanging particle indices i and j, and $\sigma \circ \Re_{ij} = \Re_{\sigma(i)\ \sigma(j)} \circ \sigma$.

From a constant solution $\hat{R}$ of Yang-Baxter equation

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23} \ , \tag{13}$$

we have the monodromy construction of braid group representation (usually reducible),

$$b_\alpha = \hat{R}_{\alpha\alpha+1} \ . \tag{14}$$

In this paper, however, we use another definition of braid group relations, as follows,

$$\begin{aligned} B_{ij} B_{kl} &= B_{kl} B_{ij} \ , \quad (i, j \neq k, l) \ , \\ B_{ij} B_{jk} B_{ij} &= B_{jk} B_{ij} B_{jk} \ , \end{aligned} \tag{15}$$

and it is apparent that $b_\alpha = B_{\alpha\alpha+1}$.

2 Hopping-Symmetric *Normal* Braid Group Representations

We give the representations of braid group realized through fermi creation and annihilation operators $c_i^\pm$ (with $i = 1, 2, \cdots, N$ runs over all fermions), and classify them according to their character equations.

The general solution B_{ij} to braid group relation (15) is

$$B_{ij} = -1 + N_i + N_j + x_0 c_j^+ c_j^- + x_1 c_i^- c_j^+ + x_2 N_i N_j \ , \tag{16}$$

where $N_1 = c_i^+ c_i^-$ and x_0, x_1 and x_2 are arbitrary complex numbers. We will limit ourselves to hopping-symmetric solutions, i.e.,

$$B_{ij} = B_{ji} \ , \tag{17}$$

or equivalently $x_1 = -x_0$ in this section. This consideration is from the fact that the hopping of fermions to left and right should have the same possibility.

The general solution in equation (16) can be parameterized (baxterized[13]) by the trigonometric baxterization theories (and we will closely follow the theory developed in [14, 15, 16], because the authors are familiar to the work of the Nankai-Lanzhou group). The baxterization depends on the character equations of these braid group representations. Therefore it is useful to classify the solutions according to their numbers of eigenvalues or their character equations.

2.1 The Case of Two Eigenvalues

The solution should satisfy the character equation

$$B_{ij}^2 + C_1 B_{ij} + C_0 = 0 \ , \tag{18}$$

and we have three solutions .

2.1.1 Permutator Solution: $x_1 = -x_0 = 1$, $x_2 = 0$

so that

$$(B_{ij} - 1)(B_{ij} + 1) = 0 \ . \tag{19}$$

explicitly

$$\underline{P_{ij} = -1 + N_i + N_j - c_i^+ c_j^- + c_i^- c_j^+} \ . \tag{20}$$

This solution will be remembered as P_{ij}, which is the *permutator* (or pure *hopping* hamiltonian) of this system. It can be verified that

$$\begin{aligned} P_{ij} &= P_{ji} \ , \quad P_{ij}^2 = 1 \ , \\ P_{ij} P_{jk} &= P_{jk} P_{ik} = P_{ik} P_{ij} \ , \end{aligned} \tag{21}$$

and

$$\begin{aligned} P_{ij} P_{kl} &= P_{kl} P_{ij} \ , \quad (i, j \neq k, l) \\ P_{ij} P_{ik} P_{ij} &= P_{jk} \ . \end{aligned} \tag{22}$$

An important property of P_{ij} is that it interchanges the fermi operators at sites i and j:

$$P_{ij} c_i^\pm P_{ij} = c_j^\pm \ . \tag{23}$$

2.1.2 $x_1 = -x_0 = \pm 1$, $x_2 = -2$

Explicitly

$$B_{ij} = -1 + N_i + N_j \pm (-c_i^+ c_j^- + c_i^- c_j^+) - 2 N_i N_j \ , \tag{24}$$

which are nonlinear solutions. These two solutions are special cases of a more general solution with three eigenvalues, as can be seen in the following, and so will not be discused in detail.

2.2 The Case of Three Eigenvalues

The solutions that satisfy the following character equation have three eigenvalues

$$B_{ij}^3 + C_2 B_{ij}^2 + C_1 B_{ij} + C_0 = 0 \ . \tag{25}$$

We have three of such solutions when

$$C_1 = C_0 = -x_0^2 \ , \ C_2 = 1 \ , \tag{26}$$

and

$$x_2 = -2 \quad \text{or} \quad -1 \pm x_0 \ . \tag{27}$$

That is, we may have a free parameter x_0. The eigenvalues, denoted λ_i $(i = 1, 2, 3)$, are $-1, x_0, -x_0$ respectively. The character equation (25) may be rewritten as

$$(B_{ij} + 1)(B_{ij} - x_0)(B_{ij} + x_0) = 0 \ . \tag{28}$$

When $x_2 = -2$ and $x_0 = \pm 1$, the above solution identifies with (24).

2.3 The Case of Four Eigenvalues

The solution to the character equation

$$B_{ij}^4 + C_3 B_{ij}^3 + C_2 B_{ij}^2 + C_1 B_{ij} + C_0 = 0 \tag{29}$$

gives

$$C_0 = x_0^2(1 + x_2) \ , \quad C_1 = x_0^2 x_2 \ , \quad C_2 = -x_0^2 - 1 - x_2 - 1 \ , \quad C_3 = -x_2 \ . \tag{30}$$

The four eigenvalues are $-1, 1 + x_2, \pm x_0$. This is a general solution where there are two free parameters, x_0, x_2. Note that here $x_2 \neq 0, -2$ or the character equation (29) reduces to (28) or (19).

3 Trigonometric Baxterization

First, we give a brief review of the parameterization theory of the braid group representations to get solutions of the Yang-Baxter equation with spectral parameter:

$$\hat{R}_{12}(x)\hat{R}_{23}(xy)\hat{R}_{12}(y) = \hat{R}_{23}(y)\hat{R}_{12}(xy)\hat{R}_{23}(x) \ , \tag{31}$$

where $\hat{R}_{12}(x) = \hat{R}(x) \otimes 1$ and $\hat{R}_{23}(x) = 1 \otimes \hat{R}(x)$.

Given that a braid group representations with m distinct eigenvalues λ_i with $i = 1, 2, \cdots, m$ and obeys a character equation (reduction relation)

$$\prod_{i=1}^{m} (B - \lambda_i) = 0 \ , \tag{32}$$

then the braid group representation can be expanded via the projectors

$$P_i = \prod_{j \neq i} \frac{(B - \lambda_j)}{(\lambda_i - \lambda_j)} \ , \tag{33}$$

i.e.,

$$B = \sum_{i=1}^{m} \lambda_i P_i \ . \tag{34}$$

The trigonometric baxterization gives the solution of Yang-Baxter equation as follows

$$\hat{R}(x) = \sum_{i=1}^{m} \Lambda_i(x) P_i \ , \tag{35}$$

where $\Lambda_i(x)$ are m functions to be determined, upon the requirement called *standard initial* condition

$$\hat{R}(1) \propto 1 \ . \tag{36}$$

Explicitly

$$\Lambda_i(x) = \prod_{j=1}^{i-1} \left(1 + x \frac{\bar{\lambda}_j}{\bar{\lambda}_{j+1}}\right) \prod_{j=i}^{m-1} \left(x + \frac{\bar{\lambda}_j}{\bar{\lambda}_{j+1}}\right) \ , \tag{37}$$

note that $\bar{\lambda}_i$ with $i = 1, 2, \cdots, m$ take distinct values in set $\{\lambda_1, \lambda_1, \cdots, \lambda_m\}$.

