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DISCRETIZATION OF QUANTUM MINKOWSKI SPACE–TIME

Zhan XU

P.O.Box 2735, Beijing 100080, The People's Republic of China

Telefax : (086)-1-2562587 Telephone : 2568348

Telex : 22040 BAOAS CN Cable : 6158

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Discretization of Quantum Minkowski Space-time [1]

Zhan XU

CCAST (World Lab), P. O. Box 8730, Beijing 100080
and
Institute of Theoretical Physics, Academia Sinica, Beijing 100080
and
Department of Physics and Institute of Modern Physics,
Tsinghua University, Beijing 100084 [2]

ABSTRACT

The coordinates of quantum Minkowski space-time, which are elements of an algebra, can be treated as operators. Their representation in Hilbert space is then investigated. It turns out that two of the four dimensions of the space-time are hidden, and eigenvalues of the rest two coordinates are discrete, even become a lattice in some region.

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[2] Mailing address.

The relativistic quantum field theories constructed over the conventional space-time are successful in describing the microscopic particle world above the length scale $\sim 10^{-18}$ m (or below the energy scale ~ 1 TeV). However, the difficulties emerging in loop diagrams in them may manifest some essential defects. Perhaps they are not applicable when the scale of physics was down to the sub-microscopic world [1]. In this aspect the quantum Minkowski space-time was suggested [1–8] which might be a candidate for the description of the sub-microscopic space-time. Quantum Minkowski space is a vector representation of the q-deformed (*i.e.* quantum) Lorentz group $\mathrm{SL}_q(2,\mathcal{C})$ [1,3,5,6,9]. When $q \to 1$, it approaches to the conventional one. In order to formulate physics in this space and get some predictions from it, study of its structure is thus necessary.

1. Quantum Lorentz Group and Quantum Minkowski Space

A quantum matrix $M = (M^a{}_b)(a,b=1,2)$ is said to be in $\mathrm{SL}_q(2,\mathcal{C})$ if

$$\begin{cases} \hat{R}_{12}M_1M_2 = M_1M_2\hat{R}_{12}, \\ \hat{R}_{12}M_1\overline{M}_2 = \overline{M}_1M_2\hat{R}_{12}, \\ \epsilon_{ab}M^a{}_cM^b{}_d = \epsilon_{cd}, \quad (a,b,c,d=1,2) \end{cases} \tag{1}$$

in which

$$\hat{R} = (\hat{R}^{ab}{}_{cd}) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad (q \text{ is real and positive}) \tag{2}$$

$$\epsilon_{ab} = \begin{pmatrix} 0 & -q^{-\frac{1}{2}} \\ q^{\frac{1}{2}} & 0 \end{pmatrix}, \quad (\text{the "spinor metric"}) \tag{3}$$

and

$$\overline{M} = (M^\dagger)^{-1} = (M^{-1})^\dagger, \quad (\text{in } \mathrm{SL}_q(2,\mathcal{C}), \text{ too}), \qquad M^\dagger = (M^*)^T.$$

Matrix $\hat{R}$ satisfies the Yang-Baxter equation (YBE):

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{4}$$

A q-spinor $u^a(a=1,2)$ is said to belong to the $(\frac{1}{2},0)$ representation of $\mathrm{SL}_q(2,\mathcal{C})$ if it transforms under action of $M \in \mathrm{SL}_q(2,\mathcal{C})$ as

$$u'^a = M^a{}_bu^b, \quad (a,b=1,2)$$

and its $*$-conjugation $\overline{u}_{\dot{a}} \equiv (u^a)^*$, $(a = 1,2)$ is said to belong to the $(0, \frac{1}{2})$ representation of $\mathrm{SL}_q(2,\mathcal{C})$ and transforms as

$$\overline{u}^{\dot{a}} = \overline{M}^{\dot{a}}{}_{\dot{b}}\overline{u}^{\dot{b}}, \quad (a,b = 1,2)$$

where $\overline{u}^{\dot{a}} = \overline{u}_{\dot{b}}\epsilon^{\dot{b}\dot{a}}$. Quasi-bosonic q-spinor z^a obeys the commutation relation

$$z_1 z_2 = q^{-1}\hat{R}_{12} z_1 z_2, \tag{5}$$

and quasi-fermionic q-spinor ϕ^a obeys

$$\phi_1\phi_2 = -q\hat{R}_{12}\phi_1\phi_2. \tag{6}$$

Their conjugation spinors obey similar relations. When $q \to 1$, z^a becomes commutative and ϕ^a anti-commutative.

