2 Dimensional Cyclic Representation of 
$U_q \hat{s}l_2$ and 8 Vertex Ising Model

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Abstract

We explicitly construct a two dimensional cyclic representation of $U_q(\hat{sl}_2)$ and its intertwiner. The representation is not evaluation representation. The intertwiner turns out to be $R$-matrix for eight vertex Ising model.

Much attentions is being paid to the search for a quantum group like structure whose intertwiner $R$-matrix for different irreducible representations corresponds to the elliptic 8-vertex solutions of the Yang-Baxter equation[1]. Recently such a quantum group interpretation has been found for the free fermion 8-vertex model in magnetic field, introduced in[2]. Its Boltzmann weights matrix[3, 4] acts as an intertwiner for the affinization of a quantum Hopf deformation of the Clifford algebra in 2 dimensions, $CH_2(2)$.[5]. In this paper, we will be concerned with the 2-dimensional cyclic representations of $U_q(\hat{sl}_2)$, i.e., the representations when $q = i$. We show that the intertwiner turns out to be the $R$-matrix for 8-vertex Ising model.

1 The Algebra $U_q(\hat{sl}_2)$

First let's agree upon the notations. The Cartan matrix for affine $U_q(\hat{sl}_2)$ is

\[
\begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}
\]

which is generated by $\{e_i, f_i, K_i^+, K_i^-, z_i\}$, with the algebraic relations

\[
\begin{align*}
[(e_i, e_j)] & = 0, \quad \forall e_i \in U_q(\hat{sl}_2), \\
K_i K_j^+ & = K_j^+ K_i, \quad K_i = 1, \\
K_i K_j^{+q} & = K_j^{+q} K_i, \quad \forall i, j, \\
K_i e_j & = q^{\alpha_{ij}} e_j K_i, \\
K_i f_j & = q^{-\alpha_{ij}} f_j K_i, \\
[e_i, f_j] & = \delta_{ij} K_i - K_i^{-1}, \\
\sum_{\nu} (-1)^{\nu} \left[ \begin{array}{c} \nu \\ \nu + 1 \end{array} \right] q_{\nu} & = 0, \\
\sum_{\nu} (-1)^{\nu} \left[ \begin{array}{c} \nu \\ \nu + 1 \end{array} \right] q_{\nu}^{i} & = 0
\end{align*}
\]

where the central elements $z_i$s due to quantum double construction. The co-multiplication as a Hopf structure map is defined as follows

\[
\Delta(e_i) = e_i \otimes 1 + z_i K_i \otimes e_i, \\
\Delta(f_i) = f_i \otimes K_i^{-1} + z_i^+ \otimes f_i, \\
\Delta(K_i) = K_i^{\pm q} \otimes K_i, \\
\Delta(z_i) = z_i^{\pm q} \otimes z_i.
\]

The above definition gives a family of algebras with different value of $q$.

1.1 Algebra $U_q(\hat{sl}_2)$ at $q = i$

In the rest of this paper, we will be concerned with a special member of this family, i.e., the one with $q = i$. For convenience we make an isomorphic transformation of generator $f_i$, and we introduce new central elements $\phi_i$ and
Cl, just for the convenience of intertwiner construction, so that the algebraic relations read

\[
[z_i, z_j] = [z_i, z_k] = 0, \quad \forall z_i, z_j, z_k \in U_q, \quad E_i K_j = K_j E_i, \quad K_i K_j = K_j K_i, \\
K_i^p K_j^q = K_j^q K_i^p, \quad \forall i, j, \quad K_i^0 = K_i, \quad K_i^1 = K_i, \quad K_i^{p+1} = K_i K_i^p, \quad \forall i, j.
\]

It can be seen that \(E_i\) and \(E_j\) do not commute, nor do \(F_i\) and \(F_j\). The relations in (2) imply

\[
\begin{align*}
[E_i^m, E_j] &= (-1)^m E_i^{m-1}[E_i, E_j] E_i^{m-1}, & (i \neq j, m \geq 1), \\
[F_i^m, F_j] &= (-1)^{-m} E^m [F_i, F_j] E^{-m}, & (i \neq j, m \geq 1), \\
[F_i^m, E_j] &= (-1)^{-m} E^m [F_i, E_j] E^{-m}, \quad \forall i, j, \quad m \geq 1.
\end{align*}
\]

with Serre relations changed into

\[
[E_i^m, E_i] = [F_i^m, F_i] = 0, \quad (i \neq j).
\]

1.2 The Quotient Algebra \(U_q\)

However in the following we will take a quotient such that

\[
[E_i, z] = [F_i, z] = 0, \quad \forall z \in U_q.
\]

This gives us a new algebra \(U_q\) with central subalgebra

\[
Z = \text{gen} \{1, z_i, c_i E_i^1, F_i^1, K_i^1, i = 0, 1\}.
\]

It can be seen that at bialgebra level, \(Z\) remains to be central subalgebra.