It is easy to see that when $x = 1$

$$\Lambda_1(1) = \Lambda_2(1) = \cdots = \Lambda_m(1) = \prod_{j=1}^{m-1} \left(1 + \frac{\bar{\lambda}_j}{\bar{\lambda}_{j+1}}\right) \ , \tag{38}$$

and

$$\hat{R}(1) = \Lambda_1(1) \cdot 1 \ . \tag{39}$$

Therefore when there is a pair of eigenvalues in the set $\{\lambda_1, \lambda_1, \cdots, \lambda_m\}$, say, λ_ℓ and λ_k that have opposite values (i.e., $\lambda_\ell = -\lambda_k$), then it is possible that (35) and (37) gives a solution of Yang-Baxter equation (31) but $\hat{R}(1) = 0$, if we happen to have $\bar{\lambda}_h = \lambda_\ell$ and $\bar{\lambda}_{h+1} = \lambda_k$ and so $\bar{\lambda}_h + \bar{\lambda}_{h+1} = 0$. We call such results of baxterization *reduced*. Of course we can rescale the $\Lambda_i(x)$ function (by a factor of $(x-1)^{-1}$) to prevent the zero result of $\hat{R}(1)$, but doing so does not necessarily recover the initial condition. See the specific examples of three eigenvalues, given in next section in which we give in detail the process of baxterization of all the above braid group solutions.

3.1 The Case with Two Eigenvalues

We treat the permutator solution only, in that other two solutions will be special cases of the solution with three eigenvalues. The baxterization to the permutator solution is rational, and the result reads

$$\hat{R}_{ij}(x) = x + P_{ij} \ . \tag{40}$$

3.2 The Case with Three Eigenvalues

The undetermined functions are

$$\begin{aligned} \Lambda_1(x) &= \left(x + \tfrac{\lambda_1}{\lambda_2}\right)\left(x + \tfrac{\lambda_2}{\lambda_3}\right) \\ \Lambda_2(x) &= \left(1 + x\tfrac{\lambda_1}{\lambda_2}\right)\left(x + \tfrac{\lambda_2}{\lambda_3}\right) \\ \Lambda_3(x) &= \left(1 + x\tfrac{\lambda_1}{\lambda_2}\right)\left(1 + x\tfrac{\lambda_2}{\lambda_3}\right) \ , \end{aligned} \tag{41}$$

where λ_i with $(i = 1, 2, 3)$ take distinct values in the set $\{-1, x_0, -x_0\}$.

3.2.1 $x_1 = -x_0$, $x_2 = -2$

Solutions are found when $\lambda_1 = -1$, $\lambda_2 = x_0$ and $\lambda_3 = -x_0$, or $\lambda_1 = -1$, $\lambda_2 = -x_0$ and $\lambda_3 = x_0$, respectively

$$\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right)(x - x_0) + (-2N_iN_j + N_i + N_j - 1)(xx_0 - 1) \ , \tag{42}$$

and

$$\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right)(x + x_0) + (-2N_iN_j + N_i + N_j)(xx_0 + 1) - (xx_0 + 1) \ . \tag{43}$$

It should be noted that the baxterizations of these two solutions are *reduced*, in that they do not satisfy the initial condition (we have rescaled the solution to prevent zero result). This situation

often happens in trigonometric baxterization, when there are a pair of eigenvalues of braid group solution with opposite values.

Another solution which satisfies initial condition is found when $\lambda_1 = x_0$, $\lambda_2 = -1$ and $\lambda_3 = -x_0$, namely

$$\underline{\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right) x_0 \left(1 - x^2\right) + \left(-2N_iN_j + N_i + N_j\right)(1-x)(xx_0^2+1) + \left(x^2x_0^2 - 1\right)} \ , \quad (44)$$

and in the following, we will refer it the XXZ-solution of braid group. This solution, as can be seen in the following, gives rise to an integrable fermion model which is turned into the spin-$\frac{1}{2}$ XXZ model, under Jordan-Wigner transformation.

3.2.2 $x_1 = -x_0$, $x_2 = -1 + x_0$

We have now two *reduced* results

- When $\lambda_1 = -1$, $\lambda_2 = x_0$ and $\lambda_3 = -x_0$, we have solution

$$\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right)(x - x_0) + (N_i + N_j - 1)(xx_0 - 1) + N_iN_j(1 - x_0)(x+1) \ . \quad (45)$$

- When $\lambda_1 = -1$, $\lambda_2 = -x_0$ and $\lambda_3 = x_0$, we have

$$\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right)(x + x_0) + (N_i + N_j - 1)(xx_0 + 1) + N_iN_j(1 - x_0)(x-1) \ . \quad (46)$$

3.2.3 $x_1 = -x_0$, $x_2 = -1 - x_0$

We have now two *reduced* results

- When $\lambda_1 = -1$, $\lambda_2 = x_0$ and $\lambda_3 = -x_0$, we have solution

$$\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right)(x - x_0) + (N_i + N_j - 1)(xx_0 - 1) + N_iN_j(1 + x_0)(1 - x) \ , \quad (47)$$

It is worth noticing that this solution has another simple *initial* value, namely

$$\hat{R}_{ij} = -P_{ij} \ , \quad (48)$$

and so

$$R_{ij} \propto 1 \ . \quad (49)$$

Actually

$$R_{ij}(x) = (x_0 - x) + (N_i + N_j - N_iN_j - 1)(x-1)(x_0+1) \ . \quad (50)$$

- When $\lambda_1 = -1$, $\lambda_2 = -x_0$ and $\lambda_3 = x_0$, we have

$$\hat{R}_{ij}(x) = \left(c_i^+ c_j^- - c_i^- c_j^+\right)(x + x_0) + (N_i + N_j - 1)(xx_0 + 1) - N_iN_j(1 + x_0)(x+1) \ . \quad (51)$$

The solution with four eigenvalues resists the trigonometric baxterization. We get only some *reduced* results, and we will give their asymmetric counterparts in the next section. The authors believe the parameterized form of this solution with standard initial condition should involve elliptic function.

4 Hopping-Asymmetric *Normal* Solutions and Baxterization

In this section, we will extend the above obtained solutions by eliminating the artificial constraint of hopping-symmetry (17). When $x_0 = -x_1$ the results in the following will reproduce the results in the previous sections. We separate the discussions into two subsections regarding to the numbers of eigenvalues.