There are two fundamental fusion of $\hat{R}$:

$$\begin{aligned} \hat{\mathrm{R}}_{(12)(34)} &= q^{-1}\hat{R}_{23}\hat{R}_{12}\hat{R}_{34}(\hat{R}^{-1})_{23}, \\ \hat{\mathrm{S}}_{(12)(34)} &= \hat{R}_{23}\hat{R}_{12}(\hat{R}^{-1})_{34}(\hat{R}^{-1})_{23}. \end{aligned} \tag{7}$$

They also satisfy YBE respectively and with each other the crossed YBE, *i.e.*

$$\begin{aligned} &\hat{\mathrm{R}}_{12}\hat{\mathrm{R}}_{23}\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{23}\hat{\mathrm{R}}_{12}\hat{\mathrm{R}}_{23}, \quad \hat{\mathrm{S}}_{12}\hat{\mathrm{S}}_{23}\hat{\mathrm{S}}_{12} = \hat{\mathrm{S}}_{23}\hat{\mathrm{S}}_{12}\hat{\mathrm{S}}_{23}, \\ &\hat{\mathrm{R}}_{12}\hat{\mathrm{R}}_{23}\hat{\mathrm{S}}_{12} = \hat{\mathrm{S}}_{23}\hat{\mathrm{R}}_{12}\hat{\mathrm{R}}_{23}, \quad \hat{\mathrm{S}}_{12}\hat{\mathrm{S}}_{23}\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{23}\hat{\mathrm{S}}_{12}\hat{\mathrm{S}}_{23}, \\ &\hat{\mathrm{S}}_{12}\hat{\mathrm{R}}_{23}\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{23}\hat{\mathrm{R}}_{12}\hat{\mathrm{S}}_{23}, \quad \hat{\mathrm{R}}_{12}\hat{\mathrm{S}}_{23}\hat{\mathrm{S}}_{12} = \hat{\mathrm{S}}_{23}\hat{\mathrm{S}}_{12}\hat{\mathrm{R}}_{23}, \end{aligned} \tag{8}$$

and they are "orthogonal":

$$(E + q\hat{\mathrm{R}})(E - \hat{\mathrm{S}}) = 0. \quad (E:\ 16 \times 16 \text{ unit matrix}) \tag{9}$$

Fusion of the spinor metric is

$$H_{a\dot{a}b\dot{b}} = q\epsilon_{ac}\epsilon_{\dot{a}\dot{c}}\hat{R}^{\dot{c}c}{}_{b\dot{b}}. \tag{10}$$

All the above matrices play fundamental role in the bi-spinor representation of $\mathrm{SL}_q(2,\mathcal{C})$. A bi-spinor $u_1\overline{w}_2$ transforms as

$$(u_1\overline{w}_2)' = M_1\overline{M}_2(u_1\overline{w}_2), \quad (M \in \mathrm{SL}_q(2,\mathcal{C}))$$

and $M_1\overline{M}_2$ satisfies

$$\begin{cases} \hat{\mathrm{R}}_{(12)(34)}M_1\overline{M}_2M_3\overline{M}_4 = M_1\overline{M}_2M_3\overline{M}_4\hat{\mathrm{R}}_{(12)(34)}, \\ \hat{\mathrm{S}}_{(12)(34)}M_1\overline{M}_2M_3\overline{M}_4 = M_1\overline{M}_2M_3\overline{M}_4\hat{\mathrm{S}}_{(12)(34)}, \\ H_{a\dot{a}b\dot{b}}M^a{}_cM^b{}_d\overline{M}^{\dot{a}}{}_{\dot{c}}\overline{M}^{\dot{b}}{}_{\dot{d}} = H_{c\dot{c}d\dot{d}}. \end{cases} \tag{11}$$

Bi-spinor $u^a\overline{w}^{\dot{a}}$ $(a, \dot{a} = 1,2)$ can be transformed into q-vector v^μ $(\mu = 0,+,3,-)$ by virtue of a matrix $K^\mu{}_{a\dot{a}}$:

$$v^\mu = K^\mu{}_{a\dot{a}}u^a\overline{w}^{\dot{a}}, \tag{12}$$

where $K^\mu{}_{a\dot{a}}$ is determined by the left eigen-equation of $\hat{R}$:

$$\begin{cases} K^0{}_{ab}\hat{R}^{ab}{}_{cd} = -q^{-1}K^0{}_{cd}, & (\text{singlet}) \\ K^m{}_{ab}\hat{R}^{ab}{}_{cd} = qK^m{}_{cd}. & (m = +,3,-,\ \text{triplet}) \end{cases}$$