2 \(U_q\) and Its Cyclic Representations

To construct the 2-dimensional cyclic representation, we introduce the Weyl algebra generated by two operators \(X\) and \(Z\), with the following relations

\[
X^2 = Z^2 = 1, \quad ZX = -XZ.
\]

The matrix representation we are to apply is

\[
Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

The Weyl realization of the algebra after the change of the parameters is given as follows

\[
\begin{align*}
F_i &= (i - XZ_{ij})/(2a_0 a_1 a_2), \\
F_0 &= (iX_{ij} - Z_{ij} X_{ij})/(2a_0 a_1 a_2), \\
E_i &= i(X_{ij} - Z_{ij} X_{ij})/2, \\
E_0 &= i(X_{ij} - Z_{ij} X_{ij})/2, \\
K_1 &= a_1 Z_{ij}, \\
K_0 &= -i a_2 Z_{ij}, \\
E_{ij} &= a_1 (a_2 (a_0 a_1 X_{ij} + a_1 a_2 X_{ij}))(a_0 a_1 a_2), \\
E_{ij} &= a_0 (a_1 (a_2 (a_0 a_1 X_{ij} + a_1 a_2 X_{ij}))(a_0 a_1 a_2)), \\
K_1 &= a_1 Z_{ij}, \\
K_0 &= -i a_2 Z_{ij}, \\
K_{ij} &= a_1 (a_2 (a_0 a_1 X_{ij} + a_1 a_2 X_{ij}))(a_0 a_1 a_2), \\
K_{ij} &= a_0 (a_1 (a_2 (a_0 a_1 X_{ij} + a_1 a_2 X_{ij}))(a_0 a_1 a_2)), \\
\end{align*}
\]

where

\[
a_0 = \sqrt{a_0 a_1 a_2}, \quad a_1 = \sqrt{a_0 a_1 a_2}.
\]

We stress that this representation is evaluation representation in that \(K_0 K_1 = 1\), and

\[
E_{ij} = (k_0 a_0 a_1 a_2 X_{ij}) E_{ij}, \quad F_{ij} = \frac{k_0}{a_1 a_2} E_{ij}.
\]

3 Coproducts

The coproduct map on each generator is defined as follows

\[
\begin{align*}
\Delta(E_i) &= E_i \otimes 1 + z_i K_i \otimes E_i, \\
\Delta(F_i) &= F_i \otimes 1 + z_i^2 K_i \otimes F_i, \\
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(z_i) &= z_i \otimes z_i, \\
\Delta(c_i) &= c_i K_i \otimes 1 - c_i 1 \otimes K_i, \\
\Delta(c_i) &= \frac{K_i^2 \otimes c_i - K_i \otimes K_i^2}{K_i \otimes K_i - 1 \otimes 1}.
\end{align*}
\]
Coproducts of generators are given explicitly in the following, where we neglect global coefficients that are not concerned with our construction of intertwiner,

$$\Delta E_1 = X_1y_1 + Z_1X_1y_1\mu_{11,10} - Z_1X_1y_1\mu_{11,11} - Z_2X_1y_1\mu_{11,12} - Z_2X_1y_1\mu_{11,13}$$
$$\Delta E_2 = X_1y_1\mu_{11,11} + Z_1X_1y_1\mu_{11,11} - Z_1X_1y_1\mu_{11,12} + Z_2X_1y_1\mu_{11,12} + Z_2X_1y_1\mu_{11,13}$$
$$\Delta E_3 = X_1y_1\mu_{11,12} + Z_1X_1y_1\mu_{11,12} - Z_1X_1y_1\mu_{11,13} + Z_2X_1y_1\mu_{11,13}$$

where $Z_1 = Z \otimes 1$, $Z_2 = 1 \otimes Z$ and similar for $X_1$ and $X_2$.

4 Intertwiner

- The curve $C_{10}$ is described by $x_{10}, y_{10}, \mu_{10}$.
- The curve $C_{11}$ is described by $x_{11}, y_{11}, \mu_{11}$.
- The curve $C_{20}$ is described by $x_{20}, y_{20}, \mu_{20}$.
- The curve $C_{21}$ is described by $x_{21}, y_{21}, \mu_{21}$.

A coproduct representation is on product curve $V = V_{10,11} \times V_{20,21}$ where $V_{10,11} = C_{10} \times C_{11}$ and $V_{20,21} = C_{20} \times C_{21}$. Therefore $V = (C_{10} \times C_{11}) \times (C_{20} \times C_{21})$.

