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2-D Cyclic Representations of $U_q(sl(2))$, C-G Coefficients and A Free Fermion 8-Vertex Model

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Abstract

We construct explicitly the two dimensional cyclic representation of quantum algebra $U_q(sl(2))$ with $q = i$ and z central extension. The representation is irreducible and we write down explicitly the Clebsch-Gordan coefficients and the intertwiner of tensor representations in different orders. This intertwiner turns out to be an 8-vertex R-matrix that satisfies free-Fermion condition and therefore gives an integrable model.

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The representations[1] of $U_q(sl(2))$ at q a root of unity are shown to be related to the chiral Potts models[2, 3, 4, 5]. The cyclic representations of $U_q(sl(2))$ (or its affine counterpart) are analyzed in [4, 6, 7], when q is N-th root of unity and N is an odd integer.

In this paper, we will be concerned with the 2-dimensional cyclic representations of $U_q(sl(2))$, i.e., the representations when $q = i$, or $q^4 = 1$. We will also show that the representations are completely reducible and give explicitly the Clebsch-Gordan (C-G) coefficients. The tensor representations with spectral varieties in different orders are isomorphic provided the spectral parameters sit on an algebraic curve. The intertwiner as a similarity transformation map between the two isomorphic tensor representations is shown to satisfy the free-Fermion condition[8, 9] and therefore gives Boltzman weights of an integrable free-Fermion model.

In [10], it is shown that the six-vertex Ising model is the intertwiner of $U_q(\widehat{sl}(2))$ at $q^4 = 1$, and in [11], it is shown that the six-vertex Ising model is the intertwiner of $U_q(sl(2))$. The present paper generalizes the analysis on the 2-dimensional cyclic representations and extends existing results to 8-vertex case.

1 The Algebra $U_q(sl(2))$ at $q = i$

The unital algebra $U_q(sl(2))$ is generated by e, f, K, z with the following defining relations

$$ef - q^2 fe = 1 - K^2 \ , \ eK = q^2 Ke \ , \ fK = q^{-2} Kf \ , \ [z, \bullet] = 0 \ , \ \forall \bullet \in U_q(sl(2)).$$

The Casimir element is

$$C = K^{-1}\left[fe + \frac{1}{q^2 - 1}\left(q^{-2}K^2 + 1\right)\right] .$$

The co-product operations are defined as follows

$$\Delta e = e \otimes 1 + zK \otimes e \ , \ \Delta f = f \otimes 1 + z^{-1} K \otimes f \ ,$$
$$\Delta K = K \otimes K \ , \ \Delta z = z \otimes z \ .$$

When q is p-th root of unity, the center of $U_q(sl(2))$ contains not only the Casimir operator and the central element z (appears due to Drinfeld

quantum double theory), but also the elements e^p, f^p and K^p, if p is odd, or the elements $e^{p/2}$, $f^{p/2}$ and $K^{p/2}$ if p is even.

In this letter we will focus on the special case of $p = 4$, or $q = i$. Explicitly we have the following algebraic relations (please note that a constant factor $2i$ has been absorbed into e or f),

$$\{e, f\} = 1 - K^2, \quad \{e, K\} = \{f, K\} = [z, \bullet] = 0 \ .$$

The Casimir element is

$$C = K^{-1}\left(fe + \frac{1}{2}(K^2 - 1)\right).$$

and hereafter we denote this algebra by U.

We can classify all the representations by the eigenvalues of the central elements. And we fix the eigenvalues of e^2, f^2, K^2 and z to be $\hat{\epsilon}$, $\hat{\phi}$, κ^2 and z respectively. And we introduce two more parameters ϵ and ϕ for convenience, such that

$$\epsilon = \frac{\hat{\epsilon}}{1 - z^2\kappa^2} \ , \quad \phi = \frac{\hat{\phi}}{1 - z^{-2}\kappa^2} \ .$$

Therefore the Casimir element takes the eigenvalue $c = -\frac{1}{2\kappa}\Delta$, where Δ is a constant (please note that it is not the coproduct map),

$$\Delta = \sqrt{(\kappa^2 - 1)^2 (1 - 4\epsilon\phi) + (z - z^{-1})^2} \ .$$

We denote the above representation by π_ξ, where ξ is the set of parameters $\{\vec{\epsilon}, \vec{\phi}, \kappa, z\}$.