And $\hat{\mathrm{R}}, \hat{\mathrm{S}}$ can also be transformed into their vector's counterparts:

$$\mathcal{F}^{\mu\nu}{}_{\kappa\lambda} = K^\mu{}_{a\dot{a}}K^\nu{}_{b\dot{b}}\mathrm{F}^{a\dot{a}b\dot{b}}{}_{c\dot{c}d\dot{d}}(K^{-1})^{c\dot{c}}{}_\kappa(K^{-1})^{d\dot{d}}{}_\lambda, \tag{13}$$

in which F represents $\hat{\mathrm{R}}, \hat{\mathrm{S}}$ and $\mathcal{F}$ represents $\hat{\mathcal{R}}, \hat{\mathcal{S}}$, respectively, and $H_{a\dot{a}b\dot{b}}$ transformed into

$$g_{\mu\nu} = H_{a\dot{a}b\dot{b}}(K^{-1})^{a\dot{a}}{}_\mu(K^{-1})^{b\dot{b}}{}_\nu. \tag{14}$$

Under application of $\mathrm{SL}_q(2,\mathcal{C})$ the q-vector v^μ transforms as

$$v'^\mu = L^\mu{}_\nu v^\nu, \quad (\mu,\nu = 0,+,3,-)$$

where

$$L^\mu{}_\nu = K^\mu{}_{a\dot{a}}M^a{}_b\overline{M}^{\dot{a}}{}_{\dot{b}}(K^{-1})^{b\dot{b}}{}_\nu \quad (M \in \mathrm{SL}_q(2,\mathcal{C})) \tag{15}$$

is the quantum Lorentz matrix satisfying

$$\begin{cases} \hat{\mathcal{R}}_{12}L_1L_2 = L_1L_2\hat{\mathcal{R}}_{12}, \\ \hat{\mathcal{S}}_{12}L_1L_2 = L_1L_2\hat{\mathcal{S}}_{12}, \\ g_{\mu\nu}L^\mu{}_\kappa L^\nu{}_\lambda = g_{\kappa\lambda}, \end{cases} \tag{16}$$

and hence

$$g_{\mu\nu}v^\mu v^\nu = \text{inv}. \tag{17}$$

Therefore $g_{\mu\nu}$ is the vector's metric.

The quasi-bosonic q-vector x^μ, regarded as the "coordinate" of quantum space-time, obeys the following commutation relations,

$$x_1x_2 = \hat{\mathcal{S}}_{12}x_1x_2, \tag{18}$$

which are covariant under quantum Lorentz transformation. Explicitly they are

$$\begin{cases} x^0x^+ - x^+x^0 = 0, \\ x^0x^3 - x^3x^0 = 0, \\ x^0x^- - x^-x^0 = 0, \\ qx^+x^3 - q^{-1}x^3x^+ = (q-q^{-1})x^0x^+, \\ qx^3x^- - q^{-1}x^-x^3 = (q-q^{-1})x^0x^-, \\ x^+x^- - x^-x^+ = -(q-q^{-1})(x^0-x^3)x^3, \end{cases} \tag{19}$$

therefore x^0 (the quantum time) is the linear center of this algebra. Furthermore the invariant quantity

$$J = g_{\mu\nu}x^\mu x^\nu = (x^0)^2 - qx^+x^- - q^{-1}x^-x^+ - (x^3)^2 \tag{20}$$

is the quadratic center, *i.e.*,

$$Jx^\mu - x^\mu J = 0. \tag{21}$$

At the same time we can introduce conjugation relations in this algebra such as

$$(x^0)^* = x^0, \quad (x^+)^* = q^{-1}x^-, \quad (x^3)^* = x^3, \quad (x^-)^* = qx^+, \tag{22}$$

and then

$$J^* = J, \tag{23}$$

which are covariant, too. This makes the algebra a C^*-algebra. When $q \to 1$, all the x^μs commute with each other, and if we put

$$x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm \mathrm{i}x^2), \quad (x^+)^* = x^-, \quad (\mathrm{i}^2 = -1)$$

then

$$J = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = \text{inv.}$$

which means the conventional Minkowski space-time is recovered.