1. Curve conditions

$$\rho_2^2 = \frac{x_{22}^2 - k}{2x_{22}^3 - 1}, \quad \rho_1^{'2} = \frac{k'}{1 - k x_{22}^3},$$

where $k^2 = 1 - k^2$ and $k, k'$ are conjugate elliptic norms.

2. Local intertwiner (I): 1, 0 $\leftrightarrow$ 2, 1

$$S_{10,21} = 1 + x_1X_2, \quad s_1 = \frac{\mu_{21,10} + \mu_{12,10}}{\mu_{12,10} - \mu_{12,10}}$$

3. Local intertwiner (II): 1, 0 $\leftrightarrow$ 2, 0

$$T_{10,20} = 1 + z_2Z_1, \quad t_2 = \frac{\rho_{21,0} + \rho_{12,0}}{\rho_{12,0} - \rho_{12,0}}$$

4. Local intertwiner (III): 1, 1 $\leftrightarrow$ 2, 1

$$T_{11,21} = \frac{\rho_{11,1} + \rho_{12,1}}{\rho_{12,1} - \rho_{12,1}}, \quad t_1 = \frac{\rho_{11,1} + \rho_{12,1}}{\rho_{12,1} - \rho_{12,1}}$$

5. Local intertwiner (IV): 1, 1 $\leftrightarrow$ 2, 0

$$S_{11,20} = 1 + z_2X_1, \quad s_2 = \frac{-\rho_{20,2} - \rho_{12,2}}{-\rho_{12,2} + \rho_{12,2}}$$

Global intertwiner reads

$$R(1,2) = S_{11,20}T_{11,21}S_{11,20}.$$

We list the effect of the actions of each local intertwiner $S_{10,11}$, $T_{11,21}$, $T_{10,20}$, and $S_{11,20}$ consecutively as follows.

Coproduct of $E_1$ is mapped consecutively from $\Delta E_1$ to $\Delta E_1'$, $\Delta E_1''$ and finally $\Delta E_1'''$:

$$\Delta E_1 = X_1y_1 + Z_1X_1y_1\mu_{11,10} - Z_1X_1y_1\mu_{11,11} - Z_2X_1y_1\mu_{11,12} - Z_2X_1y_1\mu_{11,13}$$

$$\Delta E_1' = X_1y_1\mu_{11,11} + Z_1X_1y_1\mu_{11,11} - Z_1X_1y_1\mu_{11,12} + Z_2X_1y_1\mu_{11,12} + Z_2X_1y_1\mu_{11,13}$$

$$\Delta E_1'' = X_1y_1\mu_{11,12} + Z_1X_1y_1\mu_{11,12} - Z_1X_1y_1\mu_{11,13} + Z_2X_1y_1\mu_{11,13}$$

$$\Delta E_1''' = X_1y_1\mu_{11,13} + Z_1X_1y_1\mu_{11,13} - Z_1X_1y_1\mu_{11,14} - Z_2X_1y_1\mu_{11,14}$$

Coproduct of $E_2$ is mapped into:

$$\Delta E_2 = X_1y_1 + Z_1X_1y_1\mu_{21,20} + Z_2X_1y_1\mu_{21,21} + Z_2X_1y_1\mu_{21,22}$$

$$\Delta E_2' = X_1y_1\mu_{21,21} + Z_1X_1y_1\mu_{21,21} - Z_1X_1y_1\mu_{21,22} + Z_2X_1y_1\mu_{21,22}$$

$$\Delta E_2'' = X_1y_1\mu_{21,22} + Z_1X_1y_1\mu_{21,22} - Z_1X_1y_1\mu_{21,23} - Z_2X_1y_1\mu_{21,23}$$

$$\Delta E_2''' = X_1y_1\mu_{21,23} + Z_1X_1y_1\mu_{21,23} - Z_1X_1y_1\mu_{21,24} - Z_2X_1y_1\mu_{21,24}$$
This condition is satisfied because of the form of the intertwiner in equation (3).