2 The Cyclic Representations

Representation space V is spanned by

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ ,$$

respectively. The action of generators in this space reads

$$\begin{aligned}
\pi_\xi(f)\,|0\rangle &= \rho\,|1\rangle \ , & \pi_\xi(f)\,|1\rangle &= \rho^{-1}\,\hat{\phi}\,|0\rangle \ , \\
\pi_\xi(e)\,|0\rangle &= \eta\,|1\rangle \ , & \pi_\xi(e)\,|1\rangle &= \eta^{-1}\,\hat{\epsilon}\,|0\rangle \ , \\
\pi_\xi(K)\,|m\rangle &= (-)^m\kappa\,|m\rangle \ , & \pi_\xi(z)\,|m\rangle &= z\,|m\rangle \ , \qquad (m = 0, 1)
\end{aligned}$$

where

$$\eta = -\frac{z}{2\sqrt{\phi}}\frac{(\kappa^2 - 1) + \Delta}{\kappa + z}, \quad \rho = \sqrt{\phi}\left(1 - \frac{\kappa}{z}\right),$$

and therefore the central elements take their eigenvalues as follows

$$\begin{aligned}
&\pi_\xi(e)^2 = \hat{\epsilon} \cdot 1 = \epsilon(1 - z^2\kappa^2) \cdot 1 \ , && \pi_\xi(f)^2 = \hat{\phi} \cdot 1 = \phi(1 - z^{-2}\kappa^2) \cdot 1 \ , \\
&\pi_\xi(K)^2 = \kappa^2 \cdot 1 \ , && \pi_\xi(z) = z \ \cdot 1.
\end{aligned}$$

So we have an algebra homomorphism

$$\pi_\xi : \ U \longrightarrow \mathrm{End}\,(V) \ , \ \ \xi \in \mathbb{C}^4 \ .$$

3 Tensor Representations and C-G Rule

3.1 The Reducibility of Tensor Representations

We show explicitly that the tensor product of two 2-dimensional representations is completely reducible. The analysis will be made in the following orthogonal basis

$$|0,0\rangle = |0\rangle \otimes |0\rangle \ , \ \ |1,0\rangle = |1\rangle \otimes |0\rangle \ , \ \ |0,1\rangle = |0\rangle \otimes |1\rangle \ , \ \ |1,1\rangle = |1\rangle \otimes |1\rangle \ .$$

We take two 2-dimensional representations π_{ξ_1} and π_{ξ_2} with the values of central elements in sets $\xi_i = \left\{\vec{\epsilon}_i, \vec{\phi}_i, \kappa_i, z_i\right\}$ $(i = 1, 2)$ to form a tensor product $\pi_{\xi_{12}} = \pi_{\xi_1} \otimes \pi_{\xi_2}$. The central elements $\{\Delta_{12}(e)^2, \Delta_{12}(f)^2, \Delta_{12}(K)^2, \Delta_{12}(z)\}$ take their values in the set $\xi_{12} = \left\{\hat{\epsilon}_{12}, \hat{\phi}_{12}, \kappa_{12}^2, z_{12}\right\}$, i.e.,

$$\hat{\epsilon}_{12} = \hat{\epsilon}_1 + z_1^2\kappa_1^2\,\hat{\epsilon}_2, \qquad \hat{\phi}_{12} = \hat{\phi}_1 + z_1^{-2}\kappa_1^2\,\hat{\phi}_2, \qquad \kappa_{12} = \kappa_1\kappa_2, \qquad z_{12} = z_1 z_2,$$

or equivalently

$$\epsilon_{12} = \epsilon_2 + \frac{(\epsilon_1 - \epsilon_2)\left(1 - \kappa_1^2 z_1^2\right)}{1 - \kappa_{12}^2 z_{12}^2} \ , \qquad \phi_{12} = \phi_2 + \frac{(\phi_1 - \phi_2)\left(1 - \kappa_1^2 z_1^{-2}\right)}{1 - \kappa_{12}^2 z_{12}^{-2}} \ .$$

It can been seen that if $\phi_1 = \phi_2 = \phi$ then $\phi_{12} = \phi$, and similarly, if $\epsilon_1 = \epsilon_2 = \epsilon$ then $\epsilon_{12} = \epsilon$.