2. Representation of Quantum Space-time in Hilbert Space

Now let us construct a representation of the algebra defined by Eqs.(19–23) in Hilbert space. Indeed all the elements x^0, x^+, x^3, x^-, J in this algebra can be regarded as operators over Hilbert space, and $*$-conjugation as their hermitian conjugation. Firstly we find that $\{J, x^0, x^3\}$ are hermitian and commutative, therefore

the simultaneous eigenstates $|\theta,\tau,\sigma\rangle$ of $\{J, x^0, x^3\}$ form basis of the Hilbert space, *i.e.*

$$\begin{cases} J|\theta,\tau,\sigma\rangle = \theta|\theta,\tau,\sigma\rangle, \\ x^0|\theta,\tau,\sigma\rangle = \tau|\theta,\tau,\sigma\rangle, \\ x^3|\theta,\tau,\sigma\rangle = \sigma|\theta,\tau,\sigma\rangle, \end{cases} \tag{24}$$

and θ, τ, σ are real numbers. Then it is easy to prove that $x^\pm|\theta,\tau,\sigma\rangle$ are eigenstate of $\{J, x^0, x^3\}$, too, with the *same* eigenvalues of J and x^0 but *different* eigenvalue of x^3:

$$\begin{cases} x^+|\theta,\tau,\sigma\rangle \sim |\theta,\tau,q^2\sigma - (q^2-1)\tau\rangle, \\ x^-|\theta,\tau,\sigma\rangle \sim |\theta,\tau,q^{-2}\sigma + (1-q^{-2})\tau\rangle, \end{cases} \tag{25}$$

therefore $x^\pm$ are "movers" of x^3. x^+ moves σ (the eigenvalue of x^3) to $q^2\sigma-(q^2-1)\tau$. If $q > 1$, it is greater than σ for $\sigma > \tau$ and less than σ for $\sigma < \tau$, *i.e.*, x^+ moves $x^3 = \sigma$ *outward* starting from the static point $\sigma = \tau$ ($x^3 = x^0$), hence it is an "expander". x^- acts just oppositely, moves x^3 *inward*, and hence is a "concentrator". If $q < 1$, they all change their moving directions, thus x^+ is concentrator and x^- is expander. By applying x^+ or x^- many times on $|\theta,\tau,\sigma\rangle$, one moves it to the points with the following eigenvalues:

$$J \text{ and } x^0 \text{ invariant}, \quad x^3 = q^{2n}\sigma - q^n[n]\omega\tau, \tag{26}$$

in which $n = 0, \pm1, \pm2, \cdots$, $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$, $\omega = q - q^{-1}$, $n > 0$ means $(x^+)^n$ application and $n < 0$ means $(x^-)^{|n|}$ application. If we let

$$\begin{cases} x^+|\theta,\tau,\sigma\rangle = c^+(\theta,\tau,\sigma)|\theta,\tau,q^2\sigma - q\omega\tau\rangle, \\ x^-|\theta,\tau,\sigma\rangle = c^-(\theta,\tau,\sigma)|\theta,\tau,q^{-2}\sigma + q^{-1}\omega\tau\rangle, \end{cases} \tag{27}$$

where $c^\pm(\theta,\tau,\sigma)$ are complex numbers, and

$$\langle\theta,\tau,\sigma|\theta,\tau,\sigma\rangle = 1, \quad (\text{for any } \theta,\tau,\sigma) \tag{28}$$

we find by the algebra that

$$\begin{cases} |c^+(\theta,\tau,\sigma)|^2 = \frac{q^{-1}}{q+q^{-1}}[(\tau + q^2\sigma)(\tau - \sigma) - \theta], \\ |c^-(\theta,\tau,\sigma)|^2 = \frac{q}{q+q^{-1}}[(\tau + q^{-2}\sigma)(\tau - \sigma) - \theta]. \end{cases} \tag{29}$$

Because $|c|^2 \geq 0$ for any complex number c, only those values of (θ,τ,σ) which make the R.H.S. of Eq.(29)≥ 0 are permitted, otherwise they are forbidden. We

denote the permitted regions in (θ, τ, σ) space required by c^+ and c^- as V^+ and V^-, respectively:

$$\begin{cases} V^+ : (\tau + q^2\sigma)(\tau - \sigma) - \theta \geq 0, \\ V^- : (\tau + q^{-2}\sigma)(\tau - \sigma) - \theta \geq 0, \end{cases} \tag{30}$$

and their boundaries (equality holds in Eq.(30)) by S^+ and S^-: $S^+ = \delta V^+$, $S^- = \delta V^-$. The final permitted region V is $V = V^+ \cap V^-$, and its boundary $S = \delta V$ is partly S^+ and partly S^-. The surfaces S^+ and S^- are q-deformed saddle surfaces in the (θ, τ, σ) space, and V^+ and V^- is the inner or outer zones bounded by the surfaces. On a plane with a fixed θ, the boundary S^+ (or S^-) becomes a pair of hyperbolas, except at $\theta = 0$ where it is a pair of straight lines. When $q = 1$, V^+ and V^- are the same, therefore $V = V^+ = V^-$ and $S = S^+ = S^-$. But for $q > 1$, V^+ and V^- are different, and we find that for $\theta \leq 0$ the boundary is S^+ at both (left and right) sides and for $\theta > 0$ the boundary is half S^+ and half S^-.