$$\Delta \Phi = X_2 p_{11}(u) X_1 X_2 p_{12}(u) X_1 X_2 p_{13}(u) X_1 X_2 p_{14}(u) X_1 X_2 p_{15}(u)$$

$$+ X_2 X_2 p_{21}(u) X_1 X_2 p_{22}(u) X_1 X_2 p_{23}(u) X_1 X_2 p_{24}(u) X_1 X_2 p_{25}(u)$$

$$\Delta \Phi^I = X_2 p_{11}(u) + X_1 X_2 p_{12}(u) + X_1 X_2 p_{13}(u) + X_1 X_2 p_{14}(u) + X_1 X_2 p_{15}(u)$$

$$\Delta \Phi^II = X_2 p_{11}(u) + X_1 X_2 p_{12}(u) + X_1 X_2 p_{13}(u) + X_1 X_2 p_{14}(u) + X_1 X_2 p_{15}(u)$$

$$\Delta \Phi^IV = X_2 p_{11}(u) + X_1 X_2 p_{12}(u) + X_1 X_2 p_{13}(u) + X_1 X_2 p_{14}(u) + X_1 X_2 p_{15}(u)$$

Coprodct of $F$ is mapped into:

$$\Delta F = X_2 p_{11}(u) X_1 X_2 p_{12}(u) X_1 X_2 p_{13}(u) X_1 X_2 p_{14}(u) X_1 X_2 p_{15}(u)$$

$$+ X_2 X_2 p_{21}(u) X_1 X_2 p_{22}(u) X_1 X_2 p_{23}(u) X_1 X_2 p_{24}(u) X_1 X_2 p_{25}(u)$$

$$\Delta F^I = X_2 p_{11}(u) + X_1 X_2 p_{12}(u) + X_1 X_2 p_{13}(u) + X_1 X_2 p_{14}(u) + X_1 X_2 p_{15}(u)$$

$$\Delta F^I = X_2 p_{11}(u) + X_1 X_2 p_{12}(u) + X_1 X_2 p_{13}(u) + X_1 X_2 p_{14}(u) + X_1 X_2 p_{15}(u)$$

$$\Delta F^IV = X_2 p_{11}(u) + X_1 X_2 p_{12}(u) + X_1 X_2 p_{13}(u) + X_1 X_2 p_{14}(u) + X_1 X_2 p_{15}(u)$$

Free fermion condition is satisfied

$$R_1 R_2 + R_2 R_3 = R_4 R_5 + R_5 R_4 .$$

This condition is satisfied because of the form of the intertwiner in equation (3).

The explicit expression for the non-zero elements of the $R$-matrix reads

$$R_{i,j} = \begin{cases} 1 + tsp_{13} & i = j = 1 \1 + tsp_{12} & i = 1, j \neq 1 \1 + tsp_{32} & i = 2, j = 1 \1 + tsp_{31} & i = 2, j \neq 1 \1 + tsp_{24} & i = 3, j = 1 \1 + tsp_{43} & i = 3, j \neq 1 \1 + tsp_{25} & i = 4, j = 1 \1 + tsp_{54} & i = 4, j \neq 1 \end{cases}$$

5 8-Vertex Ising Model

To parameterize the intertwiner into elliptic functions, we set

$$\hat{x}(u) = \left( s(u) \sqrt{\beta} \right)^{-1} , \quad \hat{y}(u) = \left( s(u) \sqrt{\beta} \right)^{-1} ,$$

$$\mu(u) = \sqrt{\beta} \delta(u), \quad \nu(u) = \sqrt{\beta} \delta(u),$$

$$\hat{\mu}(u) = \sqrt{\beta} \delta(u), \quad \hat{\nu}(u) = \sqrt{\beta} \delta(u),$$

$$\nu(u) = \sqrt{\beta} \delta(u), \quad \nu(u) = \sqrt{\beta} \delta(u),$$

where we applied short notations such as $s(u) = \frac{m(u)}{m(1)}$. Please note that $w_i$ ($i = 1, 2, 3$) are independent parameters. As a special case, we further set

$$w_3 = w_2 = -2k', \quad w_1 = w_2 - 2ik' + u ,$$

we get the intertwiner as

$$\hat{R}(u) = \frac{1 + c(u)}{4} \left[ 1 + (k \mu(u) - m(u)) X \otimes X \left( 1 + \frac{m(u) - c(u)}{2m(u)} \right) \right]$$

$$\otimes \left( 1 - \frac{k \mu(u) - m(u)}{2m(u)} \right) \left[ 1 + (k \mu(u) + m(u)) X \otimes X \right] ,$$

which is nothing but the $R$-matrix for 8-vertex Ising model

$$\hat{R}(u) = \frac{1 + c(u)}{4} \left[ \begin{array}{cccc} c(u) & 0 & 0 & k \mu(u) c(u) \\ 0 & 1 & \frac{m(u)}{2m(u)} & 0 \\ 0 & \frac{m(u)}{2m(u)} & 1 & 0 \\ k \mu(u) c(u) & 0 & 0 & c(u) \end{array} \right]$$

where $g$ is a gauge parameter

$$g = -k - ik' .$$
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