4.1 The Case of Three Eigenvalues

The set of eigenvalues is $\{-1, \sqrt{-x_0x_1}, -\sqrt{-x_1x_0}\}$, and there are two free parameters in the solution of braid group, namely

$$x_2 = -2 \ \text{ or } \ x_2 = -1 \pm \sqrt{-x_1x_0} \ . \quad (52)$$

- $x_2 = -2$ and $\lambda_1 = \sqrt{-x_1x_0}$, $\lambda_2 = -1$ and $\lambda_3 = -\sqrt{-x_1x_0}$ then the result of baxterization reads

$$\begin{aligned} \underline{\hat{R}_{ij}(x)} \ = \ & \underline{\left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left(1 - x^2\right)} \\ & \underline{+ (N_i + N_j - 2N_iN_j)(x-1)(x_0x_1x - 1) - (1 + x^2x_1x_0)} \ , \end{aligned} \quad (53)$$

which satisfies the initial condition (49). When $x_1 = -x_0$ we arrive at the solution in (44). And we will refer it as the extended XXZ-solution of braid group. This solution is important in this work, because we will analyse in detail its corresponding integrable model by the method of quantum inverse scattering method in a later section. Above baxterization allow a *gauge* transformation, e.g., under which we may arrive at

$$\begin{aligned} \hat{R}_{ij}(x) \ = \ & \left(x^kx_0c_i^+c_j^- + x^{-k}x_1c_i^-c_j^+\right)\left(1 - x^2\right) \\ & + (N_i + N_j - 2N_iN_j)(x-1)(x_0x_1x - 1) - (1 + x^2x_1x_0) \ , \end{aligned} \quad (54)$$

where $k \in \mathbb{C}$ is an arbitrary complex number, or in another form,

$$\begin{aligned} \underline{R_{ij}(x)} \ = \ & \underline{(-x^kc_i^+c_j^- + x^{-k}c_i^-c_j^+)(x_0x_1 + 1)} \\ & \underline{+N_iN_j(x - x^{-1})(-x_1 + x_0)} \\ & \underline{+N_i\left[xx_0(x_1 - 1) + x^{-1}(1 + x_0)\right]} \\ & \underline{+N_j\left[xx_1(x_0 + 1) + x^{-1}(1 - x_1)\right] - (x_0x_1x + x^{-1})} \ . \end{aligned} \quad (55)$$

- $x_2 = -2$ and $\lambda_1 = -1$, $\lambda_2 = \sqrt{-x_1x_0}$ and $\lambda_3 = -\sqrt{-x_1x_0}$ then the result of baxterization reads

$$\begin{aligned} \hat{R}_{ij}(x) \ = \ & \left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left(\sqrt{-x_0x_1} - x\right) \\ & + (N_i + N_j - 2N_iN_j - 1)\left(\sqrt{-x_0x_1} + xx_0x_1\right) \ , \end{aligned} \quad (56)$$

which is *reduced*. Another solution with $\lambda_1 = -1$, $\lambda_2 = -\sqrt{-x_1x_0}$ and $\lambda_3 = \sqrt{-x_1x_0}$ identifies with (56) under the transformation $x \to -x - 1$.

- $x_2 = -1 \pm \sqrt{-x_1x_0}$ and $\lambda_1 = -1$, $\lambda_2 = \sqrt{-x_1x_0}$ and $\lambda_3 = -\sqrt{-x_1x_0}$ we get solutions

$$\begin{aligned} \hat{R}_{ij}(x) \ = \ & \left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left(-x + \sqrt{-x_0x_1}\right) + (N_i + N_j - 1)\left(\sqrt{-x_0x_1} + xx_0x_1\right) \\ & -N_iN_j\left(\sqrt{-x_1x_0} \pm x_1x_0\right)(1 \pm x) \ . \end{aligned} \quad (57)$$

Another solution with $\lambda_1 = -1$, $\lambda_2 = -\sqrt{-x_1x_0}$ and $\lambda_3 = \sqrt{-x_1x_0}$ identifies with (57) under the transformation $x \to -x$. Both are *reduced*.

As $\hat{R}_{ij} \neq \hat{R}_{ji}$ we have to specify that the solution $R_{ij}(x) = \hat{R}_{ij} P_{ij}$ and $\tilde{R}_{ij}(x) = P_{ij}\hat{R}_{ij}$ that satisfy the following equations

$$R_{ij}(x)R_{ik}(xy)R_{jk}(y) = R_{jk}(y)R_{ik}(xy)R_{ij}(x)\,,\ \tilde{R}_{ji}(x)\tilde{R}_{ki}(xy)\tilde{R}_{kj}(y) = \tilde{R}_{kj}(y)\tilde{R}_{ki}(xy)\tilde{R}_{ji}(x)\,, \quad (58)$$

provided $i < j$ when B_{ij} satisfies (15).

4.2 The Case of Four Eigenvalues

The set of eigenvalues is $\{\lambda_{1,2,3,4}\}$ and

$$\lambda_1 = \sqrt{-x_1x_0}\,,\quad \lambda_2 = -1\,,\quad \lambda_3 = (1+x_2)\,,\quad \lambda_4 = -\sqrt{-x_1x_0}\,. \quad (59)$$

We obtain four independent *reduced* solutions.

1. When $\tilde{\lambda}_1 = \lambda_1$, $\tilde{\lambda}_2 = \lambda_4$, $\tilde{\lambda}_3 = \lambda_3$ and $\tilde{\lambda}_4 = \lambda_2$, we get

$$\begin{aligned}\hat{R}_{ij}(x) \;=\;& \left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left[\sqrt{-x_1x_0} - x(x_2+1)\right](x-x_2-1)\\ &+(N_i+N_j-1)\left[\sqrt{-x_1x_0}(x_2+1)+x_1x_0x\right]\left[x(x_2+1)-1\right]\\ &-x_2N_iN_j\left[\sqrt{-x_1x_0}(x_2+1)+x_1x_0x\right](x+1)\,.\end{aligned} \quad (60)$$

2. $\tilde{\lambda}_1 = \lambda_2$, $\tilde{\lambda}_2 = \lambda_4$, $\tilde{\lambda}_3 = \lambda_1$ and $\tilde{\lambda}_4 = \lambda_3$, we get

$$\begin{aligned}\hat{R}_{ij}(x) \;=\;& -\left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left[\sqrt{-x_1x_0}(x_2+2)+(x_2+1)x^2 - x_1x_0\right]\\ &+(N_i+N_j-1)\left[\sqrt{-x_1x_0}(x_1x_0-x_2-1)+x^2(x_2+1)x_1x_0+x_1x_0\right]\\ &-x_2x_1x_0N_iN_j\left(x^2-1\right)\,.\end{aligned} \quad (61)$$

3. When $\tilde{\lambda}_1 = \lambda_2$, $\tilde{\lambda}_2 = \lambda_4$, $\tilde{\lambda}_3 = \lambda_3$ and $\tilde{\lambda}_4 = \lambda_1$, we get

$$\begin{aligned}\hat{R}_{ij}(x) \;=\;& \left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left[\sqrt{-x_1x_0} + x(x_2+1)\right](x-x_2-1)\\ &+(N_i+N_j-1)\left[\sqrt{-x_1x_0}(x_2+1)-x_1x_0x\right]\left[x(x_2+1)-1\right]\\ &-x_2N_iN_j\left[\sqrt{-x_1x_0}(x_2+1)-x_1x_0x\right](x+1)\,.\end{aligned} \quad (62)$$

4. When $\tilde{\lambda}_1 = \lambda_3$, $\tilde{\lambda}_2 = \lambda_2$, $\tilde{\lambda}_3 = \lambda_1$ and $\tilde{\lambda}_4 = \lambda_4$, we get

$$\begin{aligned}\hat{R}_{ij}(x) \;=\;& \left(x_0c_i^+c_j^- + x_1c_i^-c_j^+\right)\left(\sqrt{-x_1x_0} - x\right)\left[x(x_2+1)-1\right]\\ &+(N_i+N_j-1)\left(\sqrt{-x_1x_0}+x_1x_0x\right)(x+1)\\ &-x_2N_iN_j\left(\sqrt{-x_1x_0}+x_1x_0x\right)(x_2x+x-1)\,.\end{aligned} \quad (63)$$

5. When $\tilde{\lambda}_1 = \lambda_3$, $\tilde{\lambda}_2 = \lambda_4$, $\tilde{\lambda}_3 = \lambda_1$ and $\tilde{\lambda}_4 = \lambda_2$, we get a solution identified with solution (61) under redefinition of spectral parameter $x \to -x$. When $\tilde{\lambda}_1 = \lambda_3$, $\tilde{\lambda}_2 = \lambda_1$, $\tilde{\lambda}_3 = \lambda_2$ and $\tilde{\lambda}_4 = \lambda_4$, we get identical solution with (62).