Furthermore, when $q \neq 1$ *not all* points in the permitted zone do really appear. If we apply x^+ (or x^-) on the boundary, because

$$\begin{cases} x^+|\theta, \tau, \sigma\rangle \,|_{S^+} = 0, \\ x^-|\theta, \tau, \sigma\rangle \,|_{S^-} = 0, \end{cases} \tag{31}$$

x^+ (x^-) annihilates those states on S^+ (S^-), hence it cannot move them. On the contrary application of x^+ (x^-) on S^- (S^+) will result in a new surface. Therefore if the boundaries are acted successively by their counter-mover infinite times, they will give some families of surfaces. Only the *intersections* of those surfaces resulted from the shift of the *real* boundary represent the permitted states.

There are three cases in different regions, taking $q > 1$ as examples.

(1) $\theta = 0$ (see Fig.1)

Real boundary is $\tau = \sigma$ (of both S^+ and S^-) and $\tau = -q^2\sigma$ (of S^+). The former can be moved neither by x^+ nor by x^-. The latter can be moved by x^-, and it gives a family of straight lines in the plane:

$$\sigma = (1 - q^{-2n} - q^{-2n-2})\tau. \qquad n = 0, 1, 2, \cdots. \tag{32}$$

The line $\sigma = \tau$ is the asymptotic limit of this family when $n \to \infty$.

(2) $\theta < 0$ (see Fig.2)

Real boundary is S^+ at both sides (a pair of hyperbolas in the plane $\theta =$ const.(< 0)), and it can be moved by x^-. This gives a family of hyperbolas :

$$[\sigma + (q^{-2n} + q^{-2n-2} - 1)\tau](\sigma - \tau) + q^{-4n-2}\theta = 0, \qquad (n = 0, 1, 2, \cdots) \tag{33}$$

and they have asymptotic lines $\sigma = \tau$ and $\sigma = (1 - q^{-2n} - q^{-2n-2})\tau$, which are just ones in the plane $\theta = 0$.

(3) $\theta > 0$ (see Fig.3)

In this case the boundary is half S^+ and half S^-. Shifts of S^+ and S^- give different families of surfaces, and only their intersections represent permitted points of states. At the section of θ =const.(> 0), those points form a lattice in the $\sigma \sim \tau$ plane, with the following eigenvalues:

$$\text{in the upper half plane } (\tau > 0) \quad \begin{cases} \sigma = (\frac{q^{n+1}+q^{-n-1}}{q+q^{-1}} - q^m)\sqrt{\theta}, \\ \tau = (\frac{q^{n+1}+q^{-n-1}}{q+q^{-1}})\sqrt{\theta}, \end{cases} \tag{34}$$

the lower half plane ($\tau < 0$) is symmetric w.r.t. the origin,

where $n = 0, 1, 2, \cdots$; $m = -n, -n+2, \cdots, n-2, n$. For a given n we have $(n+1)$ nodes on the horizontal line (τ =const.). Also when going to infinity those points asymptotically fall on the straight lines at the section $\theta = 0$.

If q keeps greater than 1 and goes up, the permitted points become more scattered (see Figs.4–6). And if $q < 1$, the permitted points are exactly the same as those for q^{-1} (> 1).

3. Conclusion and Discussion

We see that in the Hilbert space representation of quantum space-time its apparent number of dimensions becomes $1 + 1$ (x^0 and x^3), and other two dimensions (x^+ and x^-) seem to be hidden. This representation is discrete in view of its eigenvalue space, even becomes a lattice in the "time-like" region ($J > 0$).

Also the above conclusion holds for other quantum (q-deformed) 4-vectors in the quantum Minkowski space, say the 4-momentum, because they obey the same commutation relation. Therefore in the quantum space-time the observable number of components of energy-momentum becomes two and their values discrete. Quantitatively they rely on the deformation parameter q, which is also observable in principle. Because the quantization of space-time depends on the scale of the

relevant phenomena, this parameter is presumably a running quantity.