Let

$$|0_{12}\rangle = \alpha\,|0_{\xi_1},0_{\xi_2}\rangle + \delta\,|1_{\xi_1},1_{\xi_2}\rangle\,,\ |1_{12}\rangle = \beta\,|1_{\xi_1},0_{\xi_2}\rangle + \gamma\,|0_{\xi_1},1_{\xi_2}\rangle\,,$$

or equivalently,

$$\begin{aligned}
|0_{\xi_1},0_{\xi_2}\rangle &= \tilde{\alpha}^{(+)}\left|0_{12}^{(+)}\right\rangle + \tilde{\alpha}^{(-)}\left|0_{12}^{(-)}\right\rangle,\ |1_{\xi_1},0_{\xi_2}\rangle = \tilde{\beta}^{(+)}\left|1_{12}^{(+)}\right\rangle + \tilde{\beta}^{(-)}\left|1_{12}^{(-)}\right\rangle,\\
|0_{\xi_1},1_{\xi_2}\rangle &= \tilde{\gamma}^{(+)}\left|1_{12}^{(+)}\right\rangle + \tilde{\gamma}^{(-)}\left|1_{12}^{(-)}\right\rangle,\ |1_{\xi_1},1_{\xi_2}\rangle = \tilde{\delta}^{(+)}\left|0_{12}^{(+)}\right\rangle + \tilde{\delta}^{(-)}\left|0_{12}^{(-)}\right\rangle,
\end{aligned}$$

The action of the generators in this representation space gives

$$\begin{aligned}
(\pi_{\xi_1}\otimes\pi_{\xi_2})(f)\left|0_{12}^{(\pm)}\right\rangle &= \rho_{12}\left|1_{12}^{(\pm)}\right\rangle\,,\\
(\pi_{\xi_1}\otimes\pi_{\xi_2})(f)\left|1_{12}^{(\pm)}\right\rangle &= (\rho_{12})^{-1}\,\hat{\phi}_{12}\left|0_{12}^{(\pm)}\right\rangle,\\
(\pi_{\xi_1}\otimes\pi_{\xi_2})(e)\left|0_{12}^{(\pm)}\right\rangle &= \eta_{12}^{(\pm)}\left|1_{12}^{(\pm)}\right\rangle,\\
(\pi_{\xi_1}\otimes\pi_{\xi_2})(e)\left|1_{12}^{(\pm)}\right\rangle &= \left(\eta_{12}^{(\pm)}\right)^{-1}\hat{\epsilon}_{12}\left|0_{12}^{(\pm)}\right\rangle\,,\\
(\pi_{\xi_1}\otimes\pi_{\xi_2})(K)\left|m_{12}^{(\pm)}\right\rangle &= (-)^k\kappa_{12}\left|m_{12}^{(\pm)}\right\rangle,\\
(\pi_{\xi_1}\otimes\pi_{\xi_2})(z)\left|m_{12}^{(\pm)}\right\rangle &= z\left|m_{12}^{(\pm)}\right\rangle\,,\quad (m=0,1)
\end{aligned}\tag{1}$$

where

$$\eta_{12}^{(\pm)} = \frac{1}{2\sqrt{\phi_{12}}}\frac{1-\kappa_{12}^2\pm\Delta_{12}}{1+z_{12}^{-1}\kappa_{12}},\qquad \rho_{12} = \sqrt{\phi_{12}}\left(1-\kappa_{12}z_{12}^{-1}\right)$$

and

$$\Delta_{12} = \sqrt{(\kappa_{12}^2-1)^2 - 4\,\hat{\phi}_{12}\hat{\epsilon}_{12}}.$$

The Casimir operator takes the values of $C_{12} = \pm\frac{1}{2\kappa_{12}}\Delta_{12}$. The 2-dimensional representations $\pi_{\xi_{12}}$ are classified into two kinds $\pi_{\xi_{12}}^{\pm}$ according to the plus/minus sign in the eigenvalue of the Casimir operator.