5 *Anomalous* Solutions and Baxterization

We assume the solution is of the following general form

$$B_{ij} = 1 + x_0\left(c_i^+c_j^- - c_i^-c_j^+\right) + x_2N_iN_j + x_3(N_i+N_j) + x_4(c_i^+c_j^+ - c_i^-c_j^-)\,, \quad (64)$$

which involves upto two-body interactions of fermions. When x_i take the values in the following table, B_{ij} are braid group representations.

# of Sol.	x_0	x_2	x_3	x_4
(i)	1	x_2	$-x_2/2$	$(x_2-2)/2$
(ii)	1	x_2	$-x_2/2$	$-(x_2-2)/2$
(iii)	-1	x_2	$-x_2/2$	$(x_2-2)/2$
(iv)	-1	x_2	$-x_2/2$	$-(x_2-2)/2$
(v)	x_0	0	$(\sqrt{2x_0^2-1}-1)/2$	$(\sqrt{2x_0^2-1}+1)/2$

Solutions (i)-(iv) have the same set of eigenvalues, which is $\{-(x_4-4)/2,\ \pm x_2/2\}$.

1. We take $\lambda_1 = x_2/2$, $\lambda_2 = -(x_4-4)/2$ and $\lambda_3 = -x_2/2$, then according to (41), we get the results of baxterization given explicitly in the following,

$$\begin{aligned}\hat{R}_{ij}(x) \;=\;& a(x)\left(N_i+N_j-2N_iN_j\right)\\ &+b(x)\left(\mathrm{sign}_1c_i^+c_j^- + \mathrm{sign}_2c_i^-c_j^+\right)\\ &+c(x)\left(\mathrm{sign}_3c_i^+c_j^+ + \mathrm{sign}_4c_i^-c_j^-\right)\\ &+d(x)\,,\end{aligned} \quad (65)$$

where functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ read

$$\begin{aligned}a(x) &= (x^2-1)(x_2-4)x_2\,,\\ b(x) &= 2(x_2x-x_2+4)(x-1)\,,\\ c(x) &= (x_2-2)(x-1)(x_2x+x_2-4)\,,\\ d(x) &= 2(x+1)(xx_2+x_2-4)\,,\end{aligned} \quad (66)$$

and the sign_i with $i = 1,2,3,4$ are given in the following table.

# of Sol.	sign_1	sign_2	sign_3	sign_4
(i)	+	−	−	+
(ii)	+	−	+	−
(iii)	−	+	−	+
(iv)	−	+	+	−

2. We take $\lambda_1 = -(x_4-4)/2$, $\lambda_2 = x_2/2$ and $\lambda_3 = -x_2/2$, then we have *reduced* solutions

$$\begin{aligned}\hat{R}_{ij}(x) \;=\;& +2\mathrm{sign}_3\left(c_i^+c_j^- - c_i^-c_j^+ + 1\right)(x+\mathrm{sign}_4)\\ &-\mathrm{sign}_2\left(c_i^+c_j^+ - c_i^-c_j^-\right)(x-\mathrm{sign}_4)(x_2-2)\\ &+(N_i+N_j-2N_iN_j)\left[x(x_2-4)+\mathrm{sign}_1x_2\right]\,,\end{aligned} \quad (67)$$

where sign_i with $i = 1,2,3,4$ are given in the following table.

# of Sol.	sign_1	sign_2	sign_3	sign_4
(i)	−	+	+	+
(ii)	−	−	+	+
(iii)	−	+	−	+
(iv)	−	−	−	+

3. We take $\lambda_1 = -(x_4-4)/2$, $\lambda_2 = -x_2/2$ and $\lambda_3 = x_2/2$, then we have *reduced* solutions in the same form as in (67) but with sign_i with $i = 1,2,3,4$ redefined in the following table.

# of Sol.	sign_1	sign_2	sign_3	sign_4
(i)	+	+	+	−
(ii)	+	−	+	−
(iii)	+	+	−	−
(iv)	+	−	−	−

Solution (v) has two eigenvalues which are

$$\frac{1}{2}\left(\sqrt{2x_0^2-1}\pm 2x_0+1\right) , \tag{68}$$

and the result of baxterization is

$$\begin{aligned}\hat{R}_{ij}(x) \;=\; & \Big[2x_0\left(c_i^+c_j^- - c_i^-c_j^+\right)+\left(c_i^+c_j^+ - c_i^-c_j^-\right)\left(\sqrt{2x_0^2-1}+1\right)\\ & +(N_i+N_j-1)\left(\sqrt{2x_0^2-1}-1\right)\Big](x-1)-2\left(\sqrt{2x_0^2-1}x+1\right) ,\end{aligned} \tag{69}$$

And it is worth-noticing that this solution is linear.

6 Quantum Algebras

6.1 Yang-Baxter Algebra

A Yang-Baxter algebra $\mathcal{T}(R)$ is an associative algebra generated by

$$\left\{1, t_j^i(\lambda)\ ,\ \ i,j=1,2,\cdots,n\right\}$$

that satisfy the following algebraic relation

$$\hat{R}(\lambda\mu^{-1})\,\hat{T}(\lambda)\otimes\hat{T}(\mu)=\hat{T}(\mu)\otimes\hat{T}(\lambda)\hat{R}(\lambda\mu^{-1})\ , \tag{70}$$

where $\hat{T}=\left(t_j^i(\lambda)\right)_{n\times n}$. Following the convention of quantum inverse scattering method, we call the space $V\otimes V$ the auxiliary space and the representation space of $\mathcal{T}(R)$ the quantum space.

The representation of Yang-Baxter algebra $\mathcal{T}(\mathcal{R})$ in linear space W is a linear map $\rho\ :\ \mathcal{T}\to \text{End}(W)$ in W, where ρ is algebra homomorphism and satisfies

$$\hat{R}\left(\lambda\mu^{-1}\right)\rho\mathcal{T}(\lambda)\otimes\rho\mathcal{T}(\mu)=\rho\mathcal{T}(\mu)\otimes\rho\mathcal{T}(\lambda)\hat{R}\left(\lambda\mu^{-1}\right)\ , \tag{71}$$

where $\rho T(\lambda)=\left(\rho t_j^i(\lambda)\right)$.