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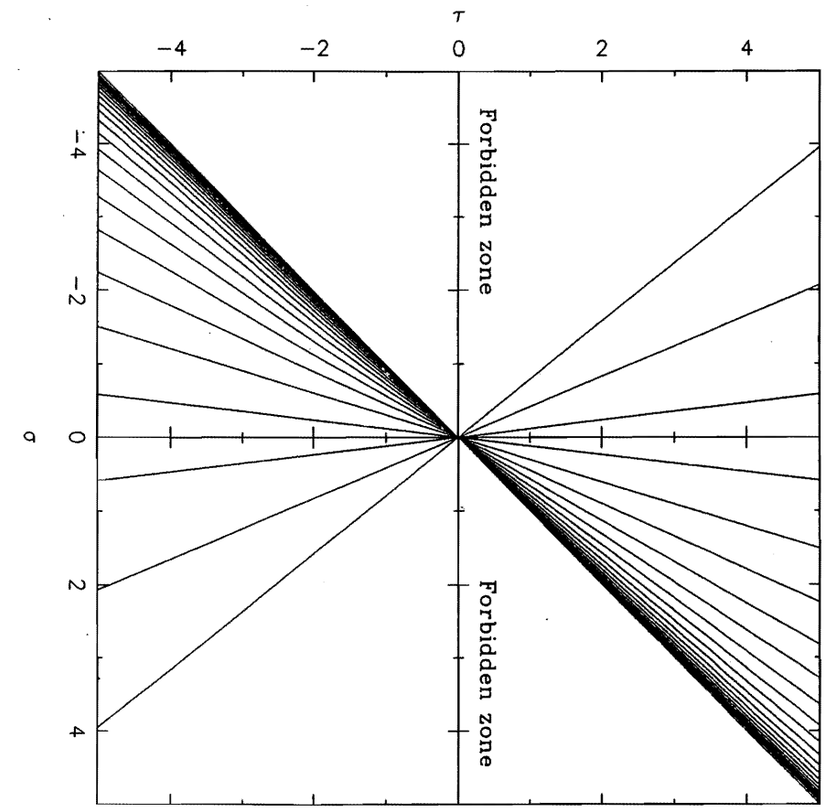

Fig1. Section of quantized space−time at θ=0. Deformation parameter q=1.125.

Fig3. Section of quantized space-time at $\theta=+1$.
Deformation parameter $q=1.125$.

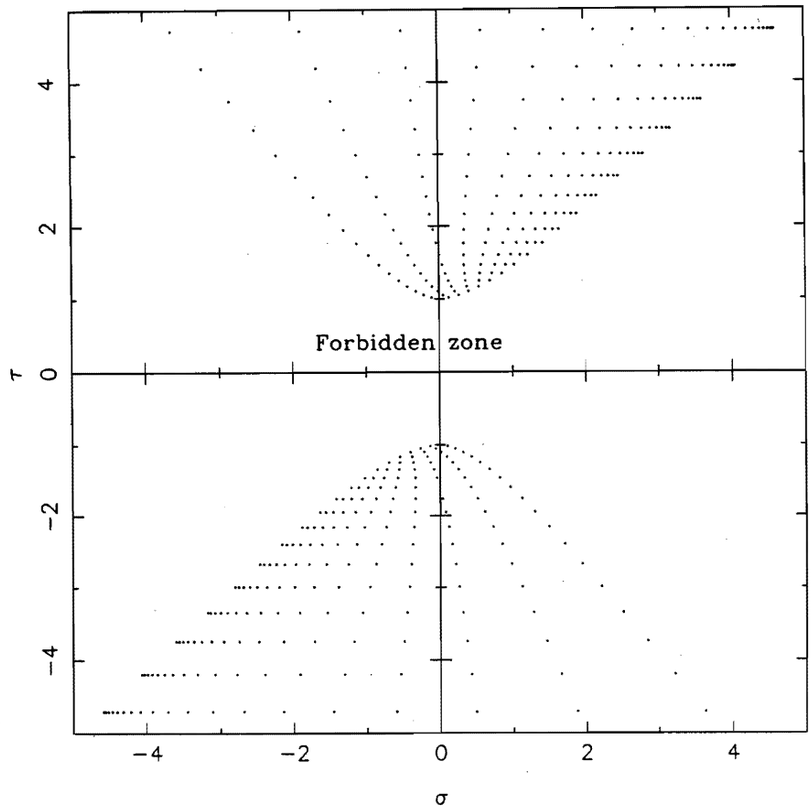

Fig2. Section of quantized space-time at $\theta=-1$.
Deformation parameter $q=1.125$.