3.2 Clebsch-Gordan Coefficients

For the construction of intertwiner, we give the explicit expression of C-G coefficients and inverse C-G coefficients. From (1), we have the following

coefficients,

$$\begin{aligned}
\alpha^{(-)} &= z_1^{-1}\kappa_1\delta^{(-)}\left(z_1^2\rho_{12}\eta_1^{-1}\,\hat{\epsilon}_1 - \rho_2^{-1}\,\hat{\phi}_2\right)\left(\eta_1\rho_{12} - \eta_{12}^{(-)}\rho_1\right)^{-1},\\
\delta^{(+)} &= z_1\kappa_1^{-1}\alpha^{(+)}\left(\rho_{12}\eta_1 - \rho_1\eta_{12}^{(+)}\right)\left(z_1^2\rho_{12}\eta_1^{-1}\,\hat{\epsilon}_1 - \eta_{12}^{(+)}\rho_1^{-1}\,\hat{\phi}_1\right)^{-1},\\
\beta^{(\pm)} &= \rho_{12}^{-1}\left(\rho_1\alpha^{(\pm)} - z_1^{-1}\kappa_1\rho_2^{-1}\,\hat{\phi}_2\,\delta^{(\pm)}\right),\\
\gamma^{(\pm)} &= \rho_{12}^{-1}\left(z_1^{-1}\kappa_1\rho_2\alpha^{(\pm)} + \rho_1^{-1}\,\hat{\phi}_1\,\delta^{(\pm)}\right),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\alpha}^{(\pm)} &= \pm\frac{\delta^{(\mp)}}{\alpha^{(+)}\delta^{(-)}-\alpha^{(-)}\delta^{(+)}}, & \tilde{\delta}^{(\pm)} &= \mp\frac{\alpha^{(\mp)}}{\alpha^{(+)}\delta^{(-)}-\alpha^{(-)}\delta^{(+)}},\\
\tilde{\beta}^{(\pm)} &= \pm\frac{\gamma^{(\mp)}}{\beta^{(+)}\gamma^{(-)}-\beta^{(-)}\gamma^{(+)}}, & \tilde{\gamma}^{(\pm)} &= \mp\frac{\beta^{(\mp)}}{\beta^{(+)}\gamma^{(-)}-\beta^{(-)}\gamma^{(+)}}.
\end{aligned}$$

where the superscipt $(\pm)$ corresponds to the representations $\pi_{\xi_{12}^{(\pm)}}$.

The non-zero C-G coefficients are given as follows

$$\begin{aligned}
\left\langle 0_{\xi_{12}^{(\pm)}}\middle|0_{\xi_1},0_{\xi_2}\right\rangle &= \tilde{\alpha}^{(\pm)}, & \left\langle 0_{\xi_{12}^{(\pm)}}\middle|1_{\xi_1},1_{\xi_2}\right\rangle &= \tilde{\delta}^{(\pm)},\\
\left\langle 1_{\xi_{12}^{(\pm)}}\middle|0_{\xi_1},1_{\xi_2}\right\rangle &= \tilde{\gamma}^{(\pm)}, & \left\langle 1_{\xi_{12}^{(\pm)}}\middle|1_{\xi_1},0_{\xi_2}\right\rangle &= \tilde{\beta}^{(\pm)},
\end{aligned}$$

and

$$\begin{aligned}
\left\langle 0_{\xi_1},0_{\xi_2}\middle|0_{\xi_{12}^{(\pm)}}\right\rangle &= \alpha^{(\pm)}, & \left\langle 1_{\xi_1},1_{\xi_2}\middle|0_{\xi_{12}^{(\pm)}}\right\rangle &= \delta^{(\pm)},\\
\left\langle 0_{\xi_1},1_{\xi_2}\middle|1_{\xi_{12}^{(\pm)}}\right\rangle &= \gamma^{(\pm)}, & \left\langle 1_{\xi_1},0_{\xi_2}\middle|1_{\xi_{12}^{(\pm)}}\right\rangle &= \beta^{(\pm)}.
\end{aligned}$$

4 The Construction of Intertwiner

We consider the following algebra homomorphisms

$$\begin{aligned}
(\pi_{\xi_1}\otimes\pi_{\xi_2}) &: U \longrightarrow \pi_{\xi_1}(U)\otimes\pi_{\xi_2}(U)\,,\\
(\pi_{\xi_2}\otimes\pi_{\xi_1}) &: U \longrightarrow \pi_{\xi_2}(U)\otimes\pi_{\xi_1}(U)\,.
\end{aligned}$$

These two representations are isomorphic provided that the spectral varieties ξ_1 and ξ_2 sit on an algebraic curve. In case this is true, we should have an

intertwiner R, or a similarity transformation matrix between the two representation spaces. In other words, we should solve the following intertwining equation,

$$R(\xi_1,\xi_2)\,(\pi_{\xi_1}\otimes\pi_{\xi_2})\,(g) = (\pi_{\xi_2}\otimes\pi_{\xi_1})\,(g)R(\xi_1,\xi_2)\ , \tag{2}$$

to give R if it exists.