A solution of Yang-Baxter equation (10) naturally gives a representation of Yang-Baxter algebra $\mathcal{T}(\mathcal{R})$: Suppose that $\{e_i\}_{i=1}^n$ is a set of basis of space V, the action of $R(\lambda)$ on space $V\otimes V$ is

$$R(\lambda)\,e_i\otimes e_j=R_{ij}^{kl}(\lambda)\,e_k\otimes e_l\ , \tag{72}$$

and the action of $t_i^k(\lambda)\ :\ V\to V$ reads

$$t_i^k(\lambda)e_j=R_{ij}^{kl}(\lambda)\,e_l\ . \tag{73}$$

Then it is obvious from equation (11) that $R(\lambda)$ itself is a representation of Yang-Baxter algebra. As (11) can be rewritten into

$$\hat{R}\left(\lambda\mu^{-1}\right)F_R(\lambda)\otimes F_R(\mu)=F_R(\mu)\otimes F_R(\lambda)\hat{R}\left(\lambda\mu^{-1}\right)\ , \tag{74}$$

where the index 3 of the original R-operator belongs to the quantum space, and $F_R(\lambda)=R(\lambda)$ will be called the fundamental representation of $\mathcal{T}(\mathcal{R})$.

6.2 Algebras $\mathcal{F}_q$ and F_q from Extended XXZ-Solution of Braid Group

Consider the solution in (55) with $k=1$ in limits of $x\to\infty$ and $x\to 0$, respectively

$$\begin{aligned}R_{ij}^+ &= -c_i^+c_j^-\left(x_0+\tfrac{1}{x_1}\right)+N_i\left(x_0-\tfrac{x_0}{x_1}\right)+N_j(1+x_0)+N_iN_j\left(\tfrac{x_0}{x_1}-1\right)-x_0\ ,\\ R_{ij}^- &= c_i^-c_j^+\left(x_1+\tfrac{1}{x_0}\right)+N_i\left(1+\tfrac{1}{x_0}\right)+N_j\left(\tfrac{1}{x_0}-\tfrac{x_1}{x_0}\right)+N_iN_j\left(\tfrac{x_0}{x_1}-1\right)-\tfrac{1}{x_0}\ ,\end{aligned} \tag{75}$$

which we call the *upper* and *lower triangular* R-operators, respectively. It is easy to see

$$R_{ij}^-(x)=\left(R_{ji}^+\right)^{-1}\ , \tag{76}$$

and the following decomposition applies

$$R_{ij}(x)=\frac{1}{x_0x}R_{ij}^++x_0xR_{ij}^-=\frac{1}{x_0x}R_{ij}^++x_0x\left(R_{ji}^+\right)^{-1}\ . \tag{77}$$

The corresponding Yang-Baxter algebra has an apparent fundamental representation, namely, $F_R(\lambda)=R(\lambda)$. Taking $x,y\to 0,\infty$ in(74), we have

$$\begin{aligned}R_{12}^+F_1^+F_2^\pm &= F_2^\pm F_1^+R_{12}^+\ ,\\ R_{12}^-F_1^-F_2^\pm &= F_2^\pm F_1^-R_{12}^-\ ,\\ R_{12}^+F_1^\pm F_2^- &= F_2^-F_1^\pm R_{12}^+\ ,\\ R_{12}^-F_1^\pm F_2^+ &= F_2^+F_1^\pm R_{12}^-\ .\end{aligned} \tag{78}$$

We assume that the Lax-operator (in *single particle picture*) has the same decomposition relation,

$$L(x)=\frac{1}{x_0x}L^++x_0xL^-\ , \tag{79}$$

and

$$L^+=c^+\ell^-+N\alpha+(1-N)\beta\ ,\quad L^-=c^-\ell^++N\gamma+(1-N)\delta\ , \tag{80}$$

where $L^\pm$ are well-defined (this decomposition is similar to the *contraction* operation in the theory of the classical Lie algebra). Then we should have

$$\begin{aligned}&R_{12}^+L_1^+L_2^+=L_2^+L_1^+R_{12}^+\ ,\quad R_{12}^-L_1^-L_2^-=L_2^-L_1^-R_{12}^-\ ,\\ &R_{12}^+L_1^+L_2^-=L_2^-L_1^+R_{12}^+\ ,\quad R_{12}^-L_1^-L_2^+=L_2^+L_1^-R_{12}^-\ ,\\ &R_{12}^+L_1^-L_2^-=L_2^-L_1^-R_{12}^+\ ,\quad R_{12}^-L_1^+L_2^+=L_2^+L_1^+R_{12}^-\ ,\end{aligned} \tag{81}$$

which reads equivalently

$$\begin{aligned}&x_1\ell^+\ell^- - x_0\ell^-\ell^+=(1+x_0x_1)\left[\delta\alpha-\gamma\beta\right]\ ,\\ &\ell^\pm\alpha=-x_1^\mp\alpha\ell^\pm\ ,\qquad \ell^\pm\beta=-x_0^\pm\beta\ell^\pm\ ,\\ &\ell^\pm\gamma=x_0^\pm\gamma\ell^\pm\ ,\qquad \ell^\pm\delta=x_1^\mp\delta\ell^\pm\ ,\\ &\alpha\beta=\beta\alpha\ ,\qquad \alpha\gamma=\gamma\alpha\ ,\\ &\alpha\delta=\delta\alpha\ ,\qquad \beta\gamma=\gamma\beta\ ,\\ &\beta\delta=\delta\beta\ ,\qquad \gamma\delta=\delta\gamma\ .\end{aligned} \tag{82}$$

This is a Yang-Baxter algebra and will be denoted $\mathcal{F}_q$, as it is obtained from fermi operator valued Yang-Baxter solutions.

The fundamental representation gives a representation for the above algebra, as we pointed out, which reads explicitly

$$\begin{aligned}&\ell^-=-c^-\left(x_0+\frac{1}{x_1}\right)\ ,\qquad \ell^+=c^+\left(\frac{1}{x_0}+x_1\right)\ ,\\ &\alpha=\left(x_0+\frac{x_0}{x_1}\right)N-\frac{x_0}{x_1}\ ,\qquad \gamma=\left(\frac{1}{x_0}-1\right)N+1\ ,\\ &\beta=(1+x_0)\,N-x_0\ ,\qquad \delta=\left(\frac{1}{x_0}-\frac{x_1}{x_0}\right)N-\frac{1}{x_0}\ .\end{aligned} \tag{83}$$

Another realization of the above algebra is

$$\begin{aligned} \tilde{c}^- &= -\left(x_0 + \frac{1}{x_1}\right)^{-1} \ell^- , & \tilde{c}^+ &= \left(\frac{1}{x_0} + x_1\right)^{-1} \ell^+ , \\ \alpha &= x_0 (-x_1)^{\tilde{N}-1} , & \gamma &= (x_0)^{-\tilde{N}} , \\ \beta &= (-x_0)^{\tilde{M}} , & \delta &= -\frac{1}{x_0} (x_1)^{1-\tilde{M}} , \end{aligned} \tag{84}$$

and

$$\tilde{c}^+ \tilde{c}^- - \frac{x_0}{x_1} \tilde{c}^- \tilde{c}^+ = (-)^{\tilde{N}} \frac{x_1^{\tilde{N}-\tilde{M}} - (-x_0)^{\tilde{M}-\tilde{N}}}{x_1 - (-x_0)^{-1}} , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm , \quad \left[\tilde{M}, \tilde{c}^\pm\right] = \mp \tilde{c}^\pm . \tag{85}$$

The fermi operator algebra is recovered if we identify that $\tilde{N} = N$ and $\tilde{M} = 1 - N$ and $N^2 = N$ when the realization in equations (83) is recovered. This is a generalization of the fermi operator algebra, and therefore we denote it F_q algebra, spanned by the generators $\tilde{c}^\pm$, $\tilde{N}$ and $\tilde{M}$.