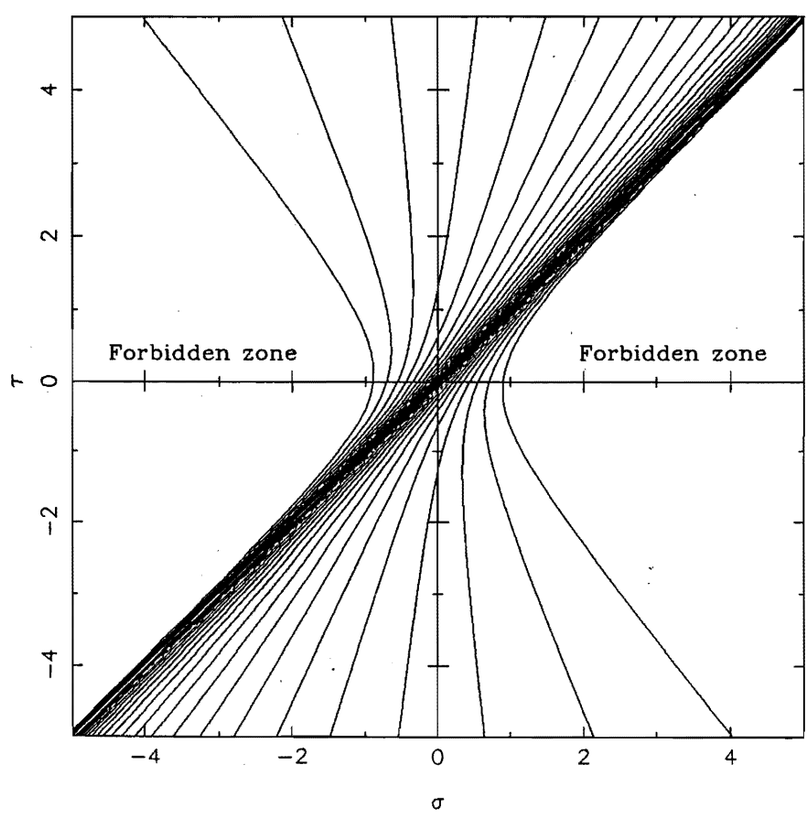

Fig5. Section of quantized space-time at $\theta=-1$.
Deformation parameter $q=1.25$.

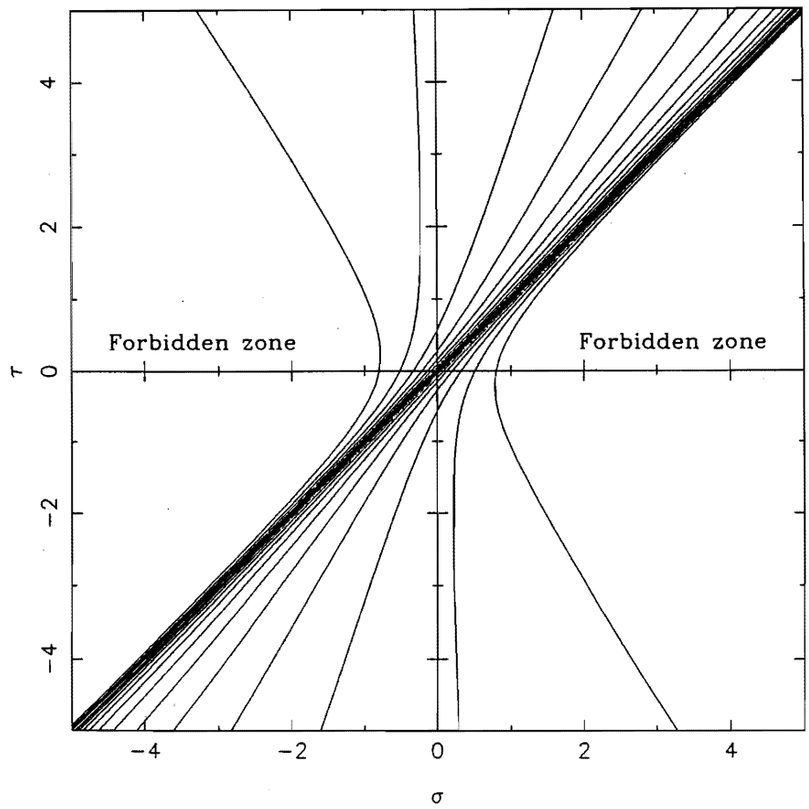

Fig4. Section of quantized space-time at $\theta=0$.
Deformation parameter $q=1.25$.