First we should consider the necessary conditions for intertwiner satisfying eqn. (2) to exist. Actually, taking trace of that equation in the indicated representations gives such conditions. When $g = e^2$ and f^2 are concerned, we have the following nontrivial constraints,

$$\left(1-\kappa_1^2 z_1^2\right)\left(1-\kappa_2^2 z_2^2\right)(\epsilon_1-\epsilon_2)=0\ ,\ \left(\kappa_1^2-z_1^2\right)\left(\kappa_2^2-z_2^2\right)(\phi_1-\phi_2)=0\ .$$

The solution is an algebraic curve with equations $\epsilon_1=\epsilon_2$ and $\phi_1=\phi_2$, and we will be working under this condition in the following discussions.

According to eqn.(2), we have the entries of intertwiner explicitly as follows

$$R^{mn}_{kl}(\xi_1,\xi_2) = \sum_{r_\xi=0_\xi,1_\xi} \langle m_{\xi_2}, n_{\xi_1}|\, r_\xi\rangle\, \langle r_\xi|\, k_{\xi_1}, l_{\xi_2}\rangle$$

and the non-zero ones read

$$\begin{aligned}
R^{00}_{00} &= \overline{\alpha}^{(+)}\widetilde{\alpha}^{(+)}+\overline{\alpha}^{(-)}\widetilde{\alpha}^{(-)}, & R^{01}_{01} &= \overline{\gamma}^{(+)}\widetilde{\gamma}^{(+)}+\overline{\gamma}^{(-)}\widetilde{\gamma}^{(-)},\\
R^{10}_{01} &= \overline{\beta}^{(+)}\widetilde{\gamma}^{(+)}+\overline{\beta}^{(-)}\widetilde{\gamma}^{(-)}, & R^{01}_{10} &= \overline{\gamma}^{(+)}\widetilde{\beta}^{(+)}+\overline{\gamma}^{(-)}\widetilde{\beta}^{(-)},\\
R^{10}_{10} &= \overline{\beta}^{(+)}\widetilde{\beta}^{(+)}+\overline{\beta}^{(-)}\widetilde{\beta}^{(-)}, & R^{11}_{11} &= \overline{\delta}^{(+)}\widetilde{\delta}^{(+)}+\overline{\delta}^{(-)}\widetilde{\delta}^{(-)},\\
R^{11}_{00} &= \overline{\delta}^{(+)}\widetilde{\alpha}^{(+)}+\overline{\delta}^{(-)}\widetilde{\alpha}^{(-)}, & R^{00}_{11} &= \overline{\alpha}^{(+)}\widetilde{\delta}^{(+)}+\overline{\alpha}^{(-)}\widetilde{\delta}^{(-)}.
\end{aligned}$$

Here we apply a map

$$\begin{aligned}
\sigma\ :\ \xi_{12} &\longrightarrow \overline{\xi}_{12}=\xi_{21},\\
x &\longmapsto \overline{x}, \qquad x=\alpha,\beta,\gamma,\delta.
\end{aligned}$$

5 A Free-Fermion 8-Vertex Model

As it is known, the intertwiner constructed in this way does not necessarily give (the Boltzman weights of) an integrable model. However, it can be

verified that our R-matrix gives an integrable eight vertex model, in that it satisfies the free fermion condition[8]

$$R^{00}_{00}R^{11}_{11}+R^{01}_{10}R^{10}_{10}=R^{11}_{00}R^{00}_{11}+R^{01}_{01}R^{10}_{10}.$$

This is an interesting result.

It has been shown that the R-matrix of an elliptic free Fermion model [9] can be obtained as intertwiner of the tensor representations of a Clifford-Hopf algebra $\widehat{CH}_q(2)$ [12, 13, 14], which is a graded algebra. While in our result, it is $U_q(sl(2))$, in the same category as the chiral Potts models. We wish to come back to the relationship between the results of Madrid group and the present paper in a future publication.

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