6.3 $\mathcal{F}_q$ and F_q as Quantum Super Bialgebras

The algebra homomorphism (coproduct) that gives a consistent map from *single particle picture* to *multiparticle picture*, or

$$\begin{aligned} \Delta_{ij} :&\quad A \longrightarrow \Delta_{ij}(A) = A_i A_j , \\ \Delta_{ij} :&\quad L^\pm \longrightarrow \Delta_{ij}(L^\pm) \propto L_i^\pm L_j^\pm , \end{aligned} \tag{86}$$

explicitly

$$\begin{aligned} \Delta_{ij} :&\quad \ell^+ \longrightarrow \Delta_{ij}(\ell^+) = \ell_i^+ \delta_j + \gamma_i \ell_j^+ , \\ \Delta_{ij} :&\quad \ell^- \longrightarrow \Delta_{ij}(\ell^-) = \ell_i^- \beta_j + \alpha_i \ell_j^- , \\ \Delta_{ij} :&\quad \alpha \longrightarrow \Delta_{ij}(\alpha) = \alpha_i \alpha_j , \\ \Delta_{ij} :&\quad \beta \longrightarrow \Delta_{ij}(\beta) = \beta_i \beta_j , \\ \Delta_{ij} :&\quad \gamma \longrightarrow \Delta_{ij}(\gamma) = \gamma_i \gamma_j , \\ \Delta_{ij} :&\quad \delta \longrightarrow \Delta_{ij}(\delta) = \delta_i \delta_j . \end{aligned} \tag{87}$$

The coproduct operation should satisfy $\Delta(1) = 1 \otimes 1$.

This algebra bears a grading property. If we give each of the generators a grading factor, namely

$$\deg(\ell^\pm) = 1 , \quad \deg(\alpha) = \deg(\beta) = \deg(\gamma) = \deg(\delta) = 0 , \tag{88}$$

and

$$(A \otimes B)(C \otimes D) = (-)^{\deg(B)\deg(C)} AC \otimes BD , \tag{89}$$

then the algebra homomorphism gives the tensor representation of $\mathcal{F}_q$. Due to this grading property, this is a super bialgebra.

We write explicitly the coproduct of $\mathcal{F}_q$ in realization (84), i.e., F_q algebra. It should be noted that this representation gives a consistent quantum bi-algebra structure only when $x_1 = -x_0$. The relations read

$$\{\tilde{c}^+, \tilde{c}^-\} = (-)^{\tilde{N}} \left[\tilde{N} - \tilde{M}\right]_{x_1} , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm , \quad \left[\tilde{M}, \tilde{c}^\pm\right] = \mp \tilde{c}^\pm , \tag{90}$$

and

$$\begin{aligned} \Delta(\tilde{N}) &= \tilde{N} \otimes 1 + 1 \otimes \tilde{N} , \\ \Delta(\tilde{M}) &= \tilde{M} \otimes 1 + 1 \otimes \tilde{M} , \\ \Delta(\tilde{c}^+) &= \tilde{c}^+ \otimes (-x_0)^{-\tilde{M}} + x_0^{-\tilde{N}} \otimes \tilde{c}^+ , \\ \Delta(\tilde{c}^-) &= \tilde{c}^- \otimes (-x_0)^{\tilde{M}} + x_0^{\tilde{N}} \otimes \tilde{c}^- , \end{aligned} \tag{91}$$

which is a super bialgebra, and the grading property is charaterized by commutation relation (89) with

$$\deg(\tilde{c}^\pm) = 1 , \quad \deg\left(\tilde{N}\right) = \deg\left(\tilde{M}\right) = 0 . \tag{92}$$

Now we take a homomorphism such that $\tilde{M} = 1 - \tilde{N}$, hence the above relations reduce into

$$\{\tilde{c}^+, \tilde{c}^-\} = (-)^{\tilde{N}} \left[2\tilde{N} - 1\right]_{x_1} , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm . \tag{93}$$

When $x_1 \to 1$ the ordinary fermion algebra is recovered (up to a scalar factor)

$$\{\tilde{c}^+, \tilde{c}^-\} = -1 , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm . \tag{94}$$

When $x_1 = x_0$ we have

$$\{\tilde{c}^+, \tilde{c}^-\} = (-)^{\tilde{M}} \frac{x_0^{\tilde{M}-\tilde{N}} - (-)^{2(\tilde{N}-\tilde{M})} x_0^{\tilde{N}-\tilde{M}}}{x_0 - x_0^{-1}} , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm , \quad \left[\tilde{M}, \tilde{c}^\pm\right] = \mp \tilde{c}^\pm . \tag{95}$$

Now we take a homomorphism such that $\tilde{M} = 1 - \tilde{N}$, hence the above relations reduce into

$$\{\tilde{c}^+, \tilde{c}^-\} = (-)^{\tilde{M}} \frac{x_0^{\tilde{M}-\tilde{N}} - (-)^{2(2\tilde{N}-1)} x_0^{2\tilde{N}-1}}{x_0 - x_0^{-1}} , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm , \quad \left[\tilde{M}, \tilde{c}^\pm\right] = \mp \tilde{c}^\pm . \tag{96}$$

and the ordinary fermion algebra is recovered when $x_0 \to 1$:

$$\{\tilde{c}^+, \tilde{c}^-\} = -1 , \quad \left[\tilde{N}, \tilde{c}^\pm\right] = \pm \tilde{c}^\pm . \tag{97}$$

7 XXZ Model and Algebraic Bethe Ansatz

7.1 Monodromy Operator

For L_j, a representation of $\mathcal{T}(R)$ on quantum space V_j, with $j = 1, 2, \cdots, N$, we have

$$\hat{R}(\lambda\mu^{-1}) L_j(\lambda) \otimes L_j(\mu) = L_j(\mu) \otimes L_j(\lambda) \hat{R}(\lambda\mu^{-1}) . \tag{98}$$

Let $\mathcal{H}$ denote $V_1 \otimes V_2 \otimes \cdots \otimes V_N$ and V_0 the auxiliary space, then $L_j(\lambda) \in \mathrm{End}(V_0 \otimes V_j)$ can be embedded into $\mathrm{End}(V_0 \otimes \mathcal{H})$ in a trivial way, and the embedded operator will also be remembered as $L_j(\lambda)$. Let monodromy operator be

$$T(\lambda) = L_1(\lambda) L_2(\lambda) \cdots L_N(\lambda) , \tag{99}$$

we have

$$\hat{R}(\lambda\mu^{-1}) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) \hat{R}(\lambda\mu^{-1}) . \tag{100}$$

Hence $T(\lambda)$ is obviously a representation of $\mathcal{T}(\mathcal{R})$. As $T(\lambda)$ is actually analogous to the *multiparticle representation* of universal enveloping Lie algebras, we will call $T(\lambda)$ a *multiparticle* representation of $\tilde{T}(\lambda)$. This *multiparticle representation* is also called *monodromy* operator, while $L_j(\lambda)$ will be called L- (Lax-) operator, according to the convention in the theory of quantum inverse scattering method.

Let

$$t(\lambda) = \mathrm{Tr}_0 T(\lambda) , \tag{101}$$

where Tr_0 denotes the trace taken in auxiliary space, i.e.,

$$\mathrm{Tr}_0 T(\lambda) = \langle 0_0 | T(\lambda) | 0_0 \rangle + \langle 1_0 | T(\lambda) | 1_0 \rangle . \tag{102}$$

Therefore from equation (100) (we have assumed that $R(\lambda)$ is invertible), we have immediately

$$[t(\lambda) , t(\mu)] = 0 . \tag{103}$$

This relation is very important in proving the integrability in the theory of quantum inverse scattering method.