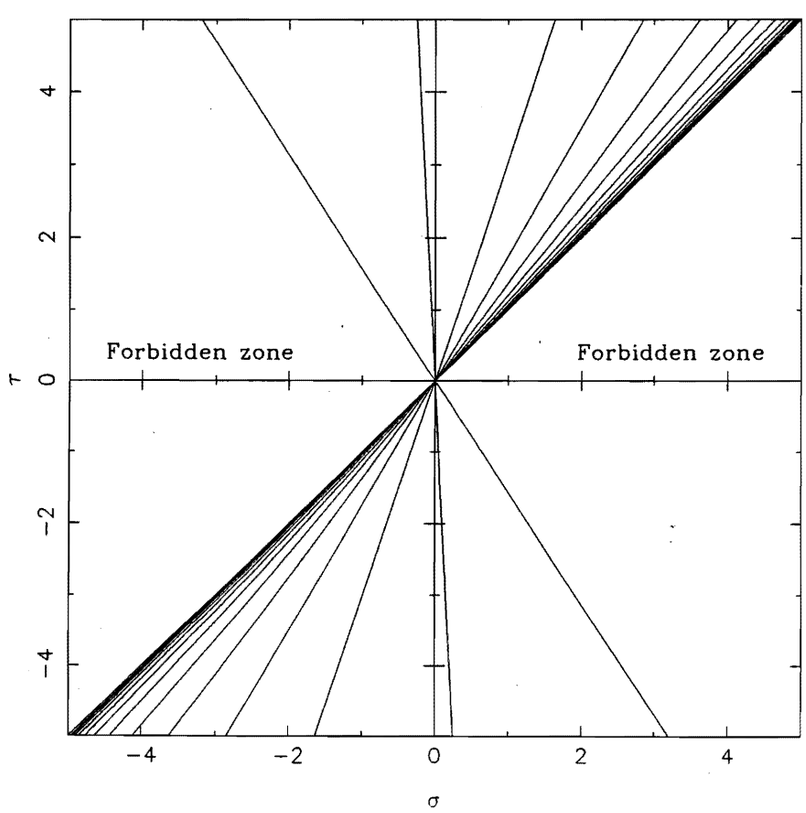
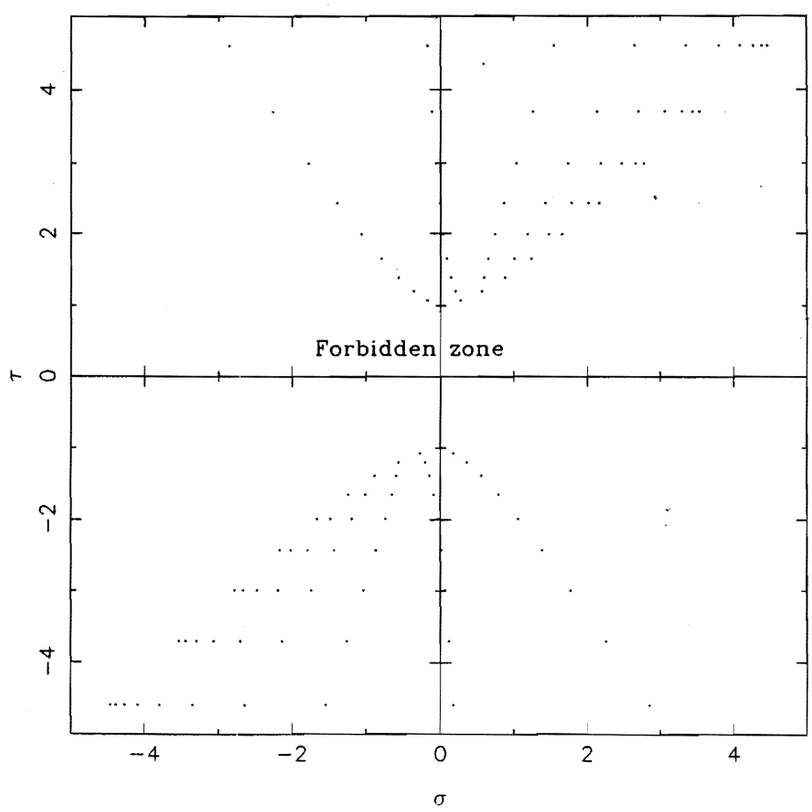
Fig6. Section of quantized space−time at $\theta=+1$.
Deformation parameter $q=1.25$.