For solution (55) with $k = 0$, the monodromy operator T is taken as

$$T_i(x) = -c_i^+ B(x) + c_i^- C(x) + N_i A(x) + (1 - N_i) D(x) \ , \tag{104}$$

then from Yang-Baxter relation

$$R_{12}(xy^{-1}) T_1(x) T_2(y) = T_2(y) T_1(x) R_{12}(xy^{-1}) \ , \tag{105}$$

we get algebraic relations for A, B, C and D with spectral parameters x and y. We list all the relations in Appendix and in the following we apply only a few of them.

7.2 Conserved Currents

From (B.9) or (B.11) we have

$$[A(y), D(x)] = [A(x), D(y)] \ , \tag{106}$$

therefore

$$t(x) = \mathrm{Tr}_0 T(x) = A(x) + D(x) \ , \tag{107}$$

with $x \in \mathbb{C}$ constitute a family of commuting quantities, i.e.,

$$[t(x), t(y)] = [A(x), D(y)] - [A(y), D(x)] = 0 \ . \tag{108}$$

7.3 Reference States

There are two candidates of the *reference state*,

$$|0_1, 0_2, \cdots, 0_N\rangle \ , \quad \text{or} \quad |1_1, 1_2, \cdots, 1_N\rangle \ , \tag{109}$$

on which the monodromy operator acts as a *triangular* operator. We will select the second state in (109) as our reference state.

It is important to observe the actions of A, B, C and D on *reference state* and for this end, let us give explicitly the action of $R_{0j}(x)$ (where 0 indicates the auxiliary particle, and i the *observed particle* on the reference state.

$$R_{0i}(x)|1_1, 1_2, \cdots, 1_N\rangle = \alpha_i c_0^+ + \beta_i N_0 + \gamma_i (1 - N_i) \ , \tag{110}$$

where

$$\alpha_i = (1 + x_1 x_0)\,|0_i\rangle \ , \quad \beta_i = -\left(1 + x^2 x_1 x_0\right) \ , \quad \gamma_i = x_1(1 - x^2)|1_i\rangle \ . \tag{111}$$

Therefore

$$R_{0i}(x)|1_i\rangle R_{0j}(x)|1_j\rangle = (\alpha_i \gamma_j + \beta_i \alpha_j)\, c_0^+ + \beta_i \beta_j N_0 + \gamma_i \gamma_j (1 - N_0) \ . \tag{112}$$

Furthermore

$$T(x)|\mathrm{ref}\rangle = \prod_{i=1}^{N} [R_{0i}(x)|1_i\rangle] = (*) c_0^+ + \prod_{k=1}^{N} \beta_k N_0 + \prod_{k=1}^{N} \gamma_k (1 - N_0) \ , \tag{113}$$

where $*$ is some nonzero operator-valued coefficient.

Compare (113) and (104) we see that

$$\begin{aligned} A(x)|\mathrm{ref}\rangle &= \Delta_+(x)|\mathrm{ref}\rangle \ , \\ D(x)|\mathrm{ref}\rangle &= \Delta_-(x)|\mathrm{ref}\rangle \ , \\ C(x)|\mathrm{ref}\rangle &= 0 \ , \end{aligned} \tag{114}$$

and

$$\Delta_+(x) = \left[-\left(1 + x^2 x_1 x_0\right)\right]^N \ , \quad \Delta_-(x) = \left[x_1\left(1 - x^2\right)\right]^N \ . \tag{115}$$

Hence $C(x)$ acts on the reference state as an annihilator, but $B(x)$ and $C(x)$ as eigen-operator. As we will show in the following, $B(x)$ acts in the role of a creation operator.

7.4 Bethe Ansatz State

The Bethe ansatz state is defined as

$$|\lambda_1 \lambda_2, \lambda_3, \cdots, \lambda_n\rangle = B(\lambda_1)\, B(\lambda_2)\, B(\lambda_3) \cdots B(\lambda_n)\, |\mathrm{ref}\rangle \ . \tag{116}$$

Noticing that $B(\lambda_1)$ and $B(\lambda_2)$ are commuting, therefore it can be seen that the spectral parameters in the Bethe ansatz states are in equal positions, i.e., the states are independent of the order of the spectral parameters. This observation is very important. In the standard way (making use of equations (B2), (B6) and (B14)), we obtain the action of $t(\lambda)$ on the Bethe ansatz states,

$$\begin{aligned} A(\lambda)|\lambda_1, \cdots, \lambda_n\rangle &= \epsilon(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n)\, |\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots, \lambda_n\rangle \\ &\quad + \sum_{j=1}^{n} M_j(\lambda; \lambda_1, \lambda_2, \cdots, \lambda_n)\, |\lambda; \lambda_1, \cdots, \hat{\lambda}_j, \cdots, \cdots \lambda_n\rangle \ , \end{aligned} \tag{117}$$

and

$$\begin{aligned} D(\lambda)|\lambda_1, \cdots, \lambda_n\rangle &= \tilde{\epsilon}(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n)\, |\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots, \lambda_n\rangle \\ &\quad + \sum_{j=1}^{n} \tilde{M}_j(\lambda; \lambda_1, \lambda_2, \cdots, \lambda_n)\, |\lambda; \lambda_1, \cdots, \hat{\lambda}_j, \cdots, \cdots \lambda_n\rangle \ , \end{aligned} \tag{118}$$

where

$$\begin{aligned} \epsilon(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n) &= \Delta_+(\lambda) \left(\prod_{k=1}^{n} \frac{a(\lambda^{-1}\lambda_k)}{d(\lambda^{-1}\lambda_k)} \right) \ , \\ \tilde{\epsilon}(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n) &= (-)^n \Delta_-(\lambda) \left(\prod_{k=1}^{n} \frac{a(\lambda\lambda_k^{-1})}{d(\lambda\lambda_k^{-1})} \right) \ , \\ M_j(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n) &= -\Delta_+(\lambda_j) \frac{c(\lambda^{-1}\lambda_j)}{d(\lambda^{-1}\lambda_j)} \left(\prod_{k=1, k\neq j}^{n} \frac{a(\lambda^{-1}\lambda_k)}{d(\lambda^{-1}\lambda_k)} \right) \ , \\ \tilde{M}_j(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n) &= (-)^{n-1} \Delta_-(\lambda_j) \frac{c(\lambda\lambda_j^{-1})}{d(\lambda\lambda_j^{-1})} \left(\prod_{k=1, k\neq j}^{n} \frac{a(\lambda\lambda_k^{-1})}{d(\lambda\lambda_k^{-1})} \right) \ , \end{aligned} \tag{119}$$

we have

$$\begin{aligned} t(\lambda)|\lambda_1, \lambda_2, \cdots, \lambda_n\rangle &= e(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n)\, |\lambda_1, \lambda_2, \cdots, \lambda_n\rangle + \\ &\quad \sum_{j=1}^{n} \left(M_j(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n) + \tilde{M}_j(\lambda; \lambda_1 \lambda_2, \cdots, \lambda_n) \right) |\lambda; \lambda_1, \cdots, \hat{\lambda}_j, \cdots, \cdots \lambda_n\rangle \ , \end{aligned} \tag{120}$$

with $e(*) = \epsilon(*) + \tilde{\epsilon}(*)$.

The algebraic Bethe ansatz requires that the Bethe ansatz states serve as the eigenstates of $t(\lambda)$, i.e., we must have (for each $j = 1, 2, \cdots, n$)

$$M_j(*) + \tilde{M}_j(*) = 0 \ , \tag{121}$$

or explicitly,

$$\frac{\Delta_+(\lambda_j)}{\Delta_-(\lambda_j)} = (-)^n \frac{c(\lambda\lambda_j^{-1})}{c(\lambda^{-1}\lambda_j)} \prod_{k=1, k\neq j}^{n} \frac{a(\lambda\lambda_k^{-1})}{a(\lambda^{-1}\lambda_k)} \prod_{k=1}^{n} \frac{d(\lambda\lambda_k^{-1})}{d(\lambda^{-1}\lambda_k)} \ , \tag{122}$$

equivalently

$$\left[e^{\eta_0-\eta_1}\frac{\cosh(\lambda_j+\eta_0+\eta_1)}{\sinh(\lambda_j)}\right]^N = \prod_{k=1,k\neq j}\frac{\cosh(\eta_0+\eta_1+\lambda-\lambda_j)}{\cosh(\eta_0+\eta_1-\lambda+\lambda_j)}\,, \tag{123}$$

by the reparameterization

$$x_0 = e^{2\eta_0}\,,\quad x_1 = e^{2\eta_1}\,,\quad \lambda\to e^{\lambda}\,,\quad \lambda_k\to e^{\lambda_k}\,. \tag{124}$$

7.5 Expression of Hamiltonian

Consider the following monodromy operator

$$T_0(\lambda) = L_1(\lambda)L_2(\lambda)\cdots L_N(\lambda)\,, \tag{125}$$

where $L_i(\lambda) = R_{0i}(\lambda)$ and the transfer operator $t(\lambda)$ reads

$$t(\lambda) = \mathrm{Tr}_0 T(\lambda) = \mathrm{Tr}\left(L_1(\lambda)L_2(\lambda)\cdots L_N(\lambda)\right)\,, \tag{126}$$

and so

$$t(1) = \mathrm{Tr}\left(P_{01}P_{02}\cdots P_{0N}\right) = P_{12}P_{13}\cdots P_{1N}\,, \tag{127}$$

and

$$[t(1)]^{-1} = P_{NN-1}\cdots P_{N2}P_{N1}\,. \tag{128}$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}T(\lambda) = \sum_{k=1}^{N}T_k(\lambda)\,, \tag{129}$$

where

$$T_k(\lambda) = L_1(\lambda)L_2(\lambda)\cdots L_k'(\lambda)\cdots L_N(\lambda)\,, \tag{130}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda) = \sum_{k=1}^{N}t_k(\lambda)\,, \tag{131}$$

with $t_k(\lambda) = \mathrm{Tr}_0 T_k(\lambda)$ and so

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)\right|_{\lambda=1} = \sum_{k=1}^{N}t_k(1)\,. \tag{132}$$

Therefore

1. When $k=1$, $t_1(1) = L_{N1}'(1)P_{N2}\cdots P_{13}P_{12}$,

$$[t(1)]^{-1}\,t_1(1) = \hat{R}_{12}'(1) = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\hat{R}_{12}(\lambda)\right|_{\lambda=1}\,. \tag{133}$$

2. When $2\le k\le N-1$ we have

$$t_k(1) = P_{12}\cdots P_{1k-1}L_{1k}'(1)\cdots P_{1k+1}P_{1N}\,, \tag{134}$$

and so

$$[t(1)]^{-1}\,t_k(1) = \hat{R}_{kk+1}'(1)\,. \tag{135}$$

3. When $k=N$ we have

$$[t(1)]^{-1}\,t_N(1) = \hat{R}_{N1}'(1)\,. \tag{136}$$

Under the periodic boundary condition, $R_{N1} = R_{NN+1}$.

The definition of Hamiltonian is

$$H = \sum_{k=1}^{N}h_{kk+1}\,, \tag{137}$$

where

$$\begin{aligned} h_{kk+1} &= \hat{R}_{kk+1}'(1) = -2\left(x_0c_k^+c_{k+1}^- + x_1c_k^-c_{k+1}^+\right) - 2x_0x_1 \\ &\quad + (N_k+N_{k+1}-2N_kN_{k+1})(x_0x_1-1)\,. \end{aligned} \tag{138}$$

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Appendix Bethe Ansatz Algebra

$$\begin{aligned}
&(B.1) && [A(x),A(y)] = 0\,,\\
&(B.2) && [B(x),B(y)] = 0\,,\\
&(B.3) && [C(x),C(y)] = 0\,,\\
&(B.4) && [D(x),D(y)] = 0\,,\\
&(B.5) && A(x)B(y)a(xy^{-1}) - A(y)B(x)c(xy^{-1}) - B(y)A(x)b(xy^{-1}) = 0\,,\\
&(B.6) && A(y)B(x)d(xy^{-1}) - B(x)A(y)a(xy^{-1}) + B(y)A(x)c(xy^{-1}) = 0\,,\\
&(B.7) && A(x)C(y)c(xy^{-1}) - A(y)C(x)a(xy^{-1}) + C(x)A(y)d(xy^{-1}) = 0\,,\\
&(B.8) && A(x)C(y)b(xy^{-1}) + C(x)A(y)c(xy^{-1}) - C(y)A(x)a(xy^{-1}) = 0\,,\\
&(B.9) && [A(y),D(x)]\,d(xy^{-1}) + [B(x)C(y) - B(y)C(x)]\,c(xy^{-1}) = 0\,,\\
&(B.10) && [A(x)D(y) - A(y)D(x)]\,c(xy^{-1}) + B(y)C(x)b(xy^{-1}) + C(x)B(y)d(xy^{-1}) = 0\,,\\
&(B.11) && [A(x),D(y)]\,b(xy^{-1}) + [C(x)B(y) - C(y)B(x)]\,c(xy^{-1}) = 0\,,\\
&(B.12) && [D(y)A(x) - D(x)A(y)]\,c(xy^{-1}) + C(y)B(x)d(xy^{-1}) + B(x)C(y)b(xy^{-1}) = 0\,,\\
&(B.13) && B(x)D(y)b(xy^{-1}) - D(x)B(y)c(xy^{-1}) + D(y)B(x)a(xy^{-1}) = 0\,,\\
&(B.14) && B(x)D(y)c(xy^{-1}) - B(y)D(x)a(xy^{-1}) - D(x)B(y)d(xy^{-1}) = 0\,,\\
&(B.15) && C(x)D(y)a(xy^{-1}) - C(y)D(x)c(xy^{-1}) + D(y)C(x)b(xy^{-1}) = 0\,,\\
&(B.16) && C(y)D(x)d(xy^{-1}) + D(x)C(y)a(xy^{-1}) - D(y)C(x)c(xy^{-1}) = 0\,,
\end{aligned}$$

where

$$\begin{aligned} a(x) &= -\left(x^2x_1x_0+1\right)\,,\\ b(x) &= \left(x^2-1\right)x_0\,,\\ c(x) &= -\left(x_1x_0+1\right)x\,,\\ d(x) &= -\left(x^2-1\right)x_1\,. \end{aligned} \tag{139}$$

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